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Research Article

H_∞ Control of Discrete-Time Singularly Perturbed Systems via Static Output Feedback

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This paper concentrates on H_∞ control problems of discrete-time singularly perturbed systems via static output feedback. Two methods of designing an H_∞ controller, which ensures that the resulting closed-loop system is asymptotically stable and meets a prescribed H_∞ norm bound, are presented in terms of LMIs. Though based on the same matrix transformation, the two approaches are turned into different optimal problems. The first result is given by an ϵ-independent LMI, while the second result is related to ϵ. Furthermore, a stability upper bound of the singular perturbation parameter is obtained. The validity of the proposed two results is demonstrated by a numerical example.

1. Introduction

Singularly perturbed systems widely exist in industrial processes, such as aircraft and racket systems, power systems, and nuclear reactor systems. These kind of systems usually embrace complicated dynamic phenomena which are characterized by slow and fast modes with multiple time-scales. This property causes high dimensionality and ill-conditioning problems. In control theory, a parameter-related state-space model is frequently used to describe a singularly perturbed system. With important practical meaning, the stability bounds of the singular perturbation parameter have been extensively studied by many researchers. In early times, a traditional method of decomposing the original system into fast and slow subsystems was frequently used, see [1, 2]. In [3], a method to testify the stability of singularly perturbed systems without the fast-slow decomposition was established. The stability bound was proved to have close relationship with the system matrix, which contributes to analyse some robust control problems. Furthermore, two algorithms to compute and improve the stability bound were developed in [4]. More stability problems are discussed in [5–10] and the references therein.

In recent years, computer science is increasingly applied to industrial processes. Therefore, discrete-time singularly perturbed systems have attracted much attention. An H_∞ control problem for uncertain discrete-time singularly perturbed systems via state feedback was studied in [11], where two methods of designing H_∞ controllers were given in terms of LMIs. Fast sampling discrete-time singularly perturbed systems were taken into consideration in [12], and the obtained results were generalized to robust controller design. In [13], a new sufficient condition which guaranteed the existence of state feedback controllers and made the closed-loop system asymptotically stable while satisfying a prescribed H_∞ norm requirement was proposed. This condition was also given in the form of LMIs but was proved to be less conservative than that in [11]. It will be more perfect if a theoretical proof was given. Interested readers can refer to [14–16] for more information of discrete-time singularly perturbed systems.

Though state feedback can achieve desired properties, it requires the availability of all state variables, which cannot be satisfied in most of the practical systems. On the other hand, dynamic output feedback usually increases the dimension of the original system. Therefore, static output feedback plays an important role in control theory considering that it is the simplest control technique in a closed-loop sense and can be easily realized with a cost not as high as that in state feedback case [17]. The primal point involved in static output feedback is the decoupling problem. A variety of approaches are developed to solve such issues. In [18], special inequality and some tuning scalars were used to transfer nonlinear matrix inequality to a linear one. A stabilizing state feedback controller gain and some matrix transformations were introduced to deal with the nonlinear inequalities in [19]. This method is effective and easy to implement. Thus, the same decoupling technique is adopted in this paper.

Based on those reasons and motivated by the above studies, we aim to design an H_∞ controller via static output feedback to stabilize a discrete-time singularly perturbed system and guarantee that the transfer function of the resulting closed-loop system satisfies a prescribed H_∞ norm bound.

The rest of this paper is organized as follows. Section 2 states the system description and some useful lemmas. In Section 3, two LMI-based methods are proposed to design a static output feedback controller for the system presented in Section 2. A numerical example is given to demonstrate the effectiveness of the proposed results in Section 4. Finally, conclusions are given in Section 5.

The following notation will be adopted throughout this paper. I denotes an identity matrix with appropriate dimension. A^T denotes the transpose of matrix A. Sym$\{A\}$ denotes $A + A^T$. For a symmetric block matrix, $(^*)$ stands for the blocks induced by symmetry.

2. Problem Formulation

In this paper, we consider a class of linear fast sampling discrete-time singularly perturbed systems of the following form:

$$
\begin{aligned}
x_{k+1} &= A_\epsilon x_k + B_{1\epsilon} w_k + B_{2\epsilon} u_k, \\
z_k &= C_1 x_k + D_{11} w_k + D_{12} u_k, \\
y_k &= C_2 x_k,
\end{aligned} \tag{1}
$$

where $x_k = \begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix}$, $A_\epsilon = \begin{bmatrix} I_{n1}+\epsilon A_{11} & \epsilon A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B_{1\epsilon} = \begin{bmatrix} \epsilon B_{11} \\ B_{12} \end{bmatrix}$, $B_{2\epsilon} = \begin{bmatrix} \epsilon B_{21} \\ B_{22} \end{bmatrix}$, $C_1 = \begin{bmatrix} C_{11}^T \\ C_{12}^T \end{bmatrix}^T$, $C_2 = \begin{bmatrix} C_{21}^T \\ C_{22}^T \end{bmatrix}^T$, $x_k \in R^n$ is the state vector, in which $x_{1k} \in R^{n_1}$, $x_{2k} \in R^{n_2}$, and $n_1 + n_2 = n$, $u_k \in R^{m_1}$ is the control input, $w_k \in R^{m_2}$ is the disturbance input which belongs to $L_2[0,\infty)$, $y_k \in R^{q_1}$ is the measurement output, $z_k \in R^{q_2}$ is the controlled output. The scalar $\epsilon > 0$ denotes the singular perturbation parameter.

In the rest of this paper, we will assume that system (1) is completely controllable and observable. We will also assume that not all of its state variables are available. Therefore, the following static output feedback control law is taken into consideration:

$$
u_k = F y_k, \tag{2}
$$

then the resulting closed-loop system can be obtained as follows:

$$
\begin{aligned}
x_{k+1} &= \widetilde{A}_\epsilon x_k + B_{1\epsilon} w_k, \\
z_k &= \widetilde{C}_1 x_k + D_{11} w_k, \\
y_k &= C_2 x_k,
\end{aligned} \tag{3}
$$

where

$$
\begin{gathered}
\widetilde{A}_\epsilon = \begin{bmatrix} I_{n1} + \epsilon \widetilde{A}_{11} & \epsilon \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \\
\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} F \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}, \\
\widetilde{C}_1 = \begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} \end{bmatrix}, \\
\begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} + D_{12} F \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}.
\end{gathered} \tag{4}
$$

The H_∞ problem studied in this paper can be described as follows: given a scalar $\gamma > 0$ and a discrete-time singularly perturbed system (1), design a static output feedback controller in the form of (2) such that the closed-loop system (3) is asymptotically stable and its transfer function from ω to z,

$$
G(z) = \widetilde{C}_1 \left(zI - \widetilde{A}_\epsilon\right)^{-1} B_{1\epsilon} + D_{11}, \tag{5}
$$

satisfies $\|G\|_\infty < \gamma$.

The following lemmas will be used in establishing our main results.

Lemma 1 (see [19]). *Consider a discrete-time transfer function $G(z) = C(zI - A)^{-1}B + D$. The following statements are equivalent:*

(i) $\|G(z)\|_\infty < 1$ *and A are stable in the discrete-time sense* ($|\lambda_i(A)| < 1$)*;*

(ii) *there exists $X = X^T > 0$ such that*

$$
\begin{bmatrix} A^T X A - X & A^T X B & C^T \\ B^T X A & B^T X B - I & D^T \\ C & D & -I \end{bmatrix} < 0 \tag{6}
$$

holds.

Lemma 2 (see [20]). *The following two statements are equivalent.*

(i) *Let A, B, and C be given such that the LMI*

$$
\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} + \text{Sym}\left\{ \begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} C^T & -I \end{bmatrix} \right\} < 0 \tag{7}
$$

is feasible in X and Y.

(ii) *Let A, B, and C be given such that the following LMI*

$$A + BC^T + CB^T < 0 \tag{8}$$

holds.

Lemma 3 (see [20]). *The following two statements are equivalent.*

(i) *Let A, B, and C be given such that the LMI*

$$\begin{bmatrix} A & B + CG^T \\ B^T + GC^T & -G - G^T \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} + \operatorname{Sym}\left\{ \begin{bmatrix} 0 \\ I \end{bmatrix} G \begin{bmatrix} C^T & -I \end{bmatrix} \right\} < 0 \tag{9}$$

is feasible in G.

(ii) *Let A, B, and C be given such that the LMIs*

$$\begin{gathered} A < 0 \\ A + BC^T + CB^T < 0 \end{gathered} \tag{10}$$

hold.

3. Main Results

In this section, two LMI-based methods of designing a static output feedback controller are proposed to ensure asymptotical stability of a closed-loop discrete-time singularly perturbed system (3). The first result is given in the form of an ϵ-independent LMI, while the second result is presented by two LMIs which are related to the singular perturbation parameter ϵ. Furthermore, we can obtain the stability upper bound of ϵ.

Before giving the results, let F_0 be a stabilizing state feedback controller gain of the system (1). In other words, the matrix F_0 is chosen to make matrix $A_{\epsilon 0} = A_\epsilon + B_{2\epsilon} F_0$ stable. Then, we introduce the following transformations which will be used in the rest of this paper:

$$A_0 = A + B_{2\epsilon} F_0, \qquad C_{20} = C_2 + D_{12} F_0, \qquad FC_2 = S + F_0, \tag{11}$$

where

$$\begin{gathered} A_0 = \begin{bmatrix} A_{110} & A_{120} \\ A_{210} & A_{220} \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ C_{20} = \begin{bmatrix} C_{210}^T \\ C_{220}^T \end{bmatrix}^T, \qquad S = \begin{bmatrix} S_1^T \\ S_2^T \end{bmatrix}^T, \qquad F_0 = \begin{bmatrix} F_{01}^T \\ F_{02}^T \end{bmatrix}^T. \end{gathered} \tag{12}$$

Theorem 4. *For a discrete-time singularly perturbed system in the form of* (1), *given a stabilizing state feedback controller with gain F_0, if there exist matrices $P_{11} > 0$, $P_{22} > 0$, and $G > 0$ and matrices P_{12}, L, X_{ij}, Y_{kj}, and $i = 1, \ldots, 5$, $k, j = 1, 2$ with appropriate dimensions, such that the following LMI holds:*

$$\begin{bmatrix} \Xi - XF_0^T T - \left(XF_0^T T\right)^T & \Phi - X - T^T F_0 Y^T & XC_2^T + T^T L \\ * & -Y - Y^T & YC_2^T \\ * & * & -G - G^T \end{bmatrix} < 0, \tag{13}$$

where

$$\begin{gathered} X = \left[X_{ij}\right], \quad Y = \left[Y_{kj}\right], \quad i = 1, \ldots, 5, \ k, j = 1, 2, \\ \Delta_{110} = P_{11} A_{110}^T + P_{12} A_{120}^T + \left(P_{11} A_{110}^T + P_{12} A_{120}^T\right)^T, \\ \Delta_{120} = P_{11} A_{210}^T + P_{12} A_{220}^T - P_{12}, \\ \Phi = \begin{bmatrix} 0 & P_{22} \\ P_{11} & P_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ T = \begin{bmatrix} 0 \\ B_{21} \\ B_{22} \\ D_{12} \\ 0 \end{bmatrix}^T, \\ \Xi = \begin{bmatrix} -P_{22} & P_{22} A_{120}^T & P_{22} A_{220}^T & P_{22} C_{120}^T & 0 \\ * & \Delta_{110} & \Delta_{120} & P_{11} C_{110}^T + P_{12} C_{120}^T & B_{11} \\ * & * & -P_{22} & 0 & B_{12} \\ * & * & * & -\gamma I & D_{11} \\ * & * & * & * & -\gamma I \end{bmatrix}, \end{gathered} \tag{14}$$

then there exists a scalar $\epsilon^* > 0$ such that for every $\epsilon \in (0, \epsilon^*]$, the system (3) *is asymptotically stable, and its transfer function satisfies $\|G\|_\infty < \gamma$ with the static output feedback controller in the form of* (2) *whose control gain is given by $F = LG^{-T}$.*

Proof. Based on Lemma 1, the closed-loop system (3) is asymptotically stable, and its transfer function satisfies $\|G\|_\infty < \gamma$ if there exists a positive definite matrix P such that the following LMI holds:

$$\begin{bmatrix} -P & P\widetilde{A}_\epsilon^T & P\widetilde{C}_1^T & 0 \\ * & -P & 0 & B_{1\epsilon} \\ * & * & -\gamma I & D_{11} \\ * & * & * & -\gamma I \end{bmatrix} < 0, \tag{15}$$

then for all $\epsilon \in (0, \epsilon^*]$, let

$$P(\epsilon) = \begin{bmatrix} \epsilon P_{11} & \epsilon P_{12} \\ * & P_{22} \end{bmatrix} > 0 \tag{16}$$

satisfy (15), and we have

$$\begin{bmatrix} -\epsilon P_{11} & -\epsilon P_{12} & \epsilon P_{11}+\epsilon^2\Delta_{11} & \epsilon\Delta_{12} & \epsilon\Gamma_1 & 0 \\ * & -P_{22} & \Delta_{21}(\epsilon) & \Delta_{22}(\epsilon) & \Gamma_2(\epsilon) & 0 \\ * & * & -\epsilon P_{11} & -\epsilon P_{12} & 0 & \epsilon B_{11} \\ * & * & * & -P_{22} & 0 & B_{12} \\ * & * & * & * & -\gamma I & D_{11} \\ * & * & * & * & * & -\gamma I \end{bmatrix} < 0, \tag{17}$$

where

$$\begin{gathered} \Delta_{11} = P_{11}\widetilde{A}_{11}^T + P_{12}\widetilde{A}_{12}^T, \qquad \Gamma_1 = P_{11}\widetilde{C}_{11}^T + P_{12}\widetilde{C}_{12}^T, \\ \Delta_{21}(\epsilon) = \epsilon\left(P_{12}^T + P_{22}\widetilde{A}_{12}^T\right) + \epsilon^2 P_{12}^T\widetilde{A}_{11}^T, \\ \Delta_{12} = P_{11}\widetilde{A}_{21}^T + P_{12}\widetilde{A}_{22}^T, \qquad \Gamma_2(\epsilon) = \epsilon P_{12}^T\widetilde{C}_{11}^T + P_{22}\widetilde{C}_{12}^T, \\ \Delta_{22}(\epsilon) = \epsilon P_{12}^T\widetilde{A}_{21}^T + P_{22}\widetilde{A}_{22}^T. \end{gathered} \tag{18}$$

Using the Schur complement to (17), we conclude that

$$\begin{bmatrix} -P_{22}+\epsilon\Theta_{11} & \epsilon P_{22}\widetilde{A}_{12}^T + \epsilon^2\Theta_{12} & P_{22}\widetilde{A}_{22}^T + \epsilon\Theta_{13} & P_{22}\widetilde{C}_{12}^T + \epsilon\Theta_{14} & 0 \\ * & \epsilon^2\left(\Delta_{11}+\Delta_{11}^T\right) + \epsilon^2\Theta_{22} & \epsilon\left(\Delta_{12}-P_{12}\right)+\epsilon^2\Theta_{23} & \epsilon\Gamma_1 + \epsilon^2\Theta_{24} & \epsilon B_{11} \\ * & * & -P_{22}+\epsilon\Theta_{33} & \epsilon\Theta_{34} & B_{12} \\ * & * & * & -\gamma I + \epsilon\Theta_{44} & D_{11} \\ * & * & * & * & -\gamma I \end{bmatrix} < 0, \tag{19}$$

where

$$\begin{gathered} \Theta_{11} = P_{12}^T P_{11}^{-1} P_{12}, \qquad \Theta_{12} = -P_{12}^T P_{11}^{-1} P_{12}\widetilde{A}_{12}^T, \\ \Theta_{13} = -P_{12}^T P_{11}^{-1} P_{12}\widetilde{A}_{22}^T, \qquad \Theta_{14} = -P_{12}^T P_{11}^{-1} P_{12}\widetilde{C}_{12}^T, \\ \Theta_{22} = \Delta_{11}^T P_{11}^{-1}\Delta_{11}, \qquad \Theta_{23} = \Delta_{11}^T P_{11}^{-1}\Delta_{12}, \\ \Theta_{24} = \Delta_{11}^T P_{11}^{-1}\Gamma_1, \qquad \Theta_{33} = \Delta_{12}^T P_{11}^{-1}\Delta_{12}, \\ \Theta_{34} = \Delta_{12}^T P_{11}^{-1}\Gamma_1, \qquad \Theta_{44} = \Gamma_1^T P_{11}^{-1}\Gamma_1. \end{gathered} \tag{20}$$

Pre- and post-multiplying (19) by $\mathrm{diag}(I, (1/\epsilon)I, I, I, I)$, we have

$$\Pi + \epsilon\Theta < 0, \tag{21}$$

where

$$\Pi = \begin{bmatrix} -P_{22} & P_{22}\widetilde{A}_{12}^T & P_{22}\widetilde{A}_{22}^T & P_{22}\widetilde{C}_{12}^T & 0 \\ * & \Delta_{11}+\Delta_{11}^T & \Delta_{12}-P_{12} & \Gamma_1 & B_{11} \\ * & * & -P_{22} & 0 & B_{12} \\ * & * & * & -\gamma I & D_{11} \\ * & * & * & * & -\gamma I \end{bmatrix},$$

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & 0 \\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} & 0 \\ * & * & \Theta_{33} & \Theta_{34} & 0 \\ * & * & * & \Theta_{44} & 0 \\ * & * & * & * & 0 \end{bmatrix}. \tag{22}$$

because ϵ is small enough, we can simplify (21) as in the following form:

$$\Pi < 0. \tag{23}$$

To turn (23) into an LMI, we will use the same idea proposed in [17]. By substituting the transformation defined before, we get another form of (23):

$$\Xi + \Phi\left(T^T S\right)^T + \left(T^T S\right)\Phi^T < 0, \tag{24}$$

where Ξ, Φ, S, and T are given as before.

Applying Lemma 2 to (24), we get

$$\begin{bmatrix} \Xi & \Phi \\ * & 0 \end{bmatrix} + \mathrm{Sym}\left\{\begin{bmatrix} X \\ Y \end{bmatrix}\begin{bmatrix} S^T T & -I \end{bmatrix}\right\} < 0. \tag{25}$$

It is obvious that (25) can be rewritten as follows:

$$\begin{bmatrix} \Xi & \Phi - X \\ * & -Y-Y^T \end{bmatrix} + \mathrm{Sym}\left\{\begin{bmatrix} XS^T \\ YS^T \end{bmatrix}\begin{bmatrix} T & 0 \end{bmatrix}\right\} < 0. \tag{26}$$

Taking the expression of S into account, we have

$$\begin{gathered} \begin{bmatrix} \Xi - XF_0^T T - \left(XF_0^T T\right)^T & \Phi - X - T^T F_0 Y^T \\ * & -Y-Y^T \end{bmatrix} \\ + \mathrm{Sym}\left\{\begin{bmatrix} XC_2^T \\ YC_2^T \end{bmatrix}\begin{bmatrix} F^T T & 0 \end{bmatrix}\right\} < 0. \end{gathered} \tag{27}$$

Using Lemma 3 to (27), we have

$$\begin{gathered} \begin{bmatrix} \Xi - XF_0^T T - \left(XF_0^T T\right)^T & \Phi - X - T^T F_0 Y^T & XC_2^T \\ * & -Y-Y^T & YC_2^T \\ * & * & 0 \end{bmatrix} \\ + \mathrm{Sym}\left\{\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} G \begin{bmatrix} F^T T & 0 & -I \end{bmatrix}\right\} < 0. \end{gathered} \tag{28}$$

By letting $L = FG^T$, we can finally get (13). This completes the proof. □

Both the static output feedback gain matrix F and the minimum H_∞ norm γ can be obtained by solving the following optimization problem:

$$\min_{P_{mj}, X_{ij}, Y_{kj}, L, G,} \gamma, \tag{OP1}$$

where $i = 1,\ldots,5$, $m \le j$, $m,k,j = 1,2$, subject to $\gamma > 0$, $P_{11} > 0$, $P_{22} > 0$, and $G > 0$ and the LMI in (13).

Note that the result in Theorem 4 is finally obtained by solving a γ-related optimal problem. While in the following part, a different approach of designing a static output feedback controller is given by solving an optimal problem in which ϵ is involved.

Theorem 5. *Given a scalar $\gamma > 0$ and a stabilizing state feedback controller with gain F_0, if there exist matrices $P_{11} > 0$, $P_{22} > 0$, $G > 0$, and $Q_{mm} > 0$, $m = 1,\ldots,4$ and matrices P_{12}, L, Y, $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{Y}$, X_i, Q_{mn}, $n < m$, $i,m,n = 1,\ldots,4$, with appropriate dimensions, such that the following set of LMIs hold:*

$$\begin{bmatrix} \widetilde{Q}_1 & \widetilde{Q}_2 & -\widetilde{X}_1 & \widetilde{X}_1 B_{21} + C_2^T L^T - F_0^T G^T \\ \widetilde{Q}_3 & \widetilde{Q}_4 & P_{12}^T - \widetilde{X}_2 & \widetilde{X}_2 B_{21} \\ -\widetilde{X}_1^T & P_{12} - \widetilde{X}_2^T & -\widetilde{Y} - \widetilde{Y}^T & \widetilde{Y} B_{21} \\ B_{21}^T \widetilde{X}_1^T + L C_2 - G F_0 & B_{21}^T \widetilde{X}_2^T & B_{21}^T \widetilde{Y}^T & -G - G^T \end{bmatrix} < 0,$$

$$\begin{bmatrix} \Pi_{11} & * & * & * & * & * \\ 0 & -\gamma I & * & * & * & * \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & * & * & * \\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} & * & * \\ -X_1^T & -X_2^T & \Phi - X_3^T & -X_4^T & -Y - Y^T & * \\ \Psi^T X_1^T - L C_2 + G F_0 & \Psi^T X_2^T & \Psi^T X_3^T & \Psi^T X_4^T & \Psi^T Y^T & -G - G^T \end{bmatrix} < 0, \quad (29)$$

where

$$\widetilde{Q}_1 = \begin{bmatrix} -Q_{11} & -Q_{21}^T \\ -Q_{21} & -Q_{22} \\ -Q_{31} & -Q_{32} \end{bmatrix}, \qquad \widetilde{Q}_2 = \begin{bmatrix} -Q_{31}^T & -Q_{41}^T \\ -Q_{32}^T & -Q_{42} \\ -Q_{33} & -Q_{43} \end{bmatrix},$$

$$\widetilde{Q}_3 = \begin{bmatrix} \left(-Q_{41} + P_{12}^T A_{110}\right)^T \\ -Q_{44}^T \end{bmatrix}^T,$$

$$\widetilde{Q}_4 = \begin{bmatrix} \left(-Q_{43} + P_{12}^T B_{11}\right)^T \\ -Q_{44}^T \end{bmatrix}^T,$$

$$\Pi_{11} = \begin{bmatrix} -\epsilon^* P_{11} & * \\ -P_{12}^T & -P_{22} \end{bmatrix}, \qquad \Pi_{32} = \begin{bmatrix} P_{22} B_{12} \\ P_{11} B_{11} + P_{12} B_{12} \\ D_{11} \end{bmatrix},$$

$$\Pi_{33} = \begin{bmatrix} -P_{22} & * & * \\ -P_{12} & -\epsilon^* P_{11} & * \\ 0 & 0 & -\gamma I \end{bmatrix},$$

$$\Pi_{41} = \begin{bmatrix} Q_{11} & Q_{21}^T \\ Q_{21} & Q_{22} \\ Q_{31} & Q_{32} \\ Q_{41} & Q_{42} \end{bmatrix}, \qquad \Pi_{43} = \begin{bmatrix} Q_{41}^T & 0 & 0 \\ Q_{42}^T & 0 & 0 \\ Q_{43}^T & 0 & 0 \\ Q_{44} & 0 & 0 \end{bmatrix},$$

$$\Pi_{31} = \begin{bmatrix} P_{12}^T + P_{22} A_{210} & P_{22} A_{220} \\ \epsilon^* P_{11} + P_{11} A_{110} + P_{12}^T A_{210} & P_{11} A_{120} + P_{12} A_{220} \\ C_{110} & C_{120} \end{bmatrix},$$

$$\Phi = \begin{bmatrix} P_{22} & 0 & 0 \\ P_{12} & P_{11} & 0 \\ 0 & 0 & I \end{bmatrix},$$

$$\Pi_{44} = \begin{bmatrix} -\epsilon^* Q_{11} & * & * & * \\ -\epsilon^* Q_{21} & -\epsilon^* Q_{22} & * & * \\ -\epsilon^* Q_{31} & -\epsilon^* Q_{32} & -\epsilon^* Q_{33} & * \\ -\epsilon^* Q_{41} & -\epsilon^* Q_{42} & -\epsilon^* Q_{43} & -\epsilon^* Q_{44} \end{bmatrix},$$

$$\Pi_{42} = \begin{bmatrix} Q_{31}^T \\ Q_{32}^T \\ Q_{33} \\ Q_{43} \end{bmatrix}, \qquad \Psi = \begin{bmatrix} B_{22} \\ B_{21} \\ D_{12} \end{bmatrix}, \quad (30)$$

then for any singular perturbation parameter $\epsilon \in (0, 1/\epsilon^*]$, by introducing a static output feedback controller in the form of (2), the closed-loop system (3) is asymptotically stable, and its transfer function satisfies $\|G\|_\infty < \gamma$ with the controller gain $F = G^{-1} L$.

Proof. According to Lemma 1, we know that the closed-loop system (3) is asymptotically stable, and its transfer function is less than γ if there exists a positive definite matrix P such that the following LMI holds:

$$\begin{bmatrix} -P & * & * & * \\ 0 & -\gamma I & * & * \\ P\widetilde{A}_\epsilon & P B_{1\epsilon} & -P & * \\ \widetilde{C}_1 & D_{11} & 0 & -\gamma I \end{bmatrix} < 0. \quad (31)$$

For every singular perturbation parameter $\epsilon \in (0, 1/\epsilon^*]$, let

$$P(\epsilon) = \begin{bmatrix} \frac{1}{\epsilon} P_{11} & * \\ P_{12}^T & P_{22} \end{bmatrix} > 0 \quad (32)$$

satisfy (31), and we get

$$\begin{bmatrix} \frac{1}{\epsilon}P_{11} & * & * & * & * & * \\ -P_{12}^T & -P_{22} & * & * & * & * \\ 0 & 0 & -\gamma I & * & * & * \\ \frac{1}{\epsilon}P_{11}+P_{11}\widetilde{A}_{11}+P_{12}\widetilde{A}_{21} & P_{11}\widetilde{A}_{12}+P_{12}\widetilde{A}_{22} & P_{11}B_{11}+P_{12}B_{12} & -\frac{1}{\epsilon}P_{11} & * & * \\ P_{12}^T+\epsilon P_{12}^T\widetilde{A}_{11}+P_{22}\widetilde{A}_{21} & \epsilon P_{12}\widetilde{A}_{12}+P_{22}\widetilde{A}_{22} & \epsilon P_{12}^T B_{11}+P_{22}B_{12} & P_{12}^T & P_{22} & * \\ \widetilde{C}_{11} & \widetilde{C}_{12} & D_{11} & 0 & 0 & -\gamma I \end{bmatrix} < 0. \quad (33)$$

Pre- and post-multiplying (33) by

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \quad (34)$$

and its transpose, respectively, then for every $\epsilon \in (0, 1/\epsilon^*]$, we have

$$\begin{bmatrix} \Theta_{11}\left(\frac{1}{\epsilon}\right) & * \\ \Theta_{21}\left(\frac{1}{\epsilon}\right) & \Theta_{22}\left(\frac{1}{\epsilon}\right) \end{bmatrix} + \begin{bmatrix} \epsilon\Omega & 0 \\ 0 & 0 \end{bmatrix} < 0, \quad (35)$$

where

$$\Omega = \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ P_{12}^T\widetilde{A}_{11} & P_{12}^T\widetilde{A}_{12} & P_{12}^T B_{11} & 0 \end{bmatrix},$$

$$\Theta_{11}\left(\frac{1}{\epsilon}\right) = \begin{bmatrix} -\frac{1}{\epsilon}P_{11} & * & * & * \\ -P_{12}^T & -P_{22} & * & * \\ 0 & 0 & -\gamma I & * \\ P_{12}^T+P_{22}\widetilde{A}_{21} & P_{22}\widetilde{A}_{22} & P_{22}B_{12} & -P_{22} \end{bmatrix},$$

$$\Theta_{21}\left(\frac{1}{\epsilon}\right) = \begin{bmatrix} \frac{1}{\epsilon}P_{11}+\Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & -P_{12} \\ \widetilde{C}_{11} & \widetilde{C}_{12} & D_{11} & 0 \end{bmatrix}, \quad (36)$$

$$\Theta_{22}\left(\frac{1}{\epsilon}\right) = \begin{bmatrix} -\frac{1}{\epsilon}P_{11} & * \\ 0 & -\gamma I \end{bmatrix},$$

$$\Upsilon_{11} = P_{11}\widetilde{A}_{11} + P_{12}\widetilde{A}_{21},$$

$$\Upsilon_{12} = P_{11}\widetilde{A}_{12} + P_{12}\widetilde{A}_{22},$$

$$\Upsilon_{13} = P_{11}B_{11} + P_{12}B_{12}.$$

If there exists a positive definite matrix $Q = [Q_{ij}]$, $i, j = 1, \ldots, 4$ satisfies

$$\Omega - Q < 0, \quad (37)$$

and for all $\epsilon \in (0, 1/\epsilon^*]$,

$$\begin{bmatrix} \Theta_{11}(\epsilon^*) & * & * \\ \Theta_{21}(\epsilon^*) & \Theta_{22}(\epsilon^*) & * \\ Q & 0 & -\epsilon^* Q \end{bmatrix} < 0, \quad (38)$$

then we can easily conclude that

$$\begin{bmatrix} \Theta_{11}\left(\frac{1}{\epsilon}\right) & * & * \\ \Theta_{21}\left(\frac{1}{\epsilon}\right) & \Theta_{22}\left(\frac{1}{\epsilon}\right) & * \\ Q & 0 & -\frac{1}{\epsilon}Q \end{bmatrix} < 0. \quad (39)$$

Applying the Schur complement to (39), we get

$$\begin{bmatrix} \Theta_{11}\left(\frac{1}{\epsilon}\right) & * \\ \Theta_{21}\left(\frac{1}{\epsilon}\right) & \Theta_{22}\left(\frac{1}{\epsilon}\right) \end{bmatrix} + \begin{bmatrix} \epsilon Q & 0 \\ 0 & 0 \end{bmatrix} < 0. \quad (40)$$

Taking (37) into consideration, we have (35).

To turn (37) and (40) into LMIs, we still adopt the same technique used in [17]. By introducing the transformations defined before, we can rewrite (37) and (40) into the following form:

$$\begin{bmatrix} \widetilde{Q}_1 & \widetilde{Q}_2 \\ \widetilde{Q}_3 & \widetilde{Q}_4 \end{bmatrix} + \mathrm{Sym}\left\{\begin{bmatrix} 0 \\ P_{12}^T \end{bmatrix} \begin{bmatrix} B_{21}S & 0 \end{bmatrix}\right\} < 0,$$

$$\begin{bmatrix} \Pi_{11} & * & * & * \\ 0 & -\gamma I & * & * \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & * \\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} \end{bmatrix} \quad (41)$$

$$+ \mathrm{Sym}\left\{\begin{bmatrix} 0 \\ 0 \\ \Phi \\ 0 \end{bmatrix} \Psi \begin{bmatrix} S & 0 & 0 & 0 \end{bmatrix}\right\} < 0.$$

Applying Lemma 2 to (41), we have

$$\begin{bmatrix} \widetilde{Q}_1 & \widetilde{Q}_2 & 0 \\ \widetilde{Q}_3 & \widetilde{Q}_4 & P_{12}^T \\ 0 & P_{12} & 0 \end{bmatrix} + \text{Sym}\left\{\begin{bmatrix} \widetilde{X}_1 \\ \widetilde{X}_2 \\ \widetilde{Y} \end{bmatrix}\left[B_{21}S \;\; 0 \;\; -I\right]\right\} < 0, \quad (42)$$

$$\begin{bmatrix} \Pi_{11} & 0 & \Pi_{31}^T & \Pi_{41}^T & 0 \\ 0 & -\gamma I & \Pi_{32}^T & \Pi_{42}^T & 0 \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & \Pi_{43}^T & \Phi \\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} & 0 \\ 0 & 0 & \Phi^T & 0 & 0 \end{bmatrix} \quad (43)$$

$$+ \text{Sym}\left\{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ Y \end{bmatrix}\Psi\left[S \;\; 0 \;\; 0 \;\; 0 \;\; -I\right]\right\} < 0.$$

It is obvious that (42) and (43) can be rewritten as

$$\begin{bmatrix} \widetilde{Q}_1 & \widetilde{Q}_2 & -\widetilde{X}_1 \\ \widetilde{Q}_3 & \widetilde{Q}_4 & P_{12}^T - \widetilde{X}_2 \\ -\widetilde{X}_1^T & P_{12} - \widetilde{X}_2^T & -\widetilde{Y} - \widetilde{Y}^T \end{bmatrix} \quad (44)$$

$$+ \text{Sym}\left\{\begin{bmatrix} \widetilde{X}_1 B_{21} \\ \widetilde{X}_2 B_{21} \\ \widetilde{Y} B_{21} \end{bmatrix}\left[S \;\; 0 \;\; 0\right]\right\} < 0,$$

$$\begin{bmatrix} \Pi_{11} & 0 & \Pi_{31}^T & \Pi_{41}^T & -X_1 \\ 0 & -\gamma I & \Pi_{32}^T & \Pi_{34}^T & -X_2 \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & \Pi_{43}^T & \Phi - X_3 \\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} & -X_4 \\ -X_1^T & -X_2^T & \Phi^T - X_3^T & -X_1^T & -Y - Y^T \end{bmatrix} \quad (45)$$

$$+ \text{Sym}\left\{\begin{bmatrix} X_1\Psi \\ X_2\Psi \\ X_3\Psi \\ X_4\Psi \\ Y\Psi \end{bmatrix}\left[S \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]\right\} < 0,$$

where Ψ is given as before.

Taking Lemma 3 into account, we know that (43) and (44) are equivalent to

$$\begin{bmatrix} \widetilde{Q}_1 & \widetilde{Q}_2 & -\widetilde{X}_1 & \widetilde{X}_1 B_{21} \\ \widetilde{Q}_3 & \widetilde{Q}_4 & P_{12}^T - \widetilde{X}_2 & \widetilde{X}_2 B_{21} \\ -\widetilde{X}_1^T & P_{12} - \widetilde{X}_2^T & -\widetilde{Y} - \widetilde{Y}^T & \widetilde{Y} B_{21} \\ B_{21}^T \widetilde{X}_1^T & B_{21}^T \widetilde{X}_2^T & B_{21}^T \widetilde{Y}^T & 0 \end{bmatrix} \quad (46)$$

$$+ \text{Sym}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} G_1\left[S \;\; 0 \;\; 0 \;\; -I\right]\right\} < 0,$$

$$\begin{bmatrix} \Pi_{11} & 0 & \Pi_{31}^T & \Pi_{41}^T & -X_1 & X_1\Psi \\ 0 & -\gamma I & \Pi_{32}^T & \Pi_{34}^T & -X_2 & X_2\Psi \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & \Pi_{43}^T & \Phi - X_3 & X_3\Psi \\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} & -X_4 & X_4\Psi \\ -X_1^T & -X_2^T & \Phi^T - X_3^T & -X_4^T & -Y - Y^T & Y\Psi \\ \Psi^T X_1^T & \Psi^T X_2^T & \Psi^T X_3^T & \Psi^T X_4^T & Y\Psi^T & 0 \end{bmatrix} \quad (47)$$

$$+ \text{Sym}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} G_2\left[S \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; -I\right]\right\} < 0,$$

respectively.

From the above analysis, we learn that if there exists $G_1 = G_2 = G$ which satisfies both (46) and (47), then (29) can be finally concluded by considering the expression of S. This completes the proof. □

We can obtain the static output feedback controller gain matrix F, as well as the stability upper bound of the singular perturbation parameter which we record as $1/\epsilon^*$ by solving the following optimal problem:

$$\min_{P_{ij}, Q_{mn}, X_k, Y, \widetilde{X}_l, \widetilde{Y}, L, G} \epsilon^*, \quad \text{(OP2)}$$

where $i \leq j$, $i, j, l = 1, 2$, $m, n, k = 1, \ldots, 4$, subject to $\epsilon^* > 0$, $P_{ii} > 0$, $Q_{nn} > 0$, and $G > 0$ and the LMIs in (29).

Remark 6. The result in Theorem 4 finally turns into (OP1), which is solved by optimizing γ, the minimum H_∞ norm of the closed-loop system (3). While results in Theorem 5 are obtained by solving a different optimal problem (OP2), which optimizes the stability upper bound $1/\epsilon^*$ with a fixed performance index γ.

4. A Numerical Example

In this section, a numerical example is presented to illustrate the effectiveness of the proposed results.

Example 7. Consider a discrete-time singularly perturbed system described by (1) with

$$A_{\epsilon} = \begin{bmatrix} 1+0.2129\epsilon & 1.8140\epsilon \\ -0.1814 & 0.8179 \end{bmatrix},$$
$$B_{1\epsilon} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \qquad B_{2\epsilon} = \begin{bmatrix} 0.1874\epsilon \\ 0.1812 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad D_{11} = \begin{bmatrix} 0.01 \\ 0 \\ 0 \end{bmatrix}, \tag{48}$$
$$D_{12} = \begin{bmatrix} 0.31 \\ 0 \\ 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0.4394 \\ 0.1372 \end{bmatrix}^T.$$

First of all, by solving the optimal problem (OP1), we can get the introduced static output feedback controller gain

$$F = -0.1078 \tag{49}$$

and the minimum H_∞ norm of the closed-loop system

$$\gamma = 0.8557. \tag{50}$$

Then, referring to (OP2), we can solve the corresponding LMIs with $\gamma = 1$. The obtained static output feedback controller gain is

$$F = -23.3354, \tag{51}$$

while the stability upper bound of the singular perturbation parameter is

$$\frac{1}{\epsilon^*} = 0.2775. \tag{52}$$

Remark 8. Though the performance of the closed-loop system is not as good as the state feedback case, static output feedback controller plays more important role in implemental sense with proper performance.

5. Conclusion

In this paper, H_∞ control problems for fast sampling singularly perturbed systems via static output feedback have been discussed. Rather than adopting the traditional design method of decomposing the original system into fast and slow subsystems, two LMI-based sufficient conditions have been given to guarantee the existence of static output feedback controllers and the asymptotical stability of the closed-loop system with a transfer function whose H_∞ norm is less than γ. With LMI toolbox in matlab platform, the obtained LMI results can be solved easily. The proposed methods simplify the controller design procedure.

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