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Research Article

The Automorphism Group of the Lie Ring of Real Skew-Symmetric Matrices

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Denote by $\mathfrak{A}_n$ the set of all $n \times n$ skew-symmetric matrices over the field of real numbers, which forms a Lie ring under the usual matrix addition and the Lie multiplication as $[A, B] = AB - BA$, $A, B \in \mathfrak{A}_n$. In this paper, we characterize the automorphism group of the Lie ring $\mathfrak{A}_n$.

1. Introduction and Main Result

A Lie ring is defined as a nonassociative ring with multiplication that is anticommutative and satisfies the Jacobi identity. More specifically, we can define a Lie ring L to be an abelian group with an operation $[,]$ that has the following properties:

(i) biadditive:

$$[x + y, z] = [x, z] + [y, z], \qquad [z, x + y] = [z, x] + [z, y], \tag{1}$$

for all $x, y, z \in L$;

(ii) the Jacobi identity:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \tag{2}$$

for all $x, y, z \in L$;

(iii) for all x in L,

$$[x, x] = 0. \tag{3}$$

It is well known that a Lie algebra can be viewed as a Lie ring. So, the theory of Lie ring can be used in the theory of the Lie algebra. Recall that an automorphism of a Lie ring $(L, [,])$ is a bijective map ϕ form L onto itself such that $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in L$. There are a lot of papers that studied the automorphism groups of some fixed Lie rings (or, more for the Lie algebras), see [1–7].

Note that any associative ring L can be made into a Lie ring $(L, +, [,])$ by defining a bracket operator $[x, y] = xy - yx$. Let $\mathbb{R}$ be the real number field and denote by $\mathbb{R}^{\times}$ the rest $\mathbb{R} \setminus \{0\}$. Let M_n be the algebra of all $n \times n$ matrices over $\mathbb{R}$. We denote by $\mathfrak{A}_n$ the subset of M_n consisting of all $n \times n$ skew-symmetric matrices, that is,

$$\mathfrak{A}_n = \left\{ \left[a_{ij}\right] \in M_n : \left[a_{ij}\right]^t = -\left[a_{ij}\right] \right\}. \tag{4}$$

It is well known that the set $\mathfrak{A}_n$ forms a Lie ring under the usual matrix addition and the Lie multiplication as $[A, B] = AB - BA$, $A, B \in \mathfrak{A}_n$. In the same way, we know that M_n or T_n (the set of all $n \times n$ upper triangular matrices) as well as forms a Lie ring.

Hua [8] gives the form of any automorphism of the Lie ring M_n over a skew field by using the fundamental theorem of geometry of matrices; Dolinar [1] studies the automorphism of the Lie ring of triangular matrices T_n over any field. Jacobson [9] considers the Lie algebra $\mathfrak{A}_n$ over any algebraically closed field; he gives the form of the automorphism of the Lie algebra $\mathfrak{A}_n$ for the case $n \geq 5$ if n is odd or $n \geq 10$ if n is even. Now, let us see a general result on isomorphism of some Lie rings as follows.

Proposition 1 (see [10, 11]). *Let A' and A be prime rings with involutions of the first kind and of characteristic not 2.*

Let K' and K denote, respectively, the skew elements of A' and A. Assume that the dimension of the central closure of A' over $C_{A'}$ is different from 1, 4, 9, 16, 25, and 64. Then, any Lie isomorphism θ of K' onto K can be extended uniquely to an associative isomorphism of $\langle K' \rangle$ onto $\langle K \rangle$, the associative subrings generated by K' and K, respectively.

Note that the Lie ring $\mathfrak{A}_n$ is a particular class of the previous setting of skew elements of M_n. So, the previous proposition in fact partially solved the problem to characterize the automorphism group of $\mathfrak{A}_n$. However, the problem is still open when n takes any positive integer.

The purpose of this paper is to characterize $\mathrm{Aut}\mathfrak{A}_n$, the automorphism group of the Lie ring $\mathfrak{A}_n$, for $n \geq 2$. Our main result is the following.

Theorem 2. *Suppose that $n \geq 2$ is an integer, then $\phi \in \mathrm{Aut}\mathfrak{A}_n$ if and only if there is a real orthogonal matrix Q such that*

$$\phi(X) = QXQ^t, \quad \forall X \in \mathfrak{A}_n. \tag{5}$$

Further, one has $\mathrm{Aut}\mathfrak{A}_n \cong O_n(R)$, where $O_n(R)$ is the real orthogonal group.

2. Preliminary Results

Now, let us start this section by denoting some notations. Denote by $[n/2]$ the maximal integer number no more than $n/2$. Let $E_{ij}^{(n)}$ be the $n \times n$ matrix which has 1 in the (i, j) entry and is 0 elsewhere. Set $D = E_{12}^{(2)} + E_{21}^{(2)}$, and denote by $I_n = \Sigma_{i=1}^n E_{ii}^{(n)}$ the $n \times n$ identity matrix. Note that the notation I_0 means that the matrix vanished. Let $J = E_{11}^{(2)} - E_{22}^{(2)}$, $K = E_{12}^{(2)} - E_{21}^{(2)}$.

Suppose that $\mathfrak{S} \subset \mathfrak{A}_n$. We call $\mathfrak{S}$ to be commutative if $[x, \mathfrak{S}] = 0$, for all $x \in \mathfrak{S}$, and call $\mathfrak{S}$ to be maximal commutative if $\mathfrak{S}$ is not only commutative but $[y, \mathfrak{S}] \neq 0$, for all $y \notin \mathfrak{S}$. Clearly, the maximal commutative subset is a subring of $\mathfrak{A}_n$. Suppose that $A \in \mathfrak{A}_n$. Set

$$\mathfrak{C}(A) := \{X \in \mathfrak{A}_n : [A, X] = 0\}. \tag{6}$$

We denote by $A \oplus B$ and $A \otimes B$ the direct sum and the Kronecker product of A and B, respectively.

Definition 3. A matrix $A \in \mathfrak{A}_n$ is called regular if it satisfies the following conditions: (i) when n is an even number, there is an orthogonal matrix Q, and the real numbers $\varepsilon_1, \ldots, \varepsilon_{n/2} \in \mathbb{R}$ with different absolute values, such that

$$A = Q \operatorname{diag}(\varepsilon_1 K, \ldots, \varepsilon_{n/2} K) Q^t. \tag{7}$$

(ii) When n is an odd number, there is an orthogonal matrix Q, and the nonzero real numbers $\varepsilon_1, \ldots, \varepsilon_{[n/2]} \in \mathbb{R}$ with different absolute values, such that

$$A = Q \operatorname{diag}(\varepsilon_1 K, \ldots, \varepsilon_{[n/2]} K \oplus 0) Q^t. \tag{8}$$

Now, a subring $\mathfrak{h}$ of $\mathfrak{A}_n$ is called a regular subring if $\mathfrak{h}$ is maximal commutative and there is a regular matrix in $\mathfrak{h}$.

For $x \in \mathfrak{A}_n$, denote $\mathfrak{C}(x) = \{A \in \mathfrak{A}_n : [A, x] = 0\}$.

Lemma 4 (see [12, 2.5.14]). *Suppose that $A \in \mathfrak{A}_n$. Then, there are an orthogonal matrix Q and real numbers $a_1, \ldots, a_{[n/2]}$ such that*

$$A = Q \operatorname{diag}(a_1 K \oplus \cdots \oplus a_{[n/2]} K \oplus 0) Q^t. \tag{9}$$

Lemma 5 (see [13, 14]). *Let $\mathbb{F}$ be any field, and let $K_n(\mathbb{F})$ denote the space of all $n \times n$ alternate matrices over $\mathbb{F}$. Then, ϕ is an additive surjective mapping of $K_n(\mathbb{F})$ $(n \geq 2)$ to itself that preserves rank 2 matrices if and only if ϕ is of the following forms:*

(i) *$n \geq 4$, $\phi((a_{ij})) = \alpha P^t(f(a_{ij}))P$, for all $(a_{ij}) \in K_n(\mathbb{F})$, where $\alpha \in \mathbb{F} \setminus \{0\}$, P is an $n \times n$ invertible matrix, and f is a field automorphism of $\mathbb{F}$;*

(ii) *when $n = 4$, ϕ is of the form*

$$\phi((a_{ij})) = \alpha P^t (f(a_{ij}))^* P, \quad \forall (a_{ij}) \in K_4(\mathbb{F}), \tag{10}$$

where α, P, and f have the same meaning as before, and $(a_{ij}) \mapsto (a_{ij})^*$ is either the identity map or the map:

$$\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{bmatrix}. \tag{11}$$

In the next text, we always assume that $\phi \in \mathrm{Aut}\mathfrak{A}_n$ is arbitrary.

Lemma 6. *Suppose that $\mathfrak{h}$ is a regular subring of $\mathfrak{A}_n$. Then, there is an orthogonal matrix Q and maps $\eta_i : \mathfrak{A}_n \to \mathbb{R}$, $i = 1, \ldots, [n/2]$ such that*

$$QXQ^t = \eta_1(X) K \oplus \cdots \oplus \eta_{[n/2]}(X) K \oplus 0, \quad \forall X \in \mathfrak{h}. \tag{12}$$

Proof. For every $X \in \mathfrak{h}$, note that QXQ^t is commutative with every regular matrix in $\mathfrak{h}$. So, one can obtain the conclusion by Lemma 4. □

Corollary 7. *Suppose that $\mathfrak{h}$ is a regular subring of $\mathfrak{A}_n$, and $H \in \mathfrak{h}$ is a regular matrix. Then,*

$$X \in \mathfrak{h} \Longleftrightarrow [X, H] = 0. \tag{13}$$

Lemma 8. *Suppose that $\mathfrak{h}_1$, $\mathfrak{h}_2$ are both regular subrings of $\mathfrak{A}_n$, and that there is a regular matrix $H \in \mathfrak{h}_1 \cap \mathfrak{h}_2$. Then, $\mathfrak{h}_1 = \mathfrak{h}_2$.*

Proof. Note that a regular subring is maximal; the conclusion follows by Corollary 7. □

Lemma 9. *Both maps ϕ and ϕ^{-1} preserve the regular subring. Expressly, for $H \in \mathfrak{A}_n$, one has that H is a regular matrix if and only if $\phi(H)$ is so.*

Proof. Take any regular subring $\mathfrak{h}$ of $\mathfrak{A}_n$ and a regular matrix $H \in \mathfrak{h}$. By Lemma 6, we can assume that $Q\phi(H)Q^t = \varepsilon_1 K \oplus \cdots \oplus \varepsilon_{[n/2]} K \oplus 0$, where Q is an orthogonal matrix.

Suppose that $A \in \mathfrak{A}_n$ satisfying $Q\phi(A)Q^t = 1K \oplus 2K \oplus \cdots \oplus [n/2]K \oplus 0$. Then, $\phi(A)$ is a regular matrix in $\mathfrak{A}_n$. Since $[\phi(H), \phi(A)] = 0$, $[H, A] = 0$. This means that $A \in \mathfrak{h}$. Let $\mathfrak{h}_1$ be a regular subring containing $\phi(A)$. By Lemma 8, we only need to prove that $\phi(\mathfrak{h}) = \mathfrak{h}_1$. Take any $\phi(X) \in \mathfrak{h}_1$. Then, we see by Lemma 6 that there are $x_1, \ldots, x_{[n/2]} \in \mathbb{R}$ such that

$$Q\phi(X)Q^t = x_1K \oplus \cdots \oplus x_{[n/2]}K \oplus 0. \tag{14}$$

So, $[\phi(X), \phi(H)] = 0$, and $[X, H] = 0$. Hence, $X \in \mathfrak{h}$. This shows that $\mathfrak{h}_1 \subset \phi(\mathfrak{h})$. Note that $\mathfrak{h}_1$ is maximal, so we obtain that $\phi(\mathfrak{h}) = \mathfrak{h}_1$.

Now, we prove that ϕ preserves the regular matrix. Otherwise, suppose that $\phi(H)$ is not a regular matrix, then we will get a contradiction. By the definition, we see that one of the following cases holds.

Case 1. n is odd and there is $\varepsilon_i = 0$.

Case 2. There is some $\varepsilon_i \in \{\pm\varepsilon_j\}$.

If Case 1 happens, we assume without loss the generality that $\varepsilon_1 = 0$. We take $X \in \mathfrak{A}_n$ such that

$$Q\phi(X)Q^t = E_{1n}^{(n)} - E_{n1}^{(n)}. \tag{15}$$

If Case 2 happens, we assume without loss the generality that $\varepsilon_2 \in \{\pm\varepsilon_1\}$. When $\varepsilon_1 = \varepsilon_2$, we take $X \in \mathfrak{A}_n$ such that

$$Q\phi(X)Q^t = K \otimes I_2 \oplus 0. \tag{16}$$

When $\varepsilon_1 = -\varepsilon_2$, we take $X \in \mathfrak{A}_n$ such that

$$Q\phi(X)Q^t = K \otimes J \oplus 0. \tag{17}$$

On one hand, it is clear that $[\phi(X), \phi(A)] \neq 0$, so we have $\phi(X) \notin \mathfrak{h}_1$. On the other hand, $[\phi(X), \phi(H)] = 0$; hence, $[X, H] = 0$. Thus, $X \in \mathfrak{h}$, and so $\phi(X) \in \phi(\mathfrak{h}) = \mathfrak{h}_1$; this is impossible. Note that ϕ is an automorphism; we see that ϕ^{-1} also preserves the regular matrix. The proof is completed. □

Lemma 10. *Suppose that $n \geq 5$ and $A, B \in \mathfrak{A}_n$. If* rank $A = 2$ *and $B \notin \mathbb{R}A$, then there is $C \in \mathfrak{A}_n$ such that*

$$[A, C] = 0, \qquad [B, C] \neq 0. \tag{18}$$

Proof. We can assume without loss the generality by Lemma 4 that $A = aK \oplus 0$, $a \neq 0$. Hence, we have $\mathfrak{C}(A) = 0 \oplus \mathfrak{A}_{n-2}$. If any matrix C cannot satisfy the conclusion, then one has $[B, 0 \oplus \mathfrak{A}_{n-2}] = 0$. Note that $n \geq 5$, so we have $n - 2 \geq 3$. This implies that $B \in \mathbb{R}K \oplus 0$, which contradicts with $B \notin \mathbb{R}A$. □

Lemma 11. *Let $A \in \mathfrak{A}_n$. Then, $\phi(\mathfrak{C}(A)) = \mathfrak{C}(\phi(A))$.*

Proof. As $[A, \mathfrak{C}(A)] = 0$, we deduce that $[\phi(A), \phi(\mathfrak{C}(A))] = 0$. Farther, we have $\phi(\mathfrak{C}(A)) \subset \mathfrak{C}(\phi(A))$. The desired result follows from the following:

$$\begin{aligned}\mathfrak{C}(\phi(A)) &= \left(\phi\phi^{-1}\right)\mathfrak{C}(\phi(A)) = \phi\left(\phi^{-1}\left(\mathfrak{C}(\phi(A))\right)\right) \\ &\subset \phi\left(\mathfrak{C}\left(\phi^{-1}(\phi(A))\right)\right) = \phi(\mathfrak{C}(A)).\end{aligned} \tag{19}$$

□

Lemma 12. *Suppose that $A \in \mathfrak{A}_4$ is not a regular matrix. Then, $\phi(\mathbb{R}A) = \mathbb{R}\phi(A)$.*

Proof. It follows from Lemma 6 that there is an orthogonal matrix Q such that

$$\phi(rA) = Q\left(\varepsilon_1(r)K \oplus \varepsilon_2(r)K\right)Q^t, \quad \forall r \in \mathbb{R}. \tag{20}$$

Since A is not a regular matrix and so is rA, we see that $\varepsilon_2(r) \in \{\pm\varepsilon_1(r)\}$, for all $r \in \mathbb{R}$. If $\varepsilon_1(1) = \varepsilon_2(1)$, then we will see that $\varepsilon_1 = \varepsilon_2$. Otherwise, there is $r_0 \in \mathbb{R}^\times$ such that $\varepsilon_1(r_0) = -\varepsilon_2(r_0)$, and so $Q(K \otimes I_2)Q^t \in \mathfrak{C}(\phi(A))$. But we know that $Q(K \otimes I_2)Q^t \notin \mathfrak{C}(\phi(r_0A))$, this, together with Lemma 11, gives that

$$\mathfrak{C}(\phi(A)) = \phi(\mathfrak{C}(A)) = \phi(\mathfrak{C}(r_0A)) = \mathfrak{C}(\phi(r_0A)). \tag{21}$$

This is impossible. Similarly, we can show that if $\varepsilon_1(1) = -\varepsilon_2(1)$, then $\varepsilon_1 = -\varepsilon_2$, and then we get the conclusion. □

Lemma 13. *Suppose that $n \geq 5$ and $A \in \mathfrak{A}_n$ such that* rank $A = 2$. *Then, $\phi(\mathbb{R}A) = \mathbb{R}\phi(A)$.*

Proof. If there is $B \notin \mathbb{R}A$ such that $\phi(B) \in \mathbb{R}\phi(A)$, then by Lemma 10 we can choose $C \in \mathfrak{A}_n$ such that $[A, C] = 0$, $[B, C] \neq 0$. Thus, $[\phi(A), \phi(C)] = 0$; $[\phi(B), \phi(C)] \neq 0$. But we see that $\phi(B) \in \mathbb{R}\phi(A)$; this is impossible. Furthermore,

$$\mathbb{R}\phi(A) \subset \phi(\mathbb{R}A). \tag{22}$$

For any nonzero real number r, we replace A by rA in the previous equation. It follows that

$$\mathbb{R}\phi(\mathbb{R}A) \subset \phi(\mathbb{R}A) \subset \mathbb{R}\phi(\mathbb{R}A), \tag{23}$$

that is, $\phi(\mathbb{R}A) = \mathbb{R}\phi(\mathbb{R}A)$. Note that ϕ is additive. So, $\phi(\mathbb{R}A)$ is a subspace.

Suppose that

$$\phi(rA) = Q\operatorname{diag}\left(\varepsilon_1(r)K \oplus \cdots \oplus \varepsilon_{[n/2]}(r)K \oplus 0\right)Q^t, \tag{24}$$

where Q is an orthogonal matrix, $\varepsilon_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, [n/2]$. We first prove the following.

Assertion. If there is an index i_0 such that $\varepsilon_{i_0}(r) \neq 0$, for all $r \in \mathbb{R}^\times$, then $\phi(\mathbb{R}A) \subset \mathbb{R}\phi(A)$.

In fact, for any given $r \in \mathbb{R}^\times$, suppose that $\varepsilon_{i_0}(r) = \rho\varepsilon_{i_0}(1)$, then $\rho \neq 0$. Now, we assume that the assertion is not true; then there is some index s such that $\varepsilon_s(r) \neq \rho\varepsilon_s(1)$. Then, we see by $\phi(A) \in \phi(\mathbb{R}A)$, $\phi(rA) \in \phi(\mathbb{R}A)$, and the fact $\phi(\mathbb{R}A)$ is a space that $\rho\phi(A) - \phi(rA) \in \phi(\mathbb{R}A)$. This tells us that there is some $c \in \mathbb{R}$ such that $\rho\phi(A) - \phi(rA) = \phi(cA)$. Thus, $\varepsilon_s(c) = \rho\varepsilon_s(1) - \varepsilon_s(r) \neq 0$, and so we have $c \neq 0$. But we know that $\varepsilon_{i_0}(c) = \rho\varepsilon_{i_0}(1) - \varepsilon_{i_0}(r) = 0$, which contradicts with the conditions of the assertion. This gives that $\phi(rA) = \rho\phi(A)$. The assertion is proved.

As A is not a regular matrix, one has that $\phi(rA)$ is not a regular matrix too. Next, the proof of the lemma is divided into the following cases with respect to n.

Case 1. When n is odd, note that $A \neq 0$, so we can assume without loss the generality that $\varepsilon_1(1) \neq 0$. If for some

$r_0 \in \mathbb{R}^\times$ such that $\varepsilon_1(r_0) = 0$, then it follows by Lemma 11 that $Q(E_{1n}^{(n)} - E_{n1}^{(n)})Q^t \in \mathfrak{C}(\phi(r_0 A)) = \mathfrak{C}(\phi(A))$, which is a contradiction. Now, the lemma follows by using the previous assertion for the index $i_0 = 1$.

Case 2. When n is even, assume without loss of the generality that $\varepsilon_1(1) \in \{\pm\varepsilon_2(1)\}$. If $\varepsilon_1(1) = \varepsilon_2(1)$, then $\varepsilon_1 = \varepsilon_2$. In fact, if there is some $r_0 \in \mathbb{R}^\times$ such that $\varepsilon_1(r_0) \neq \varepsilon_2(r_0)$, then one has by Lemma 11 that $Q(K \otimes I_2 \oplus 0)Q^t \in \mathfrak{C}(\phi(A)) = \mathfrak{C}(\phi(r_0 A))$ is a contradiction. If $\varepsilon_1(1) = -\varepsilon_2(1)$, then $\varepsilon_1 = -\varepsilon_2$. In fact, if there is some $r_0 \in \mathbb{R}^\times$ such that $\varepsilon_1(r_0) \neq -\varepsilon_2(r_0)$, then we see by Lemma 11 that $Q(K \otimes J \oplus 0)Q^t \in \mathfrak{C}(\phi(A)) = \mathfrak{C}(\phi(r_0 A))$; this is impossible.

When $\varepsilon_1(1) \neq 0$, if there is $r_0 \in \mathbb{R}^\times$ such that $\varepsilon_1(r_0) = 0$, then $\varepsilon_2(r_0) = 0$. Thus, $\mathfrak{A}_4 \oplus 0 \subset \mathfrak{C}(\phi(r_0 A)) = \mathfrak{C}(\phi(A))$, which is a contradiction. Now, we get the lemma by using the previous assertion for the index $i_0 = 1$.

When $\varepsilon_1(1) = 0$, if there is $r_0 \in \mathbb{R}^\times$ such that $\varepsilon_1(r_0) \neq 0$, then since $\varepsilon_2(1) = \varepsilon_1(1) = 0$, $\mathfrak{A}_4 \oplus 0 \subset \mathfrak{C}(\phi(A)) = \mathfrak{C}(\phi(r_0 A))$. This is absurd. As $A \neq 0$, it is clear that $n \geq 6$. Hence, we can assume without loss of the generality that $\varepsilon_3(1) \neq 0$. If for some $r_0 \in \mathbb{R}^\times$ such that $\varepsilon_3(r_0) = 0$, then we have by Lemma 11 that

$$Q\left(\left(E_{15}^{(5)} - E_{51}^{(5)}\right) \oplus 0\right) Q^t \in \mathfrak{C}\left(\phi\left(r_0 A\right)\right) = \mathfrak{C}\left(\phi\left(A\right)\right), \quad (25)$$

which is a contradiction. The lemma can be shown by using the previous assertion for the index $i_0 = 3$. □

Corollary 14. *Suppose that $n \geq 5$ and $W \leq \mathfrak{A}_n$ is a subspace with bases which are formed by rank 2 matrices. Then, we have*

$$\dim W = \dim \phi\left(W\right). \quad (26)$$

Proof. Suppose that rank 2 matrices $e_1, \ldots, e_s$ form bases of W. Then,

$$\mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_s = W. \quad (27)$$

It follows immediately that

$$\phi\left(\mathbb{R}e_1\right) + \cdots + \phi\left(\mathbb{R}e_s\right) = \phi\left(W\right). \quad (28)$$

If there is i such that $\phi(\mathbb{R}e_i) \cap \Sigma_{j\neq i}\phi(\mathbb{R}e_j) \neq 0$, then we can choose $\lambda_1, \ldots, \lambda_s \in \mathbb{R}$, not all zero, such that $\phi(\lambda_i e_i) = \Sigma_{j\neq i}\, \phi(\lambda_j e_j)$, which is absurd. We see by Lemma 13 that $\phi(\mathbb{R}e_i) = \mathbb{R}\phi(e_i)$, for all $i = 1, \ldots, s$. Thus,

$$\mathbb{R}\phi\left(e_1\right) \oplus \cdots \oplus \mathbb{R}\phi\left(e_s\right) = \phi\left(W\right). \quad (29)$$

The proof is completed. □

Lemma 15. *Suppose that $n \geq 5$ and $A \in \mathfrak{A}_n$ is of rank 2. Then,*

$$\dim \mathfrak{C}\left(\phi\left(A\right)\right) = \frac{1}{2}\left(n-2\right)\left(n-3\right) + 1. \quad (30)$$

Proof. Note that $\dim \mathfrak{C}(A) = \dim \mathfrak{A}_{n-2} + 1 = (1/2)(n-2)(n-3) + 1$ and $\mathfrak{C}(A)$ has bases which are formed by rank 2 matrices. This, together with Corollary 14, proves the conclusion. □

Lemma 16. *Suppose that $a_1, \ldots, a_s$ are positive real numbers, which are different from one another. Let $x = a_1 I_{n_1} \otimes K \oplus -a_1 I_{n_{-1}} \otimes K \oplus \cdots \oplus a_s I_{n_s} \otimes K \oplus -a_s I_{n_{-s}} \otimes K \oplus 0$. Then, we have*

$$\begin{aligned} \dim \mathfrak{C}\left(x\right) = {} & \left(\Sigma_i\left(n_i + n_{-i}\right)\right)^2 + 2^{-1} \\ & \times \left(n - 2\Sigma_i\left(n_i + n_{-i}\right)\right)\left(n - 2\Sigma_i\left(n_i + n_{-i}\right) - 1\right). \end{aligned} \quad (31)$$

In particular, if we let $d = \Sigma_i(n_i + n_{-i})$, then

$$\dim \mathfrak{C}\left(x\right) \leq d^2 + 2^{-1}\left(n - 2d\right)\left(n - 2d - 1\right), \quad (32)$$

and the equation holds if and only if $s = 1$.

Proof. It follows by a direct computation. □

Lemma 17. *Suppose that $\phi \in \mathrm{Aut}\mathfrak{A}_n$ preserves the rank 2 matrix subset of $\mathfrak{A}_n$. Then, there is a real orthogonal matrix Q such that*

$$\phi\left(X\right) = QXQ^t, \quad \forall X \in \mathfrak{A}_n. \quad (33)$$

Proof. The proof under the case $n = 2$ is obvious. It is not difficult to see that, if $n = 3$, then a surjective map preserving rank 2 matrices still is of the form (i) of Lemma 5. Next, we assume that $n \geq 3$ and assume that ϕ has the form (i) of Lemma 5. For distinct i, j, k, it is clear that

$$\begin{aligned} \left[E_{ij} - E_{ji}, E_{ik} - E_{ki}\right] &= E_{kj} - E_{jk}, \\ \left[E_{ij} - E_{ji}, E_{jk} - E_{kj}\right] &= E_{ik} - E_{ki}. \end{aligned} \quad (34)$$

Consider the image of ϕ; then, it follows by the form (i) of Lemma 5 that

$$\begin{aligned} & \alpha\left(\left(E_{ij} - E_{ji}\right) P^t P \left(E_{ik} - E_{ki}\right)\right. \\ & \quad \left. - \left(E_{ik} - E_{ki}\right) P^t P \left(E_{ij} - E_{ji}\right)\right) = E_{kj} - E_{jk}, \\ & \alpha\left(\left(E_{ij} - E_{ji}\right) P^t P \left(E_{jk} - E_{kj}\right)\right. \\ & \quad \left. - \left(E_{jk} - E_{kj}\right) P^t P \left(E_{ij} - E_{ji}\right)\right) = E_{ik} - E_{ki}. \end{aligned} \quad (35)$$

Hence, by a direct computation and the arbitrariness of i, j, k, it follows that $P^t P = \alpha^{-1} I_n$. Clearly, $\alpha > 0$. Note that $\mathbb{R}$ is the field of real numbers, so we have $f = 1$. Let $Q = \sqrt{\alpha}P$; then, the conclusion is obtained.

When $n = 4$ and ϕ is of the form (ii) of Lemma 5, then we let $i = 1$, $j = 2$, and $k = 3$. Thus, we have by taking the images under ϕ for the previous two equations that $1 = 0$, which is a contradiction. So, the form (ii) of Lemma 5 does not occur. □

3. The Proof of the Main Result

The proof of the main theorem is divided into the following three propositions.

Proposition 18. *Suppose that $n = 2$ or 3 and $\phi \in \mathrm{Aut}\mathfrak{A}_n$. Then, there is an orthogonal matrix Q such that*

$$\phi\left(X\right) = QXQ^t, \quad \forall X \in \mathfrak{A}_n. \quad (36)$$

Proof. Since ϕ is bijective, ϕ preserves the rank 2 matrices of $\mathfrak{A}_2$ or $\mathfrak{A}_3$. If $n = 2$, the conclusion is clear. If $n = 3$, then we also can get the conclusion by Lemma 17. □

Proposition 19. *Suppose that $\phi \in \mathrm{Aut}\mathfrak{A}_4$. Then, there is an orthogonal matrix Q such that*

$$\phi(X) = QXQ^t, \quad \forall X \in \mathfrak{A}_4. \tag{37}$$

Proof. It is clear that $[K, D] = 2J$, $[J, K] = 2D$, and $[J, D] = 2K$. Note that $K\oplus 0$ is regular, so we can assume that $\phi(K\oplus 0) = Q(aK \oplus bK)Q^t$, where $a \neq \pm b$ and Q is an orthogonal matrix. Without loss of generality, one can assume that

$$\phi(K \oplus 0) = aK \oplus bK. \tag{38}$$

Since the regular subring containing the nonregular matrix $\phi(I_2 \otimes K)$ is determined by $\phi(K \oplus 0)$, there are $c \in \mathbb{R}^\times$ and $\varepsilon \in \{\pm 1\}$ such that

$$\phi(I_2 \otimes K) = c(K \oplus \varepsilon K). \tag{39}$$

Therefore,

$$\phi(0 \oplus K) = \phi(I_2 \otimes K) - \phi(K \oplus 0) = (c - a) K \oplus (\varepsilon c - b) K. \tag{40}$$

Suppose that

$$\phi(K \otimes I_2) = \begin{bmatrix} X_1 & Y_1 \\ -Y_1^t & Z_1 \end{bmatrix}, \tag{41}$$

where X_1 is a 2×2 matrix. It follows by $[I_2 \otimes K, K \otimes I_2] = 0$ that

$$[c(K \oplus \varepsilon K), \phi(K \otimes I_2)] = 0. \tag{42}$$

So, we have $KX = XK$, $KZ = ZK$, $KY_1 = \varepsilon Y_1 K$, and $KY^t = \varepsilon Y^t K$. Note that

$$[K \oplus 0, [K \oplus 0, K \otimes I_2]] = -K \otimes I_2. \tag{43}$$

Thus,

$$\left[\begin{bmatrix} aK & 0 \\ 0 & bK \end{bmatrix}, \left[\begin{bmatrix} aK & 0 \\ 0 & bK \end{bmatrix}, \begin{bmatrix} X_1 & Y_1 \\ -Y_1^t & Z_1 \end{bmatrix}\right]\right] = -\begin{bmatrix} X_1 & Y_1 \\ -Y_1^t & Z_1 \end{bmatrix}. \tag{44}$$

This, together with $KY_1 = \varepsilon Y_1 K$, gives that $(a - \varepsilon b)^2 = 1$, $X_1 = 0$, $Z_1 = 0$. We deduce that

$$\phi(K \otimes I_2) = \begin{bmatrix} 0 & Y_1 \\ -Y_1^t & 0 \end{bmatrix}. \tag{45}$$

Similarly, we see by $[I_2 \otimes K, D \otimes K] = 0$ and $[K \oplus 0, [K \oplus 0, D \otimes K]] = -D \otimes K$ that

$$\phi(D \otimes K) = \begin{bmatrix} 0 & Y_2 \\ -Y_2^t & 0 \end{bmatrix}, \tag{46}$$

where $Y_2 K = \varepsilon K Y_2$. We also have by $[K \otimes I_2, D \otimes K] = 2J \otimes K$ that

$$\begin{aligned} \phi(J \otimes K) &= 2^{-1}\left[\begin{bmatrix} 0 & Y_1 \\ -Y_1^t & 0 \end{bmatrix}, \begin{bmatrix} 0 & Y_2 \\ -Y_2^t & 0 \end{bmatrix}\right] \\ &= 2^{-1}\begin{bmatrix} Y_2 Y_1^t - Y_1 Y_2^t & 0 \\ 0 & Y_2^t Y_1 - Y_1^t Y_2 \end{bmatrix}. \end{aligned} \tag{47}$$

Note that $Y_i K = \varepsilon K Y_i$, so we can assume that

$$Y_1 = \begin{bmatrix} y_1 & y_2 \\ -\varepsilon y_2 & \varepsilon y_1 \end{bmatrix}, \qquad Y_2 = \begin{bmatrix} y_3 & y_4 \\ -\varepsilon y_4 & \varepsilon y_3 \end{bmatrix}. \tag{48}$$

Further,

$$\begin{aligned} Y_2 Y_1^t - Y_1 Y_2^t &= 2\varepsilon (y_1 y_4 - y_2 y_3) K, \\ Y_2^t Y_1 - Y_1^t Y_2 &= -2 (y_1 y_4 - y_2 y_3) K, \\ \phi(J \otimes K) &= (y_1 y_4 - y_2 y_3)(\varepsilon K \oplus -K). \end{aligned} \tag{49}$$

On the other hand, we know by $J \otimes K = K \oplus 0 - 0 \oplus K$ that

$$\phi(J \otimes K) = (2a - c) K \oplus (2b - \varepsilon c) K. \tag{50}$$

By a direct computation with (49) and (50), we have $2a - c = c - 2\varepsilon b = \varepsilon(y_1 y_4 - y_2 y_3)$ and then $a + \varepsilon b = c$. Noting that $(a - \varepsilon b)^2 = 1$, we can assume without loss of the generality that $a - \varepsilon b = 1$ (for the case $a - \varepsilon b = -1$, the proof is similar). We deduce that

$$\phi(J \otimes K) = J \otimes K. \tag{51}$$

If $\varepsilon = -1$, then $\phi(I_2 \otimes K) = c(K \oplus -K) \in \mathbb{R}\phi(J \otimes K)$, which contradicts Lemma 12. This tells us that $\varepsilon = 1$.

Again by $[K \oplus 0, K \otimes I_2] = D \otimes K$, we get $Y_2 = KY_1$. Suppose that

$$\phi(K \otimes J) = \begin{bmatrix} X_3 & Y_3 \\ -Y_3^t & Z_3 \end{bmatrix}. \tag{52}$$

It follows by $[J \otimes K, K \otimes J] = 0$ that

$$\left[J \otimes K, \begin{bmatrix} X_3 & Y_3 \\ -Y_3^t & Z_3 \end{bmatrix}\right] = 0. \tag{53}$$

Therefore, we have $KY_3 + Y_3 K = 0$. Note that $[I_2 \otimes K, [I_2 \otimes K, K \otimes J]] = -4K \otimes J$. This tells us that $c^2 = 1$. We can assume without loss of the generality that $c = 1$. Then, we get that $a = 1$ and $b = 0$. Thus,

$$\phi(K \oplus 0) = K \oplus 0. \tag{54}$$

For any but fixed $r \in \mathbb{R}^\times$, we assert that $\phi(rK \oplus 0) \in \mathbb{R}K \oplus 0$.

In fact, firstly, by Lemma 12, we can assume that $\phi(rI_2 \otimes K) = sI_2 \otimes K$, $\phi(rK \oplus 0) = uK \oplus vK$. So, we have that $\phi(0 \oplus rK) = (s - u)K \oplus (s - v)K$.

Secondly, noting that $[I_2 \otimes K, rK \otimes I_2] = 0$ and $[rK \oplus 0, [K \oplus 0, K \otimes I_2]] = -rK \otimes I_2$, we deduce that

$$\phi(rK \otimes I_2) = (u - v)\,\phi(K \otimes I_2). \tag{55}$$

Furthermore, we see by $rJ\otimes K = (1/2)[rK\otimes I_2, [K\oplus 0, K\otimes I_2]]$ and $rJ\otimes K = rK\oplus 0 - 0\oplus rK$ that

$$\begin{gathered}\phi(rJ\otimes K) = (u-v)J\otimes K,\\ \phi(rJ\otimes K) = (2u-s)K\oplus(2v-s)K.\end{gathered}\tag{56}$$

Thus, $(2u-s) = u-v = (s-2v)$. It follows that $u+v=s$.

Finally, due to $[rK\oplus 0, [K\oplus 0, K\otimes J]] = -rK\otimes J$ and $[rI_2\otimes K, [I_2\otimes K, K\otimes J]] = -4rK\otimes J$, one can obtain that

$$\phi(rK\otimes J) = s\phi(K\otimes J) = (u-v)\phi(K\otimes J).\tag{57}$$

This tells us that $u-v=s$, and so we have that $u=s$, $v=0$. In other words, $\phi(rK\oplus 0) = uK\oplus 0$, which proves the assertion. Now, we prove that ϕ preserves the set of rank 2 matrices on $\mathfrak{A}_4$. By applying Lemma 17, we finish the proof. □

Proposition 20. *Suppose that $n\geq 5$ and $\phi\in \mathrm{Aut}\mathfrak{A}_n$. Then, there is an orthogonal matrix Q such that*

$$\phi(X) = QXQ^t,\quad \forall X\in\mathfrak{A}_n.\tag{58}$$

Proof. Take any rank 2 matrix $A\in\mathfrak{A}_n$. By Lemma 4, we can assume that

$$\begin{aligned}\phi(A) = Q\Big(&a_1I_{n_1}\otimes K\oplus -a_1I_{n_{-1}}\otimes K\oplus\cdots\oplus a_sI_{n_s}\otimes K\oplus\\ &-a_sI_{n_{-s}}\otimes K\oplus 0\Big)Q^t.\end{aligned}\tag{59}$$

Let $\Sigma_i(n_i+n_{-i}) = d$. Now, we assert that $d=1$ and so that the rank of $\phi(A)$ is 2; that is, we will assert that ϕ is a preserver of rank 2 on $\mathfrak{A}_n$; then, we can finish the proof by Lemma 17.

It follows by Lemmas 15 and 16 that

$$\frac{1}{2}(n-2)(n-3)+1\leq d^2+2^{-1}(n-2d)(n-2d-1).\tag{60}$$

Moreover, we see that $(d-1)(3d-2n+4)\geq 0$. Hence, we have either $d\leq 1$ or $d\geq 3^{-1}(2n-4)$. The former means that $d=1$, as desired. If the latter holds, then it is clear that

$$n\geq 2d\geq 2\cdot 3^{-1}(2n-4).\tag{61}$$

In this case, we deduce that $n\leq 8$ and $n\neq 7$. Hence, the remainder of the proof is the cases (i) $n=5$, $d=2$, (ii) $n=6$, $d=3$, and (iii) $n=8$, $d=4$.

Suppose that $B = Q(0\oplus K\oplus 0)Q^t$. We consider the rank of $\phi(B)$.

When $\operatorname{rank}\phi(B) = 2$, it is clear that there is an orthogonal matrix P such that $\phi(A) = P(\varepsilon I_p\otimes K\oplus -\varepsilon I_q\otimes K\oplus 0)P^t$ and $\phi(B) = P(\eta K\oplus 0)P^t$. Without loss of the generality, we can assume that $p\neq 0$. Note that $\eta\neq 0$. If $\eta\neq -2\varepsilon$, then one has $\varepsilon+\eta\neq -\varepsilon$. Let $C = Q(K\otimes I_2\oplus 0)Q^t$. As $[A+B, C] = 0$, we can find a matrix $X\in\mathfrak{A}_{n-4}$ such that

$$\phi(C) = P(0\oplus X)P^t.\tag{62}$$

If $\eta = -2\varepsilon$, then $\varepsilon-\eta\neq -\varepsilon$. Let $C = Q(K\otimes J\oplus 0)Q^t$. Since $[A-B, C] = 0$, there is a matrix $X\in\mathfrak{A}_{n-4}$ such that

$$\phi(C) = P(0\oplus X)P^t.\tag{63}$$

Thanks to $[B,[B,C]] = -C$, we deduce $\phi(C) = 0$, which is a contradiction.

When $\operatorname{rank}\phi(B)\neq 2$, then for the previous three cases of n and d, one always has $\operatorname{rank}\phi(B) = \operatorname{rank}\phi(A)$. Note that $\phi(A)$ and $\phi(B)$ are in a common regular subring, and $s=1$. It follows by Lemma 6 that there is an orthogonal matrix P such that $\phi(A) = P(\varepsilon_1K\oplus\cdots\oplus\varepsilon_dK\oplus 0)P^t$ and $\phi(B) = P(\eta_1K\oplus\cdots\oplus\eta_dK\oplus 0)P^t$, where $\eta_i\in\{\pm\eta_1\}$, $\varepsilon_i\in\{\pm\varepsilon_1\}$. Due to $\dim\mathfrak{C}(A+B) = \dim\mathfrak{C}(A-B)$, we see by Lemma 11 that

$$\begin{aligned}\dim\mathfrak{C}(\phi(A)+\phi(B)) &= \dim\mathfrak{C}(\phi(A)-\phi(B))\\ &= \dim\mathfrak{C}(A-B) = \dim\mathfrak{A}_{n-4}+4.\end{aligned}\tag{64}$$

Case 1. $n=5$. We first prove that $\operatorname{rank}(\phi(A)\pm\phi(B))\neq 2$.

If $\operatorname{rank}(\phi(A)+\phi(B)) = 2$, then we may as well assume that $\phi(A) = \varepsilon P(K\oplus K\oplus 0)P^t$ and $\phi(B) = \varepsilon P(K\oplus -K\oplus 0)P^t$. Let $E = K\otimes I_2\oplus 0$, $F = K\otimes J\oplus 0$. It is easy to see that $[E,F] = 0$. Now, we want to show that $[\phi(E),\phi(F)]\neq 0$, which is a contradiction. Note the following:

$$\begin{aligned}&[A,[A,E]] = -E, &&[A,[A,F]] = -F,\\ &[B,[B,E]] = -E, &&[B,[B,F]] = -F.\end{aligned}\tag{65}$$

So, we know that both $\phi(E)$ and $\phi(F)$ satisfy an equation about the matrix $X = [x_{ij}]\in\mathfrak{A}_5$ as follows:

$$\begin{gathered}[\phi(A),[\phi(A),X]] = -X,\\ [\phi(B),[\phi(B),X]] = -X.\end{gathered}\tag{66}$$

That is,

$$\begin{bmatrix} 0 & 0 & 2\varepsilon^2(x_{24}-x_{13}) & -2\varepsilon^2(x_{14}+x_{23}) & -\varepsilon^2x_{15}\\ 0 & 0 & -2\varepsilon^2(x_{14}+x_{23}) & 2\varepsilon^2(x_{13}-x_{24}) & -\varepsilon^2x_{25}\\ 2\varepsilon^2(x_{13}-x_{24}) & 2\varepsilon^2(x_{14}+x_{23}) & 0 & 0 & -\varepsilon^2x_{35}\\ 2\varepsilon^2(x_{14}+x_{23}) & 2\varepsilon^2(x_{24}-x_{13}) & 0 & 0 & -\varepsilon^2x_{45}\\ \varepsilon^2x_{15} & \varepsilon^2x_{25} & \varepsilon^2x_{35} & \varepsilon^2x_{45} & 0\end{bmatrix} = -X,$$

$$\begin{bmatrix} 0 & 0 & -2\varepsilon^2\left(x_{24}+x_{13}\right) & 2\varepsilon^2\left(x_{23}-x_{14}\right) & -\varepsilon^2 x_{15} \\ 0 & 0 & 2\varepsilon^2\left(x_{14}-x_{23}\right) & -2\varepsilon^2\left(x_{13}+x_{24}\right) & -\varepsilon^2 x_{25} \\ 2\varepsilon^2\left(x_{24}+x_{13}\right) & 2\varepsilon^2\left(x_{23}-x_{14}\right) & 0 & 0 & -\varepsilon^2 x_{35} \\ 2\varepsilon^2\left(x_{14}-x_{23}\right) & 2\varepsilon^2\left(x_{13}+x_{24}\right) & 0 & 0 & -\varepsilon^2 x_{45} \\ \varepsilon^2 x_{15} & \varepsilon^2 x_{25} & \varepsilon^2 x_{35} & \varepsilon^2 x_{45} & 0 \end{bmatrix} = -X. \tag{67}$$

Hence, we get that

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & x_{15} \\ 0 & 0 & 0 & 0 & x_{25} \\ 0 & 0 & 0 & 0 & x_{35} \\ 0 & 0 & 0 & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{bmatrix}, \tag{68}$$

and $\varepsilon^2 = 1$. Note that $[A+B, E] = 0$, $[A-B, F] = 0$. After taking the image, we can assume by $\varepsilon \neq 0$ that

$$\phi(E) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & -a & -b & 0 \end{bmatrix},$$
$$\phi(F) = \begin{bmatrix} 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -c & -d & 0 & 0 & 0 \end{bmatrix}. \tag{69}$$

Again by $[E, F] = 0$, we see that

$$\left[\phi(E), \phi(F)\right] = \begin{bmatrix} 0 & 0 & ac & bc & 0 \\ 0 & 0 & ad & bd & 0 \\ -ac & -ad & 0 & 0 & 0 \\ -bc & -bd & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 0. \tag{70}$$

We deduce that $ac = 0$, $bc = 0$. It follows by $\phi(E) \neq 0$ that $c = 0$. Due to $ad = 0$, $bd = 0$, one has $d = 0$. This tells us that $\phi(F) = 0$, which is a contradiction. Similarly, we know that $\operatorname{rank}(\phi(A) - \phi(B)) \neq 2$.

Since $n = 5$, it is clear that $\operatorname{rank}(\phi(A) \pm \phi(B)) = 4$. When

$$\phi(A) = \varepsilon P (K \oplus K \oplus 0) P^t,$$
$$\phi(B) = \eta P (K \oplus -K \oplus 0) P^t, \tag{71}$$

we have that $\varepsilon \neq \eta$. Note that $A \pm B$ is not a regular matrix; hence, $\phi(A) \pm \phi(B)$ is not too. Further, one has $\varepsilon + \eta \in \{\pm(\varepsilon - \eta)\}$. This implies that $\varepsilon = 0$ or $\eta = 0$, which is impossible. Similarly, we deduce that

$$\phi(A) = \varepsilon P (K \oplus -K \oplus 0) P^t,$$
$$\phi(B) = \eta P (K \oplus K \oplus 0) P^t, \tag{72}$$

which is also a contradiction.

Case 2. $n = 6, 8$. We first prove that $\operatorname{rank}(\phi(A) \pm \phi(B)) \neq 2$. Otherwise, if $\operatorname{rank}(\phi(A) + \phi(B)) = 2$, then we have

$$\dim \mathfrak{C}\left(\phi(A) + \phi(B)\right) = \dim \mathfrak{A}_{n-2} + 1 \neq \dim \mathfrak{A}_{n-4} + 4, \tag{73}$$

which is a contradiction. In a similar way, we get $\operatorname{rank}(\phi(A) - \phi(B)) \neq 2$.

If $n = 6$, we assert that $\operatorname{rank}(\phi(A) \pm \phi(B)) \neq 4$. In fact, if $\operatorname{rank}(\phi(A) + \phi(B)) = 4$, then by $\eta_i \in \{\pm\eta_1\}$, $\varepsilon_i \in \{\pm\varepsilon_1\}$, we deduce that $\eta_i \in \{\pm\varepsilon_1\}$. Without loss of the generality, we can assume that $\varepsilon_i = \eta_i$, $i = 1, 2$, and $\varepsilon_j = -\eta_j$, $j = 3$. Hence, we see that $\operatorname{rank}(\phi(A) - \phi(B)) = 2$, which is impossible. Similarly, we deduce that $\operatorname{rank}(\phi(A) - \phi(B)) \neq 4$.

Next, we prove when $n = 6$ that $\operatorname{rank}(\phi(A) \pm \phi(B)) \neq 6$. Otherwise, by (64) we can assume without loss of the generality that $\varepsilon_2 + \eta_2 \in \{\pm(\varepsilon_1 + \eta_1)\}$ and $\varepsilon_3 + \eta_3 \notin \{\pm(\varepsilon_1 + \eta_1)\}$. Note that $\eta_i \in \{\pm\eta_1\}$, $\varepsilon_i \in \{\pm\varepsilon_1\}$, so we have $\varepsilon_2 - \eta_2 \in \{\pm(\varepsilon_1 - \eta_1)\}$ and $\varepsilon_3 - \eta_3 \notin \{\pm(\varepsilon_1 - \eta_1)\}$. Thus,

$$\mathfrak{C}\left(\phi(A+B)\right) = \mathfrak{C}\left(\phi(A-B)\right). \tag{74}$$

But it is clear that $\mathfrak{C}(A+B) \neq \mathfrak{C}(A-B)$, which contradicts with $\mathfrak{C}(\phi(X)) = \phi(\mathfrak{C}(X))$, for all $X \in \mathfrak{A}_n$.

Similarly, we have when $n = 8$ that $\operatorname{rank}(\phi(A) \pm \phi(B)) \neq 6, 8$.

Finally, we prove when $n = 8$ that $\operatorname{rank}(\phi(A) \pm \phi(B)) \neq 4$. Let $Z = Q(E_{33}^{(4)} \otimes K)Q^t$. If $\operatorname{rank} \phi(Z) = 2$, then we can find a contradiction similar to the case of $\operatorname{rank} \phi(B) = 2$. Otherwise, if $\operatorname{rank} \phi(Z) = 8$, then there is an orthogonal matrix P such that

$$\phi(A) = P\left(\varepsilon_1 K \oplus \cdots \oplus \varepsilon_d K\right) P^t,$$
$$\phi(B) = P\left(\eta_1 K \oplus \cdots \oplus \eta_d K\right) P^t, \tag{75}$$
$$\phi(Z) = P\left(\lambda_1 K \oplus \cdots \oplus \lambda_d K\right) P^t,$$

where $\eta_i \in \{\pm\eta_1\}$, $\varepsilon_i \in \{\pm\varepsilon_1\}$, and $\lambda_i \in \{\pm\lambda_1\}$. It is easy to see that the three cases $\operatorname{rank}(\phi(A) \pm \phi(B)) = 4$, $\operatorname{rank}(\phi(A) \pm \phi(Z)) = 4$, and $\operatorname{rank}(\phi(Z) \pm \phi(B)) = 4$ cannot simultaneously hold. This means that $\operatorname{rank}(\phi(A) \pm \phi(B)) = 4$ is impossible.

To sum up the previous arguments, we get that $\operatorname{rank} \phi(A) = 2$. The proof is completed. □

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