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Research Article

Some Identities on the High-Order q-Euler Numbers and Polynomials with Weight 0

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We construct the Nth order nonlinear ordinary differential equation related to the generating function of q-Euler numbers with weight 0. From this, we derive some identities on q-Euler numbers and polynomials of higher order with weight 0.

1. Introduction

As a well-known definition, the Euler polynomial $E_n(x)$ is given by

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \quad (1)$$

In the special case, $x=0$, $E_n(0)=E_n$ is the nth Euler number.

From (1), we note that

$$E_0=1, \quad (E+1)^n+E_n=0, \quad \text{if } n>0, \quad (2)$$

with the usual convention of replacing E^n by E_n (see [1–16]).

In the viewpoint of the q-extension of (1) and (2), let us consider the following q-Euler number and polynomial:

$$\frac{2}{qe^t+1}e^{xt}=\sum_{n=0}^{\infty}\widetilde{E}_{n,q}(x)\frac{t^n}{n!}, \quad (3)$$

$$\widetilde{E}_{0,q}=\frac{2}{1+q}, \quad q\left(\widetilde{E}_q+1\right)^n+\widetilde{E}_{n,q}=0, \quad \text{if } n>0, \quad (4)$$

with the usual convention of replacing $\widetilde{E}_q^n$ by $\widetilde{E}_{n,q}$.

Equation (3) is called the generating function of q-Euler polynomial with weight 0. In the case $x=0$, $\widetilde{E}_{n,q}(0)=\widetilde{E}_{n,q}$ is the nth q-Euler number with weight 0 (see [5, 11]).

Throughout this paper, let q be a complex number with $|q|<1$. As $q\to 1$, we obtain (1) and (2) from (3) and (4).

The generating function of Eulerian polynomial $H_n(x \mid u)$ is defined by

$$\frac{1-u}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}H_n(x\mid u)\frac{t^n}{n!}, \quad (5)$$

where $u\in\mathbb{C}$ with $u\neq 1$. In the special case, $x=0$, $H_n(0\mid u)=H_n(u)$ is called the nth Eulerian number (see [1–3]). Sometimes that is called the nth Frobenius-Euler number (see [9–11, 15]).

From (1) and (5), we note that $H_n(x\mid -1)=E_n(x)$. From (5), we have

$$H_0(u)=1, \quad H_n(1\mid u)-uH_n(u)=(1-u)\delta_{0,n}, \quad (6)$$

where $\delta_{n,k}$ is Kronecker symbol (see [9–11]).

For $N\in\mathbb{N}$, the q-Euler polynomial of order N is defined by the generating function as follows:

$$G_q^N(t,x)=\underbrace{\left(\frac{2}{qe^t+1}\right)\times\cdots\times\left(\frac{2}{qe^t+1}\right)}_{N\text{-times}}e^{xt}=\sum_{n=0}^{\infty}\widetilde{E}_{n,q}^{(N)}(x)\frac{t^n}{n!}. \quad (7)$$

In the special case, $x=0$, $\widetilde{E}_{n,q}^{(N)}(0)=\widetilde{E}_{n,q}^{(N)}$ is called the nth q-Euler number of order N with weight 0 (see [5, 11]).

In [9], Kim derived some identities between the sums of products of Frobenius-Euler polynomials and Frobenius-Euler polynomials of higher order. The main idea is to construct nonlinear ordinary differential equations with respect to t which are closely related to the generating function of Frobenius-Euler polynomial. In [3], Choi considered nonlinear ordinary differential equations with respect to u not t.

In this paper, we construct nonlinear ordinary differential equations with respect to q. The purpose of this paper is to give some new identities on the high order q-Euler numbers and polynomials with weight 0 by using the differential equations of q.

2. Construction of Nonlinear Differential Equations

We define

$$G = G(q) = \frac{1}{qe^t + 1},$$
$$G^N(t,x) = \underbrace{G \times \cdots \times G}_{N\text{-times}} e^{xt} \quad \text{for } N \in \mathbb{N}. \tag{8}$$

From (7) and (8), we note that

$$G_q^N(t,x) = 2^N G^N(t,x) = 2^N G^N e^{xt}. \tag{9}$$

By differentiating (8) with respect to q, we get

$$G^{(1)} = \frac{dG}{dq} = -\frac{qe^t + 1 - 1}{q(qe^t+1)^2} = -\frac{G}{q} + \frac{G^2}{q},$$
$$qG^{(1)} + G = G^2. \tag{10}$$

By differentiating (10) with respect to q, we get

$$q^2 G^{(2)} + 4qG^{(1)} + 2G = 2!G^3, \tag{11}$$

where $G^{(N)} = d^N G/dq^N$.

By the derivative of (11) with respect to q, we have

$$q^3 G^{(3)} + 9q^2 G^{(2)} + 18qG^{(1)} + 3!G = 3!G^4. \tag{12}$$

Continuing this process, we get

$$(N-1)!G^N = \sum_{k=0}^{N-1} a_k(N) q^k G^{(k)}. \tag{13}$$

Let us consider the derivative of (13) with respect to q to find the coefficient $a_k(N)$ in (13).

By (10), we get

$$q\frac{d}{dq}\left((N-1)!\, G^N\right) = N!G^{N-1} qG^{(1)}$$
$$= N!G^{N-1}\left(-G + G^2\right)$$
$$= N!G^{N+1} - N(N-1)!G^N. \tag{14}$$

From (13) and (14), we get

$$N!G^{N+1} = N(N-1)!G^N + \sum_{k=0}^{N-1} k a_k(N) q^k G^{(k)} + \sum_{k=1}^{N} a_{k-1}(N) q^k G^{(k)}, \tag{15}$$

where $N!G^{N+1} = \sum_{k=0}^{N} a_k(N+1) q^k G^{(k)}$.

By comparing coefficients on both sides of (15), we obtain the following recurrence relations:

$$a_0(N+1) = N a_0(N), \qquad a_N(N+1) = a_{N-1}(N), \tag{16}$$

$$a_k(N+1) = N a_k(N) + k a_k(N) + a_{k-1}(N), \tag{17}$$

for $1 \le k \le N-1$ and $a_k(N) = 0$.

From the first part of (16), we have

$$a_0(N+1) = N a_0(N)$$
$$= N(N-1) a_0(N-1)$$
$$= \cdots = N! a_0(2). \tag{18}$$

By (10) and (13), we have

$$qG^{(1)} + G = G^2 = \sum_{k=0}^{1} a_k(2) q^k G^{(k)}$$
$$= a_0(2) G + a_1(2) qG^{(1)}. \tag{19}$$

From (18) and (19), we get

$$a_0(2) = 1, \qquad a_1(2) = 1, \qquad a_0(N) = (N-1)!. \tag{20}$$

From the second part of (16), we have

$$a_N(N+1) = a_{N-1}(N) = \cdots = a_1(2) = 1. \tag{21}$$

To find $a_k(N)$ in (13) from (17), we set

$$g(t,s) = \sum_{N \ge 1} \sum_{0 \le k \le N-1} a_k(N) \frac{t^N}{N!} s^k, \tag{22}$$

where $|t| < 1$ (see [9]).

From (17) and (22), we have

$$\sum_{N \ge 1} \sum_{0 \le k \le N-1} a_{k+1}(N+1) \frac{t^N}{N!} s^k$$
$$= \sum_{N \ge 1} \sum_{0 \le k \le N-1} N a_{k-1}(N) \frac{t^N}{N!} s^k$$
$$+ \sum_{N \ge 1} \sum_{0 \le k \le N-1} (k+1) a_{k+1}(N) \frac{t^N}{N!} s^k + g(t,s). \tag{23}$$

From the left hand side of (23), we have

$$\begin{aligned}
&\sum_{N\geq 1}\sum_{0\leq k\leq N-1} a_{k+1}(N+1)\frac{t^N}{N!}s^k \\
&\quad = \frac{1}{s}\sum_{N\geq 2}\sum_{1\leq k\leq N-1} a_k(N)\frac{t^{N-1}}{(N-1)!}s^k \\
&\quad = \frac{1}{s}\sum_{N\geq 2}\left(\sum_{0\leq k\leq N-1} a_k(N)\frac{t^{N-1}}{(N-1)!}s^k - a_0(N)\frac{t^{N-1}}{(N-1)!}\right) \\
&\quad = \frac{1}{s}\left(\sum_{N\geq 1}\sum_{0\leq k\leq N-1} a_k(N)\frac{t^{N-1}}{(N-1)!}s^k - a_0(1) - \sum_{N\geq 2} t^{N-1}\right) \\
&\quad = \frac{1}{s}\left(g_t + \frac{1}{t-1}\right),
\end{aligned} \tag{24}$$

where $g_t = \partial g/\partial t$. From the first term of the right hand side of (23), we have

$$\begin{aligned}
&\sum_{N\geq 1}\sum_{0\leq k\leq N-1} Na_{k+1}(N)\frac{t^N}{N!}s^k \\
&\quad = \frac{t}{s}\sum_{N\geq 1}\sum_{1\leq k\leq N-1} a_k(N)\frac{t^{N-1}}{(N-1)!}s^k \\
&\quad = \frac{t}{s}\left(\sum_{N\geq 1}\sum_{0\leq k\leq N-1} a_k(N)\frac{t^{N-1}}{(N-1)!}s^k - \sum_{N\geq 1}\frac{a_0(N)}{(N-1)!}t^{N-1}\right) \\
&\quad = \frac{t}{s}\left(g_t + \frac{1}{t-1}\right).
\end{aligned} \tag{25}$$

From the second term of the right hand side of (23), we have

$$\begin{aligned}
&\sum_{N\geq 1}\sum_{0\leq k\leq N-1} (k+1)a_{k+1}(N)\frac{t^N}{N!}s^k \\
&\quad = \sum_{N\geq 1}\sum_{1\leq k\leq N} ka_k(N)\frac{t^N}{N!}s^{k-1} \\
&\quad = \sum_{N\geq 1}\sum_{0\leq k\leq N-1} ka_k(N)\frac{t^N}{N!}s^{k-1} = g_s,
\end{aligned} \tag{26}$$

where $g_s = \partial g/\partial s$.

From (22)–(26), we obtain the following initial value problem quasilinear first-order partial differential equation:

$$\begin{gathered}
(t-1)g_t + sg_s = -sg - 1, \quad |t| < 1, \\
g(0,s) = 0, \quad s \in \mathbb{R}.
\end{gathered} \tag{27}$$

We consider Cauchy problem for the following first-order quasilinear partial differential equation:

$$\begin{aligned}
&P(x,y,z)z_x + Q(x,y,z)z_y \\
&\quad = R(x,y,z), \\
&z(x_0(t), y_0(t)) = z_0(t), \quad t \in I,
\end{aligned} \tag{28}$$

where I is some interval.

We know that (28) has a unique solution under some conditions as follows.

Theorem A (see [17, page 65]). *Suppose that P, Q, and R are of class C^1 in a domain Ω of $\mathbb{R}^3$ containing the point (x_0, y_0, z_0) and suppose that*

$$P(x_0,y_0,z_0)\frac{dy_0(t_0)}{dt} - Q(x_0,y_0,z_0)\frac{dx_0(t_0)}{dt} \neq 0. \tag{29}$$

Then in a neighborhood U of (x_0, y_0) there exists a unique solution of (28) at every point of initial curve contained in U.

Since (27) satisfies (29) and regularity conditions, there exists a unique solution of (27).

It is customary to write (27) in the form

$$\frac{dt}{t-1} = \frac{ds}{s} = \frac{dg}{-sg-1}, \tag{30}$$

$$t = 0, \quad s = p, \quad g = 0. \tag{31}$$

Since $dt/(t-1) = ds/s$ is separable, we get

$$u_1(t,s,g) = \frac{1-t}{s}. \tag{32}$$

u_1 is a solution of partial differential equation of (27).

From (30), we get the linear equation

$$\frac{dg}{ds} = -g - \frac{1}{s}. \tag{33}$$

By the integrating factor method, we have

$$u_2(t,s,g) = e^s g + E_i(s). \tag{34}$$

The exponential integral $E_i(s)$ is defined by

$$\begin{aligned}
E_i(s) &= \int_{-\infty}^{s}\frac{e^r}{r}dr \\
&= \gamma + \ln|s| + \sum_{n=1}^{\infty}\frac{s^n}{n\cdot n!}, \quad (s\in\mathbb{R}, s\neq 0),
\end{aligned} \tag{35}$$

where γ is Euler constant.

u_2 is another solution of partial differential equation of (27), and u_1 and u_2 are linearly independent.

From the parameterized initial conditions (31), (33), and (34), we get

$$u_2 = E_i\left(\frac{1}{u_1}\right), \qquad e^s g + E_i(x) = E_i\left(\frac{s}{1-t}\right). \tag{36}$$

Thus, from (35) and (36), we obtain the following unique solution of (27):

$$g(t,s)=e^{-s}\left(-\ln|1-t|+\sum_{n=1}^{\infty}\frac{s^n}{n\cdot n!}\left(\left(\frac{1}{1-t}\right)^n-1\right)\right). \quad (37)$$

Moreover, if we choose another initial condition

$$g(t,0)=\sum_{N\geq 1}^{\infty}a_0(N)\frac{t^N}{N!}=\sum_{N\geq 1}^{\infty}\frac{t^N}{N} \quad (38)$$

from (20) and (22), then (37) satisfies it.

We note that

$$\begin{aligned}\left(\frac{1}{1-t}\right)^n-1&=\underbrace{\left(\sum_{l_1\geq 0}t^{l_1}\right)\times\cdots\times\left(\sum_{l_n\geq 0}t^{l_n}\right)}_{n\text{-times}}-1\\&=\sum_{N\geq 1}\left(\sum_{l_1+\cdots+l_n=N}t^N\right)\\&=\sum_{N\geq 1}\binom{n+N-1}{N}t^N.\end{aligned} \quad (39)$$

By (37) and (39), we get

$$\begin{aligned}g(t,s)&=\left(\sum_{k\geq 0}\frac{(-1)^k}{k!}s^k\right)\left(\sum_{N\geq 1}\frac{t^N}{N}\right)\\&\quad+\left(\sum_{k\geq 0}\frac{(-1)^k}{k!}s^k\right)\\&\quad\times\left(\sum_{n\geq 1}\frac{s^n}{n\cdot n!}\sum_{N\geq 1}\binom{n+N-1}{N}t^N\right)\\&=\sum_{N\geq 1}\sum_{0\leq k\leq N-1}\frac{(-1)^k}{N\cdot k!}t^Ns^k+\sum_{N\geq 1}\sum_{k\geq N}\frac{(-1)^k}{N\cdot k!}t^Ns^k\\&\quad+\sum_{N\geq 1}\sum_{1\leq k\leq N-1}\left(\sum_{l=1}^{k}\frac{(-1)^{k-l}}{(k-l)!\,l\cdot l!}\binom{l+N-1}{N}\right)t^Ns^k\\&\quad+\sum_{N\geq 1}\sum_{k\geq N}\left(\sum_{l=1}^{k}\frac{(-1)^{k-l}}{(k-l)!\,l\cdot l!}\binom{l+N-1}{N}\right)t^Ns^k.\end{aligned} \quad (40)$$

It is known that

$$\begin{aligned}(-1)^l\binom{l+N-1}{N}&=\frac{l}{N}\binom{-N}{l},\\\sum_{l=0}^{k}\binom{k}{l}\binom{N}{l}&=\binom{k+N}{k}.\end{aligned} \quad (41)$$

In the case of $k\geq N$ in (40), from (41), we get

$$\begin{aligned}&\frac{(-1)^k}{N\cdot k!}+\sum_{l=1}^{k}\frac{(-1)^{k-l}}{(k-l)!l\cdot l!}\binom{l+N-1}{N}\\&\quad=(-1)^k\left(\frac{1}{N\cdot k!}+\frac{1}{N\cdot k!}\sum_{l=1}^{k}\binom{k}{l}\binom{-N}{l}\right)\\&\quad=(-1)^k\frac{1}{N\cdot k!}\left(1+\binom{k-N}{k}-1\right)=0.\end{aligned} \quad (42)$$

By (40) and (41), we get

$$\begin{aligned}g(t,s)&=\sum_{N\geq 1}\sum_{1\leq k\leq N-1}(-1)^k\frac{(N-1)!}{k!}\binom{k-N}{k}\frac{t^N}{N!}s^k\\&\quad+\sum_{N\geq 1}(N-1)!\frac{t^N}{N!}\\&=\sum_{N\geq 1}\sum_{0\leq k\leq N-1}(N-k-1)!\binom{N-1}{k}^2\frac{t^N}{N!}s^k,\end{aligned} \quad (43)$$

where $\binom{k-N}{k}=(-1)^k\binom{N-1}{k}$. Thus, by (22) and (43), we get

$$a_k(N)=(N-k-1)!\binom{N-1}{k}^2. \quad (44)$$

Therefore, by (13) and (44), we obtain the following theorem.

Theorem 1. *For $q\in\mathbb{C}$ with $|q|<1$ and $N\in\mathbb{N}$, one can consider the following nonlinear $(N-1)$th order ordinary differential equation with respect to q:*

$$\begin{aligned}G^N(q)&=\frac{1}{(N-1)!}\sum_{k=0}^{N-1}(N-k-1)!\binom{N-1}{k}^2q^kG^{(k)}\\&=\sum_{k=0}^{N-1}\frac{1}{k!}\binom{N-1}{k}q^kG^{(k)},\end{aligned} \quad (45)$$

where $G^{(k)}=d^kG^{(q)}/dq^k$ and $G^N(q)=\underbrace{G(q)\times\cdots\times G(q)}_{N\text{-times}}$. Then $G(q)=1/(qe^t+1)$ is a solution of (45).

Let us define $G^{(k)}(t,x)=G^{(k)}(q)e^{xt}$. Then we obtain the following corollary.

Corollary 2. *For $N\in\mathbb{N}$, one considers*

$$\begin{aligned}G^N(t,x)&=\frac{1}{(N-1)!}\sum_{k=0}^{N-1}(N-k-1)!\binom{N-1}{k}^2q^kG^{(k)}(t,x)\\&=\sum_{k=0}^{N-1}\frac{1}{k!}\binom{N-1}{k}q^kG^{(k)}(t,x).\end{aligned} \quad (46)$$

Then $G(t,x)=e^{xt}/(qe^t+1)$ is a solution of (46).

3. Identities on the High-Order q-Euler Numbers and Polynomials with Weight 0

From (3), (7), and (8), we get

$$G^{N}(q)=\frac{1}{2^{N}}\underbrace{\left(\frac{2}{qe^{t}+1}\right)\times\cdots\times\left(\frac{2}{qe^{t}+1}\right)}_{N\text{-times}}=\frac{1}{2^{N}}\sum_{n=0}^{\infty}\widetilde{E}_{n,q}^{(N)}\frac{t^{n}}{n!},\quad(47)$$

$$G(q)=\frac{1}{2}\frac{2}{qe^{t}+1}=\frac{1}{2}\sum_{n=0}^{\infty}\widetilde{E}_{n,q}\frac{t^{n}}{n!}.$$

From (47), we note that

$$G^{(k)}=\frac{d^{k}G(q)}{dq^{k}}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{d^{k}\widetilde{E}_{n,q}}{dq^{k}}\frac{t^{n}}{n!}.\quad(48)$$

Therefore, by (47), (48), and (45), we obtain the following theorem.

Theorem 3. *For $N\in\mathbb{N}$ and $n\in\mathbb{N}\cup\{0\}$, one has*

$$\widetilde{E}_{n,q}^{(N)}=2^{N-1}\sum_{k=0}^{N-1}\frac{1}{k!}\binom{N-1}{k}q^{k}\frac{d^{k}\widetilde{E}_{n,q}}{dq^{k}}.\quad(49)$$

From (48), we get

$$\begin{aligned}G^{(k)}(t,x)&=G^{(k)}(q)e^{xt}\\&=\left(\frac{1}{2}\sum_{n=0}^{\infty}\frac{d^{k}\widetilde{E}_{n,q}}{dq^{k}}\frac{t^{n}}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{x^{n}t^{n}}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{1}{2}\binom{n}{l}x^{n-l}\frac{d^{k}\widetilde{E}_{l,q}}{dq^{k}}\right)\frac{t^{n}}{n!}.\end{aligned}\quad(50)$$

Therefore, by (7), (47), and (50), we obtain the following corollary.

Corollary 4. *For $N\in\mathbb{N}$ and $n\in\mathbb{N}\cup\{0\}$, one has*

$$\widetilde{E}_{n,q}^{(N)}(x)=2^{N-1}\sum_{k=0}^{N-1}\frac{1}{k!}\binom{N-1}{k}q^{k}\sum_{l=0}^{n}\binom{n}{l}x^{n-l}\frac{d^{k}\widetilde{E}_{l,q}}{dq^{k}}.\quad(51)$$

From (3) and (7), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{E}_{n,q}^{(N)}\frac{t^{n}}{n!}&=\underbrace{\left(\frac{2}{qe^{t}+1}\right)\times\cdots\times\left(\frac{2}{qe^{t}+1}\right)}_{N\text{-times}}\\&=\left(\sum_{l_1=0}^{\infty}\widetilde{E}_{l_1,q}\frac{t^{l_1}}{l_1!}\right)\times\cdots\times\left(\sum_{l_N=0}^{\infty}\widetilde{E}_{l_N,q}\frac{t^{l_N}}{l_N!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l_1+\cdots+l_N=n}\frac{n!\widetilde{E}_{l_1,q}\cdots\widetilde{E}_{l_N,q}}{l_1!\cdots l_N!}\right)\frac{t^{n}}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l_1+\cdots+l_N=n}\binom{n}{l_1,\ldots,l_N}\widetilde{E}_{l_1,q}\cdots\widetilde{E}_{l_N,q}\right)\frac{t^{n}}{n!}.\end{aligned}\quad(52)$$

Therefore, by (49) and (52), we obtain the following corollary.

Corollary 5. *For $N\in\mathbb{N}$ and $n\in\mathbb{N}\cup\{0\}$, one has*

$$\sum_{l_1+\cdots+l_N=n}\binom{n}{l_1,\ldots,l_N}\widetilde{E}_{l_1,q}\cdots\widetilde{E}_{l_N,q}=2^{N-1}\sum_{k=0}^{N-1}\frac{1}{k!}\binom{N-1}{k}q^{k}\frac{d^{k}\widetilde{E}_{n,q}}{dq^{k}}.\quad(53)$$

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