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Research Article

Existence and Uniqueness of Solution to Nonlinear Boundary Value Problems with Sign-Changing Green's Function

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By using the cone theory and the Banach contraction mapping principle, the existence and uniqueness results are established for nonlinear higher-order differential equation boundary value problems with sign-changing Green's function. The theorems obtained are very general and complement previous known results.

1. Introduction

Boundary value problems (BVPs for short) for nonlinear differential equations arise in a variety of areas of applied mathematics, physics, and variational problems of control theory. The study of multipoint BVPs for second-order differential equations was initiated by Bicadze and Samarskiĭ [1] and later continued by II'in and Moiseev [2, 3] and Gupta [4]. Since then, great efforts have been devoted to nonlinear multipoint BVPs due to their theoretical challenge and great application potential. Many results on the existence of solutions for multipoint BVPs have been obtained; the methods used therein mainly depend on the fixed point theorems, degree theory, upper and lower techniques, and monotone iteration. The existence results are available in the literature [5–25] and the references therein.

Recently, by applying the fixed point theorems on cones, the authors of papers [5–7] established the existence and multiplicity of positive solutions for the nth-order three-point BVP:

$$u^{(n)}(t) + a(t) f(t, u(t)) = 0, \quad t \in (0,1),$$
$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \alpha u(\eta), \tag{1}$$

where $n \leq 2, 0 < \eta < 1$ and $0 < \alpha\eta^{n-1} < 1$. The nth-order m-point BVP

$$u^{(n)}(t) + a(t) f(t, u(t)) = 0, \quad t \in (0,1),$$
$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \tag{2}$$

has been studied in [8–10], where $n \geq 2$, $0 < \eta_1 < \eta_2 < \cdots \eta_{m-2} < 1$ and $\alpha_i > 0 (i = 1, 2, \ldots, m-2)$ with $0 < \sum_{i=1}^{m-2} \alpha_i \eta_i^{n-1} < 1$. The existence and multiplicity results of solutions were shown by using various fixed point theorems and fixed point index theory.

By using the cone theory and the Banach contraction mapping principle, the author [26] established the existence and uniqueness for singular third-order three-point boundary value problems.

The purpose of this paper is to investigate the existence and uniqueness of solution of the following higher-order differential equation boundary value problem:

$$u^{(n)}(t) + f\left(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\right) = 0, \qquad t \in J,$$
$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{3}$$

where $n \geq 2$, $f \in C(J \times \mathbb{R}^n, \mathbb{R})$, $J = (0,1)$, $\sum_{i=1}^{m-2} \alpha_i \eta_i^{n-1} \neq 1$, and $0 < \eta_1 < \cdots < \eta_{m-2} < 1$.

Here, we give the unique solution of BVP (3) under the conditions that f is mixed nonmonotone. The methods used in this paper are motivated by [26], and the arguments are based upon the cone theory and the Banach contraction mapping principle.

2. The Preliminary Lemmas

Lemma 1. *For any $f \in L(I)$, the BVP*

$$u'(t) + f(t) = 0, \quad t \in J, \tag{4}$$

$$\int_0^1 (1-t)^{n-2} u(t)\, dt = \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - t)^{n-2} u(t)\, dt \tag{5}$$

has a unique solution $u(t) = \int_0^1 G(t,s) f(s) ds$, where

$$G(t,s) = \begin{cases} -1 + \dfrac{1}{\sigma}\left[(1-s)^{n-1} - \sum_{i=1}^{m-2} \alpha_i (\eta_i - s)^{n-1}\right], & 0 \leq s \leq \eta_1,\ s \leq t, \\ \dfrac{1}{\sigma}\left[(1-s)^{n-1} - \sum_{i=1}^{m-2} \alpha_i (\eta_i - s)^{n-1}\right], & 0 \leq t \leq s \leq \eta_1, \\ -1 + \dfrac{1}{\sigma}\left[(1-s)^{n-1} - \sum_{i=j+1}^{m-2} \alpha_i (\eta_i - s)^{n-1}\right], & \eta_j \leq s \leq \eta_{j+1},\ s \leq t, \\ \dfrac{1}{\sigma}\left[(1-s)^{n-1} - \sum_{i=j+1}^{m-2} \alpha_i (\eta_i - s)^{n-1}\right], & \eta_j \leq s \leq \eta_{j+1},\ t \leq s, \\ -1 + \dfrac{(1-s)^{n-1}}{\sigma}, & \eta_{m-2} \leq s \leq t \leq 1, \\ \dfrac{(1-s)^{n-1}}{\sigma}, & \eta_{m-2} \leq s \leq 1,\ t \leq s, \end{cases}$$

$$\sigma = 1 - \sum_{i=1}^{m-2} \alpha_i \eta_i^{n-1}, \quad I = [0,1]. \tag{6}$$

Proof. First, suppose that $u \in C(I)$ is a solution to problem (4) and (5). It is easy to see by integration of (4) that

$$u(t) = u(0) - \int_0^t f(s)\, ds. \tag{7}$$

Substituting (7) into (5), we obtain

$$\begin{aligned} &\int_0^1 (1-t)^{n-2} \left[u(0) - \int_0^t f(s)\, ds\right] dt \\ &\quad = \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - t)^{n-2} \left[u(0) - \int_0^t f(s)\, ds\right] dt, \end{aligned} \tag{8}$$

and so

$$\begin{aligned} u(0) &= \left[\int_0^1 (1-t)^{n-2} dt - \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - t)^{n-2} dt\right]^{-1} \\ &\quad \times \left[\int_0^1 (1-t)^{n-2} \int_0^t f(s)\, ds\, dt - \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - t)^{n-2} \int_0^t f(s)\, ds\, dt\right] \\ &= \left(\frac{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i^{n-1}}{n-1}\right)^{-1} \\ &\quad \times \left[\int_0^1 (1-t)^{n-2} \int_0^t f(s)\, ds\, dt - \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - t)^{n-2} \int_0^t f(s)\, ds\, dt\right] \\ &= \frac{n-1}{\sigma} \left[\int_0^1 (1-t)^{n-2} \int_0^t f(s)\, ds\, dt - \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - t)^{n-2} \int_0^t f(s)\, ds\, dt\right] \\ &= \frac{n-1}{\sigma} \left[\int_0^1 f(s) \int_s^1 (1-t)^{n-2} dt\, ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} f(s) \int_s^{\eta_i} (\eta_i - t)^{n-2} dt\, ds\right] \\ &= \frac{1}{\sigma} \left[\int_0^1 f(s)(1-s)^{n-1} ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} f(s)(\eta_i - s)^{n-1} ds\right]. \end{aligned} \tag{9}$$

Substituting (9) into (7), we have

$$\begin{aligned} u(t) &= -\int_0^t f(s)\, ds \\ &\quad + \frac{1}{\sigma}\left[\int_0^1 f(s)(1-s)^{n-1} ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} f(s)(\eta_i - s)^{n-1} ds\right] \\ &= \int_0^1 G(t,s) f(s)\, ds. \end{aligned} \tag{10}$$

Conversely, suppose that $u(t) = \int_0^1 G(t,s) f(s) ds$; then it is easy to verify that (4) and (5) are satisfied. The lemma is proved. □

For any $u \in C(I)$, let

$$(I_i u)(t) = \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} u(s)\,ds, \quad i = 1, 2, \ldots, n-1,$$
$$(Fu)(t) = \int_0^1 G(t,s) f(s, (I_{n-1}u)(s), \ldots, (I_1 u)(s), u(s))\,ds, \quad t \in I. \tag{11}$$

Lemma 2. (i) *If $u \in C^{n-1}(I)$ is a solution to problem* (3), *then $v(t) = u^{(n-1)}(t) \in C(I)$ is a fixed point of F.*

(ii) *If $v \in C(I)$ is a fixed point of F, then $u(t) = (I_{n-1}v)(t) = \int_0^t ((t-s)^{n-2}/(n-2)!)\, v(s)ds \in C^{n-1}(I)$ is a solution to problem* (3).

By Lemma 1, the proof follows by routine calculations.

Let

$$h_1(t) = \max\left\{\int_0^1 |G(t,s)|\,ds, \int_0^t \int_0^1 |G(s,x)|\,dx\,ds\right\},$$
$$h_k(t) = \max\left\{\int_0^1 |G(t,s)|\,h_{k-1}(s)\,ds, \int_0^t \int_0^1 |G(s,x)|\,h_{k-1}(x)\,dx\,ds\right\}, \quad k = 2, 3, \ldots, \tag{12}$$
$$\rho(G) = \lim_{k\to\infty}\left(\sup_{t\in J} h_k(t)\right)^{-1/k}.$$

It is easy to see that $\rho(G) \geq (\sup_{t\in J} h_k(t))^{-1/k} \geq (\sup_{t,s\in J}|G(t,s)|)^{-1} > 0$.

Lemma 3 (see [27, 28]). *P is a generating cone in Banach space $(E, \|\cdot\|)$ if and only if there exists a constant $\tau > 0$ such that every element $u \in E$ can be represented in the form $u = v - w$, where $v, w \in P$ and $\|v\| \leq \tau\|u\|$, $\|w\| \leq \tau\|u\|$.*

3. Main Results

This section discusses the solution of nonlinear higher-order differential equation BVP (3).

Let $P = \{u \in C(I) \mid u(t) \geq 0, \text{for all } t \in [0,1]\}$. Obviously, P is a normal solid cone of Banach space $C(I)$, by Lemma 2.1.2 in [29], and we have that P is a generating cone in $C(I)$.

Theorem 4. *Suppose that $g \in C(J \times \mathbb{R}^{2n}, \mathbb{R})$, $f(t, x_0, x_1, \ldots, x_{n-1}) = g(t, x_0, x_0, x_1, x_1, \ldots, x_{n-1}, x_{n-1})$, and there exist positive constants $K_0, M_0, K_1, M_1, \ldots, K_{n-1}, M_{n-1}$ with*

$$\frac{K_0 + M_0}{(n-1)!} + \frac{K_1 + M_1}{(n-2)!} + \cdots + \frac{K_{n-3} + M_{n-3}}{2!} + K_{n-2} + M_{n-2} + K_{n-1} + M_{n-1} < \rho(G), \tag{13}$$

such that for any $t \in I$, $s_{01}, t_{01}, s_{02}, t_{02}, s_{11}, t_{11}, s_{12}, t_{12}, \ldots, s_{1,n-1}, t_{1,n-1}, s_{2,n-1}, t_{2,n-1} \in \mathbb{R}$ with $s_{01} \leq t_{01}, s_{02} \geq t_{02}, s_{11} \leq t_{11}, s_{12} \geq t_{12}, \ldots, s_{n-1,1} \leq t_{n-1,1}, s_{n-1,2} \geq t_{n-1,2}$, one has

$$\begin{aligned}
&-K_0(t_{01} - s_{01}) - M_0(s_{02} - t_{02}) - K_1(t_{11} - s_{11}) - M_1(s_{12} - t_{12}) - \cdots - K_{n-1}(t_{n-1,1} - s_{n-1,1}) - M_{n-1}(s_{n-1,2} - t_{n-1,2}) \\
&\quad \leq g(t, s_{01}, s_{02}, s_{11}, s_{12}, \ldots, s_{n-1,1}, s_{n-1,2}) - g(t, t_{01}, t_{02}, t_{11}, t_{12}, \ldots, t_{n-1,1}, t_{n-1,2}) \\
&\quad \leq -K_0(t_{01} - s_{01}) - M_0(s_{02} - t_{02}) - K_1(t_{11} - s_{11}) - M_1(s_{12} - t_{12}) - \cdots - K_{n-1}(t_{n-1,1} - s_{n-1,1}) - M_{n-1}(s_{n-1,2} - t_{n-1,2}),
\end{aligned} \tag{14}$$

and there exist $u_0, v_0 \in C^{n-1}(I)$, such that

$$\int_0^1 g\left(t, u_0(t), v_0(t), u_0'(t), v_0'(t), \cdots, u_0^{(n-1)}(t), v_0^{(n-1)}(t)\right) dt \tag{15}$$

converges. Then, BVP (3) *has a unique solution $I_{n-1}u^*$ in $C(I)$, and moreover, for any $u_0 \in C(I)$, the iterative sequence*

$$u_m(t) = \int_0^1 G(t,s) f(s, (I_{n-1}u_{m-1})(s), \ldots, (I_1 u_{m-1})(s), u_{m-1}(s))\,ds, \quad m = 1, 2, \ldots, \tag{16}$$

converges to u^* in $C(I)$ $(m \to \infty)$.

Remark 5. Recently, in the study of BVP (3), almost all the papers have supposed that Green's function $G(t,s)$ is nonnegative. However, the scope of α_i is not limited to $\sum_{i=1}^{m-2} \alpha_i \eta_i < 1$ in Theorem 4, so, we do not need to suppose that $G(t,s)$ is nonnegative.

Remark 6. The function f in Theorem 4 is not monotone or convex; the conclusions and the proof used in this paper are different from the known papers in essence.

Proof of Theorem 4. It is easy to see that, for any $t \in J$, $G(t,s)$ can be divided into finite partitioned monotone and bounded function on $(0,1)$, and then, by (15), we have that

$$\int_0^1 G(t,s)\, g\left(s, u_0(s), v_0(s), u_0'(s), v_0'(s), \ldots, u_0^{(n-1)}(s), v_0^{(n-1)}(s)\right) ds \tag{17}$$

converges. Let $p(t) = u_0^{(n-1)}(t)$, $q(t) = v_0^{(n-1)}(t)$; then

$$\int_0^1 G(t,s)\, g(s, (I_{n-1}p)(s), (I_{n-1}q)(s), \ldots, (I_1 p)(s), (I_1 q)(s), p(t), q(s))\,ds \tag{18}$$

converges.

For any $u, v \in C(I)$, let $x(t) = |p(t)| + |u(t)|$, $y(t) = -|q(t)| - |v(t)|$ and then $x \geq p$, $y \leq q$. By (14), we have

$$\begin{aligned}
&- K_0 (I_{n-1}x - I_{n-1}p)(t) - M_0 (I_{n-1}q - I_{n-1}y)(t) \\
&- K_1 (I_{n-2}x - I_{n-2}p)(t) - M_1 (I_{n-2}q - I_{n-2}y)(t) \\
&- \cdots - K_{n-2} (I_1 x - I_1 p)(t) - M_{n-2} (I_1 q - I_1 y)(t) \\
&- K_{n-1} (x - p)(t) - M_{n-1} (q - y)(t) \\
&\quad \leq g (t, (I_{n-1}x)(t), (I_{n-1}y)(t), \ldots, \\
&\qquad (I_1 x)(t), (I_1 y)(t), x(t), y(t)) \\
&\qquad - g (t, (I_{n-1}p)(t), (I_{n-1}q)(t), \ldots, \\
&\qquad (I_1 p)(t), (I_1 q)(t), p(t), q(t)) \\
&\quad \leq K_0 (I_{n-1}x - I_{n-1}p)(t) + M_0 (I_{n-1}q - I_{n-1}y)(t) \\
&\qquad + K_1 (I_{n-2}x - I_{n-2}p)(t) + M_1 (I_{n-2}q - I_{n-2}y)(t) \\
&\qquad + \cdots + K_{n-2} (I_1 x - I_1 p)(t) + M_{n-2} (I_1 q - I_1 y)(t) \\
&\qquad + K_{n-1} (x - p)(t) + M_{n-1} (q - y)(t) .
\end{aligned} \tag{19}$$

Hence,

$$\begin{aligned}
&\big| G(t,s)\, g (t, (I_{n-1}x)(t), (I_{n-1}y)(t), \ldots, \\
&\qquad (I_1 x)(t), (I_1 y)(t), x(t), y(t)) \\
&\quad - G(t,s)\, g (t, (I_{n-1}p)(t), (I_{n-1}q)(t), \ldots, \\
&\qquad (I_1 p)(t), (I_1 q)(t), p(t), q(t)) \big| \\
&\quad \leq |G(t,s)| \big[K_0 \left| (I_{n-1}x)(t) - (I_{n-1}p)(t) \right| \\
&\qquad + M_0 \left| (I_{n-1}q)(t) - (I_{n-1}y)(t) \right| \\
&\qquad + K_1 \left| (I_{n-2}x)(t) - (I_{n-2}p)(t) \right| \\
&\qquad + M_1 \left| (I_{n-2}q)(t) - (I_{n-2}y)(t) \right| \\
&\qquad + \cdots + K_{n-2} \left| (I_1 x)(t) - (I_1 p)(t) \right| \\
&\qquad + M_{n-2} \left| (I_1 q)(t) - (I_1 y)(t) \right| \\
&\qquad + K_{n-1} \left| x(t) - p(t) \right| + M_{n-1} \left| q(t) - y(t) \right| \big] \\
&\quad \leq |G(t,s)| \big[(K_0 + K_1 + \cdots + K_{n-1}) \| x - p \| \\
&\qquad + (M_0 + M_1 + \cdots + M_{n-1}) \| q - y \| \big] .
\end{aligned} \tag{20}$$

Following the former inequality, we can easily have that

$$\begin{aligned}
\int_0^1 & G(t,s) \big[g (s, (I_{n-1}x)(s), (I_{n-1}y)(s), \ldots, \\
&\quad (I_1 x)(s), (I_1 y)(s), x(s), y(s)) \\
&\quad - g (s, (I_{n-1}p)(s), (I_{n-1}q)(s), \ldots, \\
&\quad (I_1 p)(s), (I_1 q)(s), p(s), q(s)) \big] ds
\end{aligned} \tag{21}$$

converges, thus,

$$\begin{aligned}
\int_0^1 & G(t,s)\, g (s, (I_{n-1}x)(s), (I_{n-1}y)(s), \ldots, \\
&\quad (I_1 x)(s), (I_1 y)(s), x(s), y(s)) \, ds \\
= \int_0^1 & G(t,s)\, g (s, (I_{n-1}p)(s), (I_{n-1}q)(s), \ldots, \\
&\quad (I_1 p)(s), (I_1 q)(s), p(s), q(s)) \, ds \\
+ \int_0^1 & G(t,s) \big[g (s, (I_{n-1}x)(s), (I_{n-1}y)(s), \ldots, \\
&\quad (I_1 x)(s), (I_1 y)(s), x(s), y(s)) \\
&\quad - g (s, (I_{n-1}p)(s), (I_{n-1}q)(s), \ldots, \\
&\quad (I_1 p)(s), (I_1 q)(s), p(s), q(s)) \big] ds
\end{aligned} \tag{22}$$

is converged.

Similarly, by $x \geq u$, $y \leq v$,

$$\begin{aligned}
\int_0^1 & G(t,s)\, g (s, (I_{n-1}x)(s), (I_{n-1}y)(s), \ldots, \\
&\quad (I_1 x)(s), (I_1 y)(s), x(s), y(s)) \, ds
\end{aligned} \tag{23}$$

is converged, and we have that

$$\begin{aligned}
\int_0^1 & G(t,s)\, g (s, (I_{n-1}u)(s), (I_{n-1}v)(s), \ldots, \\
&\quad (I_1 u)(s), (I_1 v)(s), u(s), v(s)) \, ds
\end{aligned} \tag{24}$$

converges.

Define the operator $F : C(I) \times C(I) \rightarrow C(I)$ by

$$\begin{aligned}
F(u,v)(t) = \int_0^1 & G(t,s) \\
&\times g (s, (I_{n-1}u)(s), (I_{n-1}v)(s), \ldots, \\
&\quad (I_1 u)(s), (I_1 v)(s), u(s), v(s)) \, ds, \\
&\qquad \forall t \in I.
\end{aligned} \tag{25}$$

Let

$$\begin{aligned}
(A_0 u)(t) &= \int_0^1 |G(t,s)| (K_0 u)(s)\, ds, \\
(B_0 v)(t) &= \int_0^1 |G(t,s)| (M_0 v)(s)\, ds, \\
(A_i u)(t) &= \int_0^1 |G(t,s)| (K_i (I_i u))(s)\, ds, \\
&\qquad i = 1, 2, \ldots, n-1, \\
(B_i v)(t) &= \int_0^1 |G(t,s)| (M_i (I_i v))(s)\, ds, \\
&\qquad i = 1, 2, \ldots, n-1, \\
(Au)(t) &= (A_0 u + A_1 u + \cdots + A_{n-1} u)(t), \\
(Bv)(t) &= (B_0 v + B_1 v + \cdots + B_{n-1} v)(t).
\end{aligned} \tag{26}$$

By (14) and (25), for any $u_1, u_2, v_1, v_2 \in C(I)$, $u_1 \le u_2$, $v_1 \ge v_2$, we have

$$\begin{aligned}
&- A(u_2 - u_1) - B(v_1 - v_2) \\
&\quad \le F(u_1, v_1) - F(u_2, v_2) \\
&\quad \le A(u_2 - u_1) + B(v_1 - v_2), \\
&((A+B)u)(t) \\
&\quad = \int_0^1 |G(t,s)| [K_0 u + M_0 u + K_1 (I_1 u) + M_1 (I_1 u) + \cdots \\
&\qquad + K_{n-1}(I_{n-1} u) + M_{n-1}(I_{n-1} u)](s)\, ds \\
&\quad \le \left(\frac{K_0 + M_0}{(n-2)!} + \frac{K_1 + M_1}{(n-3)!} + \cdots + \frac{K_{n-3} + M_{n-3}}{1!} \right. \\
&\qquad \left. + K_{n-2} + M_{n-2} + K_{n-1} + M_{n-1} \right) \cdot \|u\| h_1(t), \\
&((A+B)^m u)(t) \\
&\quad = \int_0^1 |G(t,s)| (A+B)(A+B)^{m-1}(u)(s)\, ds \\
&\quad \le \left(\frac{K_0 + M_0}{(n-2)!} + \frac{K_1 + M_1}{(n-3)!} + \cdots + \frac{K_{n-3} + M_{n-3}}{1!} \right. \\
&\qquad \left. + K_{n-2} + M_{n-2} + K_{n-1} + M_{n-1} \right)^m \cdot \|u\| h_m(t), \\
&\qquad m = 2, 3, \ldots, \\
&\|(A+B)^m\| \le \left(\frac{K_0 + M_0}{(n-2)!} + \frac{K_1 + M_1}{(n-3)!} + \cdots \right. \\
&\qquad + \frac{K_{n-3} + M_{n-3}}{1!} + K_{n-2} + M_{n-2} \\
&\qquad \left. + K_{n-1} + M_{n-1} \right)^m \cdot \sup_{t \in J} e_m(t),
\end{aligned} \tag{27}$$

$$\begin{aligned}
r(A+B) &\le \left(\frac{K_0 + M_0}{(n-2)!} + \frac{K_1 + M_1}{(n-3)!} + \cdots \right. \\
&\quad + \frac{K_{n-3} + M_{n-3}}{1!} + K_{n-2} + M_{n-2} \\
&\quad \left. + K_{n-1} + M_{n-1} \right) \\
&\quad \times (\rho(G))^{-1} < 1.
\end{aligned} \tag{28}$$

So we can choose $\beta \in (0,1)$, which satisfies $\lim_{k \to \infty} \|(A+B)^k\|^{1/k} = r(A+B) < \beta < 1$, and so there exists a positive integer k_0 such that

$$\left\|(A+B)^k\right\| < \beta^k < 1, \quad k \ge k_0. \tag{29}$$

Since P is a generating cone in $C(I)$, from Lemma 3, there exists $\tau > 0$ such that every element $u \in C(I)$ can be represented in

$$\begin{gathered}
u = v - w, \quad v, w \in P, \\
\|v\| \le \tau \|u\|, \quad \|w\| \le \tau \|u\|;
\end{gathered} \tag{30}$$

this implies

$$-(v + w) \le u \le v + w. \tag{31}$$

Let

$$\|u\|_0 = \inf \{\|h\| \mid h \in P, -h \le u \le h\}. \tag{32}$$

By (31), we know that $\|u\|_0$ is well defined for any $u \in C(I)$. It is easy to verify that $\|\cdot\|_0$ is a norm in $C(I)$. By (30)–(32), we get

$$\|u\|_0 \le \|v + w\| \le 2\tau \|u\|, \quad \forall u \in C(I). \tag{33}$$

On the other hand, for any $h \in P$ which satisfies $-h \le u \le h$, we have $0 \le u + h \le 2h$; thus, $\|u\| \le \|u + h\| + \|-h\| \le (2N+1)\|h\|$, where N denotes the normal constant of P. Since h is arbitrary, we have

$$\|u\| \le (2N+1) \|u\|_0, \quad \forall u \in C(I). \tag{34}$$

It follows from (33) and (34) that the norms $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent. Now, for any $u, v \in C(I)$ and $h \in P$ which satisfies $-h \le u - v \le h$, let

$$\begin{aligned}
u_1 &= \frac{1}{2}(u + v - h), \\
u_2 &= \frac{1}{2}(u - v + h), \\
u_3 &= \frac{1}{2}(-u + v + h),
\end{aligned} \tag{35}$$

then $u \ge u_1$, $v \ge u_1$, $u - u_1 = u_2$, $v - u_1 = u_3$, $u_2 + u_3 = h$.

It follows from (27) that

$$-Au_2 \le F(u,u) - F(u_1,u) \le Au_2, \tag{36}$$

$$-Au_3 - Bu_2 \le F(v,u_1) - F(u_1,u) \le Au_2 + Bu_3, \tag{37}$$

$$-Bu_3 \le F(v,u_1) - F(v,v) \le Fu_3; \tag{38}$$

subtracting (37) from (36) + (38), we obtain

$$-(A+B)h \le F(u,u) - F(v,v) \le (A+B)h. \tag{39}$$

Let $G(u) = F(u,u)$; then we have

$$-(A+B)h \le G(u) - G(v) \le (A+B)h. \tag{40}$$

As A and B are both positive linear bounded operators, so $A+B$ is a positive linear bounded operator, and therefore, $(A+B)h \in P$. Hence, by mathematical induction, it is easy to know that for natural number k_0 in (29), we have

$$\begin{aligned} -(A+B)^{k_0}h &\le G^{k_0}(u) - G^{k_0}(v) \\ &\le (A+B)^{k_0}h, \quad (A+B)^{k_0}h \in P; \end{aligned} \tag{41}$$

since $(A+B)^{k_0}h \in P$, we see that

$$\left\|G^{k_0}(u) - G^{k_0}(v)\right\|_0 \le \left\|(A+B)^{k_0}\right\| \|h\|, \tag{42}$$

which implies by virtue of the arbitrariness of h that

$$\begin{aligned} \left\|G^{k_0}u - G^{k_0}v\right\|_0 &\le \left\|(A+B)^{k_0}\right\| \|u-v\|_0 \\ &\le \beta^{k_0}\|u-v\|_0. \end{aligned} \tag{43}$$

By $0 < \beta < 1$, we have $0 < \beta^{k_0} < 1$. Thus, the Banach contraction mapping principle implies that G^{k_0} has a unique fixed point u^* in $C(I)$, and so G has a unique fixed point u^* in $C(I)$; by the definition of G, F has a unique fixed point u^* in $C(I)$; then, by Lemma 2, $I_{n-1}u^*$ is the unique solution of (3). And, for any $u_0 \in C(I)$, let $u_m = F(u_{m-1}, u_{m-1})$ $(m = 1,2,\ldots)$; we have $\|u_m - u^*\|_0 \to 0$ $(k \to \infty)$. By the equivalence of $\|\cdot\|_0$ and $\|\cdot\|$ again, we get $\|u_m - u^*\| \to 0$ $(m \to \infty)$. This completes the proof. □

4. Example

In this paper, the results apply to a very wide range of functions, and we are following only one example to illustrate.

Consider the following nth-order three-point boundary value problem:

$$\begin{gathered} u^{(n)}(t) + (S_0u)(t) + (S_1u')(t) \\ + k(t)\ln(3 + |x(t)|), \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = 2u\left(\frac{1}{2}\right), \end{gathered} \tag{44}$$

where $(S_iu^{(i)})(t) = \int_0^1 h_i(t,s)u^{(i)}(s)\,ds, h_i, k \in C(I\times I, \mathbb{R}), i = 0, 1.$

Applying Theorem 4, we can find that (44) has a unique solution $I_nx^*(t) \in C^{(n)}(I)$ provided $\sup_{t,s\in I}|(h_0(t,s)/(n-2)!) + (h_1(t,s)/(n-3)!) + (k(t)/3(n-2)!)| < 1$, and moreover, for any $u_0 \in C(I)$, the iterative sequence

$$\begin{aligned} x_m(t) = \int_0^1 G(t,s)\,[S_0(I_{n-1}x_{m-1})(s) \\ + S_1(I_{n-2}x_{m-1})(s) \\ + k(s)\ln(3 + |x_{m-1}(s)|)]\,ds \end{aligned} \tag{45}$$

$(m = 1,2,\ldots)$ converges to x^* uniformly for all t in $I(m \to \infty)$.

To see that, let

$$G_1(t,s) = \begin{cases} -1 + \dfrac{2^{n-2}}{2^{n-2}-1}\left[(1-s)^{n-1} - 2\left(\dfrac{1}{2} - s\right)^{n-1}\right], & 0 \le s \le \dfrac{1}{2},\ s \le t, \\ \dfrac{2^{n-2}}{2^{n-2}-1}\left[(1-s)^{n-1} - 2\left(\dfrac{1}{2} - s\right)^{n-1}\right], & 0 \le t \le s \le \dfrac{1}{2}, \\ -1 + \dfrac{2^{n-2}}{2^{n-2}-1}(1-s)^{n-1}, & \dfrac{1}{2} \le s \le t, \\ \dfrac{2^{n-2}}{2^{n-2}-1}(1-s)^{n-1}, & \dfrac{1}{2} \le s,\ t \le s, \end{cases}$$

$$e_1^*(t) = \max\left\{\int_0^1 |G_1(t,s)|\,ds, \int_0^t\int_0^1 |G_1(s,x)|\,dx\,ds\right\}; \tag{46}$$

then $G_1(t,s)$ is Green's function of (44). It is easy to verify that $|G_1(t,s)| \le 1$, and so $\rho(G_1) \ge (\sup_{t,s\in I} e_1^*(t))^{-1} \ge 1$.

Let

$$\begin{gathered} g\left(t, u(t), v(t), u'(t), v'(t), \ldots, u^{(n-1)}(t), v^{(n-1)}(t)\right) \\ = (S_0u)(t) + (S_1u')(t) + k(t)\ln(3 + |v(t)|), \\ (K_iu)(t) = H_i^*\int_0^1 u(s)\,ds, \quad i = 0,1, \\ (M_0v)(t) = \frac{K^*}{3}\int_0^t v(s)\,ds, \\ (M_iu)(t) = 0, \quad i = 1,\ldots,n-1, \\ u_0 = v_0 = 0, \end{gathered} \tag{47}$$

where $H_i^* = \sup_{t,s\in I}|h_i(t,s)|(i = 0,1), K^* = \sup_{t\in I}|k(t)|$; then it is easy to verify that all conditions in Theorem 4 are satisfied.

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