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Review Article

A Priori Bounds in L^p and in $W^{2,p}$ for Solutions of Elliptic Equations

Sara Monsurrò and Maria Transirico

Dipartimento di Matematica, Università di Salerno, Via Ponte Don Melillo, 84084 Fisciano, Italy

Correspondence should be addressed to Sara Monsurrò; smonsurro@unisa.it

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We give an overview on some recent results concerning the study of the Dirichlet problem for second-order linear elliptic partial differential equations in divergence form and with discontinuous coefficients, in unbounded domains. The main theorem consists in an L^p-a priori bound, $P > 1$. Some applications of this bound in the framework of non-variational problems, in a weighted and a non-weighted case, are also given.

1. Introduction

The aim of this work is to give an overview on some recent results dealing with the study of a certain kind of the Dirichlet problem in the framework of unbounded domains. To be more precise, given an unbounded open subset Ω of $\mathbb{R}^n$, $n \geq 2$, we are concerned with the elliptic second-order linear differential operator in variational form

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i} + d_j\right) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c, \tag{1}$$

with coefficients $a_{ij} \in L^\infty(\Omega)$ and with the associated Dirichlet problem

$$\begin{gathered} u \in \mathring{W}^{1,2}(\Omega), \\ Lu = f, \quad f \in W^{-1,2}(\Omega). \end{gathered} \tag{2}$$

As far as we know, were Bottaro and Marina the first to approach this kind of problem who proved, in [1], an existence and uniqueness theorem for the solution of problem (2), for $n \geq 3$, assuming that

$$a_{ij} \in L^\infty(\Omega), \quad i, j = 1, \ldots, n, \tag{3}$$

$$\begin{gathered} b_i, d_i \in L^n(\Omega), \quad i = 1, \ldots, n, \\ c \in L^{n/2}(\Omega) + L^\infty(\Omega), \end{gathered} \tag{4}$$

$$c - \sum_{i=1}^{n} (d_i)_{x_i} \geq \mu, \quad \mu \in \mathbb{R}_+. \tag{5}$$

The study was later on generalized in [2] weakening the hypothesis (4) by considering coefficients b_i, d_i, and c satisfying (4) only locally and for $n \geq 2$. Further improvements have been achieved in [3], for $n \geq 3$, since the b_i, d_i, and c are taken in suitable Morrey type spaces with lower summabilities.

In [1–3], the authors also provide the bound

$$\|u\|_{W^{1,2}(\Omega)} \leq C\|f\|_{W^{-1,2}(\Omega)}, \tag{6}$$

giving explicit description of the dependence of the constant C on the data of the problem.

In two recent works, [4, 5], considering a more regular set Ω and supposing that the lower order terms coefficients are as in [3] for $n \geq 3$ and as in [2] for $n = 2$, we prove that if $f \in L^2(\Omega) \cap L^\infty(\Omega)$, then there exists a constant C, whose dependence is completely described, such that

$$\|u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \tag{7}$$

for any bounded solution u of (2) and for every $p > 2$. This can be done taking into account two different sign hypotheses, namely, (5) and the less common

$$c - \sum_{i=1}^{n} (b_i)_{x_i} \geq \mu, \quad \mu \in \mathbb{R}_+. \tag{8}$$

Successively, in [6], we deepen the study begun in [4, 5] showing that to a bounded datum $f \in L^2(\Omega)$ it corresponds a bounded solution u. This allows us to prove, by means of an approximation argument, that if f belongs to $L^2(\Omega) \cap L^p(\Omega)$, $p > 2$, then the solution is in $L^p(\Omega)$ too and verifies (7). Putting together the two preliminary L^p-estimates, $p > 2$, obtained under the different sign assumptions and adding the further hypothesis that the a_{ij} are also symmetric, by means of a duality argument, we finally obtain (7) for $p > 1$, for each sign hypothesis, assuming no boundedness of the solution and for $f \in L^2(\Omega) \cap L^p(\Omega)$.

To conclude, we provide two applications of our final L^p-bound, $p > 1$, recalling the results of [7, 8] where our estimate plays a fundamental role in the study of certain weighted and non-weighted non-variational problems with leading coefficients satisfying hypotheses of Miranda's type (see [9]). The nodal point in this analysis is the existence of the derivatives of the leading coefficients that allows us to rewrite the involved operator in variational form and avail ourselves of the above-mentioned a priori bound.

Always in the framework of unbounded domains, the study of different variational problems can be found in [10, 11]. Quasilinear elliptic equations with quadratic growth have been considered in [12]. In [13–15] a very general weighted case, with principal coefficients having vanishing mean oscillation, has been taken into account.

2. A Class of Spaces of Morrey Type

In this section we recall the definitions and the main properties of a certain class of spaces of Morrey type where the coefficients of our operators belong. These spaces generalize the classical notion of Morrey spaces to unbounded domains and were introduced for the first time in [3]; see also [16] for some details. Thus, from now on, let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$. By $\Sigma(\Omega)$ we denote the σ-algebra of all Lebesgue measurable subsets of Ω. For $E \in \Sigma(\Omega)$, χ_E is its characteristic function, $|E|$ its Lebesgue measure, and $E(x,r) = E \cap B(x,r)$ $(x \in \mathbb{R}^n,\ r \in \mathbb{R}_+)$, where $B(x,r)$ is the open ball with center in x and radius r. The class of restrictions to $\overline{\Omega}$ of functions $\zeta \in C^\infty_\circ(\mathbb{R}^n)$ is $\mathfrak{D}(\overline{\Omega})$. For $q \in [1,+\infty[$, $L^q_{\mathrm{loc}}(\overline{\Omega})$ is the class of all functions $g : \Omega \to \mathbb{R}$ such that $\zeta\, g \in L^q(\Omega)$ for any $\zeta \in \mathfrak{D}(\overline{\Omega})$.

For $q \in [1,+\infty[$ and $\lambda \in [0,n[$, the space of Morrey type $M^{q,\lambda}(\Omega)$ is made up of all the functions g in $L^q_{\mathrm{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^{q,\lambda}(\Omega)} = \sup_{\substack{\tau \in]0,1] \\ x \in \Omega}} \tau^{-\lambda/q} \|g\|_{L^q(\Omega(x,\tau))} < +\infty, \tag{9}$$

equipped with the norm defined in (9).

The closures of $C^\infty_\circ(\Omega)$ and $L^\infty(\Omega)$ in $M^{q,\lambda}(\Omega)$ are denoted by $M^{q,\lambda}_\circ(\Omega)$ and $\widetilde{M}^{q,\lambda}(\Omega)$, respectively.

The following inclusion holds true:

$$M^{q,\lambda}_\circ(\Omega) \subset \widetilde{M}^{q,\lambda}(\Omega). \tag{10}$$

Moreover,

$$M^{q,\lambda}(\Omega) \subseteq M^{q_0,\lambda_0}(\Omega) \quad \text{if } q_0 \leq q, \qquad \frac{\lambda_0 - n}{q_0} \leq \frac{\lambda - n}{q}. \tag{11}$$

We put $M^q(\Omega) = M^{q,0}(\Omega)$, $\widetilde{M}^q(\Omega) = \widetilde{M}^{q,0}(\Omega)$, and $M^q_\circ(\Omega) = M^{q,0}_\circ(\Omega)$.

Now, let us define the moduli of continuity of functions belonging to $\widetilde{M}^{q,\lambda}(\Omega)$ or $M^{q,\lambda}_\circ(\Omega)$. For $h \in \mathbb{R}_+$ and $g \in M^{q,\lambda}(\Omega)$, we set

$$F[g](h) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x\in\Omega} |E(x,1)| \leq 1/h}} \|g\chi_E\|_{M^{q,\lambda}(\Omega)}. \tag{12}$$

Given a function $g \in M^{q,\lambda}(\Omega)$, the following characterizations hold:

$$g \in \widetilde{M}^{q,\lambda}(\Omega) \Longleftrightarrow \lim_{h\to+\infty} F[g](h) = 0,$$
$$g \in M^{q,\lambda}_\circ(\Omega) \Longleftrightarrow \lim_{h\to+\infty} \left(F[g](h) + \|(1-\zeta_h)g\|_{M^{q,\lambda}(\Omega)}\right) = 0, \tag{13}$$

where ζ_h denotes a function of class $C^\infty_o(R^n)$ such that

$$0 \leq \zeta_h \leq 1, \qquad \zeta_h|_{\overline{B(0,h)}} = 1, \qquad \operatorname{supp} \zeta_h \subset B(0,2h). \tag{14}$$

Thus, if g is a function in $\widetilde{M}^{q,\lambda}(\Omega)$, a *modulus of continuity* of g in $\widetilde{M}^{q,\lambda}(\Omega)$ is a map $\widetilde{\sigma}^{q,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) \leq \widetilde{\sigma}^{q,\lambda}[g](h), \qquad \lim_{h\to+\infty} \widetilde{\sigma}^{q,\lambda}[g](h) = 0. \tag{15}$$

While if g belongs to $M^{q,\lambda}_o(\Omega)$, a *modulus of continuity* of g in $M^{q,\lambda}_o(\Omega)$ is an application $\sigma^{q,\lambda}_o[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) + \|(1-\zeta_h)g\|_{M^{q,\lambda}(\Omega)} \leq \sigma^{q,\lambda}_o[g](h), \qquad \lim_{h\to+\infty} \sigma^{q,\lambda}_o[g](h) = 0. \tag{16}$$

We finally recall two results of [4, 7], obtained adapting to our needs a more general theorem proved in [17], providing the boundedness and some embedding estimates for the multiplication operator

$$u \longrightarrow gu, \tag{17}$$

where the function g belongs to suitable spaces of Morrey type.

Theorem 1. *If $g \in M^{q,\lambda}(\Omega)$, with $q > 2$ and $\lambda = 0$ if $n = 2$, and $q \in\,]2, n]$ and $\lambda = n - q$ if $n > 2$, then the operator in* (17) *is bounded from $\overset{\circ}{W}{}^{1,2}(\Omega)$ to $L^2(\Omega)$. Moreover, there exists a constant $C \in \mathbb{R}_+$ such that*

$$\|gu\|_{L^2(\Omega)} \leq C\|g\|_{M^{q,\lambda}(\Omega)}\|u\|_{W^{1,2}(\Omega)} \quad \forall u \in \overset{\circ}{W}{}^{1,2}(\Omega), \quad (18)$$

with $C = C(n, q)$.

Let $p > 1$ and $r, t \in [p, +\infty[$. If Ω is an open subset of $\mathbb{R}^n$ having the cone property and $g \in M^r(\Omega)$, with $r > p$ if $p = n$, then the operator in (17) *is bounded from $W^{1,p}(\Omega)$ to $L^p(\Omega)$. Moreover, there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|gu\|_{L^p(\Omega)} \leq c\|g\|_{M^r(\Omega)}\|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega), \quad (19)$$

with $c = c\ (\Omega, n, p, r)$.

If $g \in M^t(\Omega)$, with $t > p$ if $p = n/2$, then the operator in (17) *is bounded from $W^{2,p}(\Omega)$ to $L^p(\Omega)$. Moreover, there exists a constant $c' \in \mathbb{R}_+$ such that*

$$\|gu\|_{L^p(\Omega)} \leq c'\|g\|_{M^t(\Omega)}\|u\|_{W^{2,p}(\Omega)} \quad \forall u \in W^{2,p}(\Omega) \quad (20)$$

with $c' = c'(\Omega, n, p, t)$.

3. The Variational Problem

Consider, in an unbounded open subset Ω of $\mathbb{R}^n$, $n \geq 2$, the second-order linear differential operator in divergence form

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i} + d_j\right) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c. \quad (21)$$

Assume that the leading coefficients satisfy the hypotheses

$$\begin{gathered} a_{ij} \in L^\infty(\Omega), \quad i, j = 1, \ldots, n, \\ \exists \nu > 0 : \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{a.e. in } \Omega,\ \forall \xi \in \mathbb{R}^n. \end{gathered} \quad (h_1)$$

For the lower order terms coefficients suppose that

$$\begin{gathered} b_i, d_i \in M_o^{2t,\lambda}(\Omega), \quad i = 1, \ldots, n, \\ c \in M^{t,\lambda}(\Omega), \\ \text{with } t > 1 \text{ and } \lambda = 0 \text{ if } n = 2, \\ \text{with } t \in \left]1, \frac{n}{2}\right] \text{ and } \lambda = n - 2t \text{ if } n > 2. \end{gathered} \quad (h_2)$$

Furthermore, let one of the following sign assumptions hold true:

$$c - \sum_{i=1}^{n} (d_i)_{x_i} \geq \mu, \quad (h_3)$$

or

$$c - \sum_{i=1}^{n} (b_i)_{x_i} \geq \mu, \quad (h_4)$$

in the distributional sense on Ω, with μ positive constant.

We are interested in the study of the Dirichlet problem

$$\begin{gathered} u \in \overset{\circ}{W}{}^{1,2}(\Omega), \\ Lu = f, \quad f \in W^{-1,2}(\Omega), \end{gathered} \quad (22)$$

(h_1)–(h_3) or (h_1), (h_2), and (h_4) being satisfied.

It is natural to associate to L the bilinear form

$$a(u, v) = \int_\Omega \left(\sum_{i,j=1}^{n}\left(a_{ij}u_{x_i} + d_j u\right)v_{x_j} + \left(\sum_{i=1}^{n} b_i u_{x_i} + cu\right)v\right)dx, \quad (23)$$

$u, v \in \overset{\circ}{W}{}^{1,2}(\Omega)$, and observe that, in view of Theorem 1, the form a is continuous on $\overset{\circ}{W}{}^{1,2}(\Omega) \times \overset{\circ}{W}{}^{1,2}(\Omega)$ and so the operator $L : \overset{\circ}{W}{}^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is continuous too.

Let us start collecting some preliminary results concerning the existence and uniqueness of the solution of problem (22), as well as some a priori estimates. For the case where assumptions (h_1)–(h_3) are taken into account and for $n = 2$, we refer to [2] while for $n \geq 3$ details can be found in [3]. If (h_1), (h_2), and (h_4) hold true, the results are proved in the more recent [5].

Theorem 2. *Under hypotheses (h_1)–(h_3) (or (h_1), (h_2), and (h_4)), problem* (22) *is uniquely solvable and its solution u satisfies the estimate*

$$\|u\|_{W^{1,2}(\Omega)} \leq C\|f\|_{W^{-1,2}(\Omega)}, \quad (24)$$

where C is a constant depending on $n, t, \nu, \mu, \|b_i - d_i\|_{M^{2t,\lambda}(\Omega)}$, $i = 1, \ldots, n$.

The next step in our analysis is to achieve an L^p-estimate, $p > 2$, for the solution of (22) (see Theorem 8). This requires some additional hypotheses on the regularity of the set and on the datum f, and some preparatory results that essentially rely on the introduction of certain auxiliary functions u_s, used for the first time by Bottaro and Marina in [1] and employed in the framework of Morrey type spaces in [3]. Let us give their definition and recall some useful properties.

Let $h \in \mathbb{R}_+ \cup \{+\infty\}$ and $k \in \mathbb{R}$, with $0 \leq k \leq h$. For each $t \in \mathbb{R}$ we set

$$\begin{gathered} G_{kh}(t) = \begin{cases} t - k & \text{if } t > k, \\ 0 & \text{if } -k \leq t \leq k, \\ t + k & \text{if } t < -k, \end{cases} \quad \text{if } h = +\infty, \\ G_{kh}(t) = G_{k\infty}(t) - G_{h\infty}(t), \quad \text{if } h \in \mathbb{R}_+. \end{gathered} \quad (25)$$

Lemma 3. *Let $g \in M_o^{q,\lambda}(\Omega)$, $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$, and $\varepsilon \in \mathbb{R}_+$. Then there exist $r \in \mathbb{N}$ and $k_1, \ldots, k_r \in \mathbb{R}$, with $0 = k_r < k_{r-1} < \cdots < k_1 < k_0 = +\infty$, such that set*

$$u_s = G_{k_s k_{s-1}}(u), \quad s = 1, \ldots, r, \quad (26)$$

one has $u_1, \ldots, u_r \in \overset{\circ}{W}{}^{1,2}(\Omega)$ *and*

$$\left\| g\chi_{\operatorname{supp}(u_s)_x} \right\|_{M^{q,\lambda}(\Omega)} \leq \varepsilon, \quad s = 1, \ldots, r, \tag{27}$$

$$|u_s| \leq |u|, \quad s = 1, \ldots, r, \tag{28}$$

$$u_1 + \cdots + u_r = u, \tag{29}$$

$$r \leq c, \tag{30}$$

with $c = c(\varepsilon, q, \|g\|_{M^{q,\lambda}(\Omega)})$ *positive constant.*

In order to prove a fundamental preliminary estimate, obtained for $p > 2$ (see Theorem 7), we need to take products involving the above defined functions u_s as test functions in the variational formulation of our problem (23). To be more precise, in the first set of hypotheses ((h_1)–(h_3)), the test functions needed are $|u|^{p-2}u_s$. The following result ensures that these functions effectively belong to $\overset{\circ}{W}{}^{1,2}(\Omega)$.

Lemma 4. *If* Ω *has the uniform* C^1*-regularity property, then for every* $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$ *and for any* $p \in]2, +\infty[$ *one has*

$$|u|^{p-2} u_s \in \overset{\circ}{W}{}^{1,2}(\Omega), \quad s = 1, \ldots, r. \tag{31}$$

Lemma 4, whose rather technical proof can be found in [4], is a generalization of a known result by Stampacchia (see [18], or [19] for details), obtained within the framework of the generalization of the study of certain elliptic equations in divergence form with discontinuous coefficients on a bounded open subset of $\mathbb{R}^n$ to some problems arising for harmonic or subharmonic functions in the theory of potential.

Once achieved (31), always in [4], we could prove the next lemma. Let u_s be the functions of Lemma 3 obtained in correspondence of a given $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$, of $g = \sum_{i=1}^n |b_i - d_i|$ and of a positive real number ε specified in the proof of Lemma 4.1 of [4]. One has the following.

Lemma 5. *Let* a *be the bilinear form defined in* (23)*. If* Ω *has the uniform* C^1*-regularity property, under hypotheses* (h_1)–(h_3)*, there exists a constant* $C \in \mathbb{R}_+$ *such that*

$$\int_\Omega |u|^{p-2} \left((u_s)_x^2 + u_s^2 \right) dx \leq C \sum_{h=1}^{s} a\left(u, |u|^{p-2} u_h\right), \qquad s = 1, \ldots, r, \quad \forall p \in]2, +\infty[, \tag{32}$$

where C *depends on* s, ν, μ.

If we consider the second set of hypotheses ((h_1), (h_2), and (h_4)), the test functions required in (23) are the products $|u_s|^{p-2}u_s$, obtained in correspondence of a fixed $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$, of $g = \sum_{i=1}^n |d_i - b_i|$ and of a positive real number ε specified in the proof of Lemma 4.1 of [5]. In this last case and if Ω has the uniform C^1-regularity property, a result of [20] applies giving that $|u_s|^{p-2}u_s \in \overset{\circ}{W}{}^{1,2}(\Omega)$, for any $p > 2$, $s = 1, \ldots, r$. Hence, in [5] we could show the result.

Lemma 6. *Let* a *be the bilinear form in* (23)*. If* Ω *has the uniform* C^1*-regularity property, under hypotheses* (h_1)*,* (h_2)*, and* (h_4)*, there exists a constant* $C \in \mathbb{R}_+$ *such that*

$$\int_\Omega |u_s|^{p-2} \left((u_s)_x^2 + u_s^2 \right) dx \leq C \sum_{h=s}^{r} a\left(u, |u_h|^{p-2} u_h\right), \qquad s = 1, \ldots, r, \ \forall p \in]2, +\infty[, \tag{33}$$

where C *depends on* s, r, ν, μ.

The two lemmas just stated put us in a position to prove the following preliminary L^p-a priori estimate, $p > 2$, in both sets of hypotheses; see also [4, 5]. We stress that here we require that both the datum f and the solution u are bounded.

Theorem 7. *Under hypotheses* (h_1)–(h_3) *or* (h_1)*,* (h_2)*, and* (h_4) *and if* Ω *has the uniform* C^1*-regularity property,* f *is in* $L^2(\Omega) \cap L^\infty(\Omega)$ *and the solution* u *of* (22) *is in* $\overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$*, then* $u \in L^p(\Omega)$ *and*

$$\|u\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}, \quad \forall p \in]2, +\infty[, \tag{34}$$

where C *is a constant depending on* $n, t, p, \nu, \mu, \|b_i - d_i\|_{M^{2t,\lambda}(\Omega)}$*,* $i = 1, \ldots, n$.

Proof. Fix $p \in]2, +\infty[$. We provide two different proofs in the cases that hypotheses (h_3) or (h_4) hold true.

Let (h_1)–(h_3) be satisfied. We consider the functions u_s, $s = 1, \ldots, r$, obtained in correspondence of the solution u and of $g = \sum_{i=1}^n |d_1 - b_i|$ and of ε as in Lemma 4.1 of [4]. In view of (29) we get

$$\int_\Omega |u|^{p-2} \left(u_x^2 + u^2 \right) dx \leq c_0 \int_\Omega |u|^{p-2} \sum_{s=1}^{r} \left((u_s)_x^2 + u_s^2 \right) dx, \tag{35}$$

with $c_0 = c_0(r)$.

Hence, (32) entails that

$$\int_\Omega |u|^{p-2} \left(u_x^2 + u^2 \right) dx \leq c_0 \sum_{s=1}^{r} C_s \sum_{h=1}^{s} a\left(u, |u|^{p-2} u_h\right) \leq C \sum_{s=1}^{r} a\left(u, |u|^{p-2} u_s\right), \tag{36}$$

with $C_s = C_s(s, \nu, \mu)$ and $C = C(r, \nu, \mu)$.

From the linearity of a, (29), and (30), we have then

$$\int_\Omega |u|^{p-2} \left(u_x^2 + u^2 \right) dx \leq C a\left(u, |u|^{p-2} u\right), \tag{37}$$

with $C = C(n, t, p, \nu, \mu, \|b_i - d_i\|_{M^{2t,\lambda}(\Omega)})$.

Using this last inequality and Hölder inequality we conclude our proof, since

$$\begin{aligned}\|u\|^p_{L^p(\Omega)} &\le \int_\Omega |u|^{p-2}\left(u_x^2+u^2\right)dx\\ &\le Ca\left(u,|u|^{p-2}u\right)=C\int_\Omega f|u|^{p-2}u\,dx\\ &\le C\int_\Omega |f|\,|u|^{p-1}dx \le C\|f\|_{L^p(\Omega)}\|u\|^{p-1}_{L^p(\Omega)}.\end{aligned}\tag{38}$$

If (h_1), (h_2), and (h_4) hold, we consider again the functions u_s, $s=1,\dots,r$, obtained in correspondence of the solution u and of g as in the previous case, and of ε as in Lemma 4.1 of [5]. In this second case, easy computations together with (29) give

$$\int_\Omega |u|^p dx \le \bar{c}_0\sum_{s=1}^{r}\int_\Omega |u_s|^p dx,\tag{39}$$

with $\bar{c}_0=\bar{c}_0(r,p)$.

Thus, from (33), we deduce that

$$\begin{aligned}\int_\Omega |u|^p dx &\le \bar{c}_0\sum_{s=1}^{r}\overline{C}_s\sum_{h=s}^{r}a\left(u,|u_h|^{p-2}u_h\right)\\ &\le \bar{c}_1\sum_{s=1}^{r}a\left(u,|u_s|^{p-2}u_s\right),\end{aligned}\tag{40}$$

with $\overline{C}_s=\overline{C}_s(s,r,\nu,\mu)$ and $\bar{c}_1=\bar{c}_1(r,p,\nu,\mu)$.

Hence, by (28) and Hölder inequality we obtain

$$\begin{aligned}\|u\|^p_{L^p(\Omega)} &\le \bar{c}_1\sum_{s=1}^{r}\int_\Omega f|u_s|^{p-2}u_s dx\\ &\le r\bar{c}_1\int_\Omega |f|\,|u|^{p-1}dx\\ &\le r\bar{c}_1\|f\|_{L^p(\Omega)}\|u\|^{p-1}_{L^p(\Omega)}.\end{aligned}\tag{41}$$

This ends the proof, in view of (30). □

In the later paper [6], estimate (34) has been improved dropping the hypotheses on the boundedness of f and u, by means of the theorem below.

Theorem 8. *Assume that hypotheses (h_1)–(h_3) or (h_1), (h_2), and (h_4) are satisfied. If the set Ω has the uniform C^1-regularity property and the datum $f\in L^2(\Omega)\cap L^p(\Omega)$, for some $p\in\,]2,+\infty[$, then the solution u of problem (22) is in $L^p(\Omega)$ and*

$$\|u\|_{L^p(\Omega)}\le C\|f\|_{L^p(\Omega)},\tag{42}$$

where C is a constant depending on $n,t,p,\nu,\mu,\|b_i-d_i\|_{M^{2t,\lambda}(\Omega)}$, $i=1,\dots,n$.

The proof, which is different according to hypothesis (h_3) or (h_4), is essentially performed into two steps. In the first step, we show some regularity results, exploiting a technique introduced by Miranda in [21]. Namely, we prove that if $u\in\overset{\circ}{W}{}^{1,2}(\Omega)$ is the solution of (22) with $f\in L^2(\Omega)\cap L^\infty(\Omega)$, then, the datum f being more regular, one also has $u\in L^\infty(\Omega)$. Thus Theorem 7 applies giving that $u\in L^p(\Omega)$ and satisfies (34). The second step consists in considering a datum $f\in L^2(\Omega)\cap L^p(\Omega)$ and then one can conclude by means of some approximation arguments; see also [16].

Finally, in [6], we prove the main result, that is, the claimed L^p-bound, $p>1$. To this aim, a further assumption on the leading coefficients is required:

$$a_{ij}=a_{ji},\quad i,j=1,\dots,n.\tag{h_0}$$

Then one has the following.

Theorem 9. *Assume that hypotheses (h_0)–(h_3) or (h_0), (h_2), and (h_4) are satisfied. If the set Ω has the uniform C^1-regularity property and the datum $f\in L^2(\Omega)\cap L^p(\Omega)$, for some $p\in\,]1,+\infty[$, then the solution u of problem (22) is in $L^p(\Omega)$ and*

$$\|u\|_{L^p(\Omega)}\le C\|f\|_{L^p(\Omega)},\tag{43}$$

where C is a constant depending on $n,t,p,\nu,\mu,\|b_i-d_i\|_{M^{2t,\lambda}(\Omega)}$, $i=1,\dots,n$.

Proof. For $p\ge 2$, Theorems 2 and 8 already prove the result. It remains to show it for $1<p<2$.

We assume that hypotheses (h_0)–(h_3) hold true. Under hypotheses (h_0), (h_2), and (h_4), a similar argument, with suitable modifications, can be used (we refer the reader to [6] for the details).

Let us define the bilinear form

$$a^*(w,v)=a(v,w),\quad w,v\in\overset{\circ}{W}{}^{1,2}(\Omega).\tag{44}$$

By (h_0) one has

$$a^*(w,v)=\int_\Omega\left(\sum_{i,j=1}^{n}\left(a_{ij}w_{x_i}+b_jw\right)v_{x_j}+\left(\sum_{i=1}^{n}d_iw_{x_i}+cw\right)v\right)dx.\tag{45}$$

Now consider the problem

$$\begin{gathered}w\in\overset{\circ}{W}{}^{1,2}(\Omega),\\ a^*(w,v)=\int_\Omega gv\,dx,\quad g\in L^2(\Omega)\cap L^{p'}(\Omega),\end{gathered}\tag{46}$$

where, since $1<p<2$, one gets $p'=p/(p-1)>2$.

As a consequence of Theorem 2 (in the second set of hypotheses) the solution w of (46) exists and is unique. Furthermore, by Theorem 8 (in the second set of hypotheses) one also has

$$\|w\|_{L^{p'}(\Omega)}\le C\|g\|_{L^{p'}(\Omega)}.\tag{47}$$

Hence, if we denote by u the solution of

$$\begin{gathered}u\in\overset{\circ}{W}{}^{1,2}(\Omega),\\ Lu=f,\quad f\in L^2(\Omega)\cap L^p(\Omega),\end{gathered}\tag{48}$$

which exists and is unique in view of Theorem 2 (in the first set of hypotheses), we obtain

$$\int_\Omega gu\,dx = a^*(w,u) = a(u,w) = \int_\Omega fw\,dx \leq \|f\|_{L^p(\Omega)}\|w\|_{L^{p'}(\Omega)} \leq C\|f\|_{L^p(\Omega)}\|g\|_{L^{p'}(\Omega)}. \tag{49}$$

Finally, taking $g = |u|^{p-1}\operatorname{sign} u$ in (49), we get the claimed result. □

4. Non-Variational Problems

In this section, we show two applications of our main estimate (43).

To this aim, let $p > 1$ and assume that

$$\Omega \text{ has the uniform } C^{1,1}\text{-regularity property.} \tag{h_0'}$$

Consider, then, the non-variational differential operator

$$\overline{L} = -\sum_{i,j=1}^n a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a, \tag{50}$$

with the following conditions on the leading coefficients:

$$\begin{gathered} a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i,j = 1,\dots,n, \\ \exists \nu > 0 : \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{a.e in } \Omega,\ \forall \xi \in \mathbb{R}^n, \\ (a_{ij})_{x_h} \in M_o^{q,\lambda}(\Omega), \quad i,j,h = 1,\dots,n, \text{ with} \\ q > 2, \qquad \lambda = 0 \quad \text{for } n = 2, \\ q \in \,]2,n[, \qquad \lambda = n - q \quad \text{for } n > 2. \end{gathered} \tag{h_1'}$$

Suppose that the lower order terms are such that

$$\begin{gathered} a_i \in M_o^r(\Omega), \quad i = 1,\dots,n, \text{ with} \\ r > 2 \text{ if } p \leq 2, \qquad r = p \text{ if } p > 2 \quad \text{for } n = 2, \\ r \geq p,\ r \geq n, \text{ with } r > p \text{ if } p = n \quad \text{for } n > 2, \end{gathered} \tag{h_2'}$$

$$\begin{gathered} a \in \widetilde{M}^t(\Omega), \text{ with} \\ t = p \quad \text{for } n = 2, \\ t \geq p,\ t \geq \frac{n}{2}, \text{ with } t > p \text{ if } p = \frac{n}{2} \quad \text{for } n > 2, \\ \operatorname*{ess\,inf}_\Omega a = a_0 > 0. \end{gathered} \tag{h_3'}$$

In view of Theorem 1, under the assumptions (h_0')–(h_3'), the operator $\overline{L} : W^{2,p}(\Omega) \to L^p(\Omega)$ is bounded.

The first application is contained in Theorem 3.2 and Corollary 3.3 of [7] (see also [22] where the case $p = 2$ is considered) and reads as follows.

Theorem 10. *Let $\overline{L}$ be defined in* (50). *If hypotheses (h_0')–(h_3') are satisfied, then there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq c\|\overline{L}u\|_{L^p(\Omega)} \quad \forall u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \tag{51}$$

with $c = c(\Omega, n, \nu, p, r, t, \|a_{ij}\|_{L^\infty(\Omega)}, \sigma_o^{q,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \widetilde{\sigma}^t[a], a_0)$.

Moreover, the problem

$$\begin{gathered} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ \overline{L}u = f, \quad f \in L^p(\Omega) \end{gathered} \tag{52}$$

is uniquely solvable.

The nodal point in achieving these results consists in the existence of the derivatives of the a_{ij}. Indeed, this consents to rewrite the operator $\overline{L}$ in divergence form and exploit (43) in order to obtain an estimate as that in (51) but for more regular functions. Then, one can prove (51) by means of an approximation argument. Estimate (51) immediately takes to the solvability of problem (52) via a straightforward application of the method of continuity along a parameter, see, for instance, [23], and by the already known solvability of an opportune auxiliary problem.

As second application of (43), we obtain, in [8], an analogous of Theorem 10, in a weighted framework. Namely, we consider a weight function ρ^s that is a power of a function ρ of class $C^2(\overline{\Omega})$ such that $\rho : \overline{\Omega} \to \mathbb{R}_+$ and

$$\begin{gathered} \sup_{x\in\Omega} \frac{|\partial^\alpha \rho(x)|}{\rho(x)} < +\infty, \quad \forall |\alpha| \leq 2, \\ \lim_{|x|\to+\infty} \left(\rho(x) + \frac{1}{\rho(x)}\right) = +\infty, \\ \lim_{|x|\to+\infty} \frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0. \end{gathered} \tag{53}$$

For instance, one can think of ρ as the function

$$\rho(x) = \left(1 + |x|^2\right)^t, \quad t \in \mathbb{R} \setminus \{0\}. \tag{54}$$

For $k \in \mathbb{N}_0$, $p \in [1, +\infty[$ and $s \in \mathbb{R}$, and given ρ satisfying (53), we define the weighted Sobolev space $W_s^{k,p}(\Omega)$ as the space of distributions u on Ω such that

$$\|u\|_{W_s^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \|\rho^s \partial^\alpha u\|_{L^p(\Omega)} < +\infty, \tag{55}$$

endowed with the norm in (55). Furthermore, we denote the closure of $C_\circ^\infty(\Omega)$ in $W_s^{k,p}(\Omega)$ by $\overset{\circ}{W}{}_s^{k,p}(\Omega)$ and put $W_s^{0,p}(\Omega) = L_s^p(\Omega)$.

In Theorems 4.2 and 5.2 of [8] we showed the following.

Theorem 11. *Let $\overline{L}$ be defined in* (50). *If hypotheses (h_0')–(h_3') are satisfied, then there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c\|\overline{L}u\|_{L_s^p(\Omega)} \quad \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,2}(\Omega), \tag{56}$$

with $c = c(\Omega, n, s, \nu, p, r, t, \|a_{ij}\|_{L^\infty(\Omega)}, \|a_i\|_{M^r(\Omega)}, \sigma_o^{q,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \widetilde{\sigma}^t[a], a_0)$.

Moreover, the problem

$$u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,2}(\Omega), \qquad \overline{L}u = f, \quad f \in L_s^p(\Omega) \tag{57}$$

is uniquely solvable.

One of the main tools in the proof of Theorem 11 is given by the existence of a topological isomorphism from $W_s^{k,p}(\Omega)$ to $W^{k,p}(\Omega)$ and from $\overset{\circ}{W}{}_s^{k,p}(\Omega)$ to $\overset{\circ}{W}{}^{k,p}(\Omega)$. This isomorphism consents to deduce by the non-weighted bound in (51) the corresponding weighted estimate in (56), taking into account also the imbedding results of Theorem 1. The existence and uniqueness of the solution of problem (57) follow then, as in the previous case, from a direct application of the method of continuity along a parameter by the solvability of a suitable auxiliary problem.

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