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Research Article

The Strong Fuzzy Henstock Integrals and Discontinuous Fuzzy Differential Equations

Yabin Shao[1,2] and Huanhuan Zhang[1]

[1] *College of Mathematics and Computer Science, Northwest University for Nationalities, Lanzhou 730030, China*
[2] *Department of Mathematics, Sichuan University, Chengdu 610065, China*

Correspondence should be addressed to Yabin Shao; yb-shao@163.com

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We generalized the existence theorems and the continuous dependence of a solution on parameters for initial problems of fuzzy discontinuous differential equation by the strong fuzzy Henstock integral and its controlled convergence theorem.

1. Introduction

The Cauchy problems for fuzzy differential equations have been studied by several authors [1–6] on the metric space (E^n, D) of normal fuzzy convex set with the distance D given by the maximum of the Hausdorff distance between the corresponding level sets. In [4], Nieto proved that the Cauchy problem has a uniqueness result if f is continuous and bounded. In [1, 3, 7–9], the authors presented a uniqueness result when f satisfies a Lipschitz condition. For a general reference to fuzzy differential equations, see a recent book by Lakshmikantham and Mohapatra [10] and references therein. In 2002, Xue and Fu [11] established the solutions to fuzzy differential equations with right-hand side functions satisfying Carathéodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous systems in which the right-hand side functions $f : [a, b] \times E^n \to E^n$ are not integrable in the sense of Kaleva [1] on certain intervals, and their solutions are not absolute continuous functions. To illustrate, we consider the following example.

Example 1. Consider the following discontinuous system:

$$x'(t) = h(t), \qquad x(0) = \widetilde{A},$$

$$g(t) = \begin{cases} 2t \sin \dfrac{1}{t^2} - \dfrac{2}{t} \cos \dfrac{1}{t^2}, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

$$\widetilde{A}(s) = \begin{cases} s, & 0 \leq s \leq 1, \\ 2 - s, & 1 < s \leq 2, \\ 0, & \text{others}, \end{cases}$$

$$h(t) = \chi_{|g(t)|} + \widetilde{A}. \tag{1}$$

Then, $h(t) = \chi_{|g(t)|} + \widetilde{A}$ is not integrable in the sense of Kaleva. However, the above system has the following solution:

$$x(t) = \chi_{|G(t)|} + \widetilde{A}t, \tag{2}$$

where

$$G(t) = \begin{cases} t^2 \sin \dfrac{1}{t^2}, & t \neq 0, \\ 0, & t = 0. \end{cases} \tag{3}$$

It is well known that the Henstock integral is designed to integrate highly oscillatory functions which the Lebesgue integral fails to do. It is known as nonabsolute integration and is a powerful tool. It is well known that the Henstock integral includes the Riemann, improper Riemann, Lebesgue, and Newton integrals [12, 13]. Though such an integral was defined by Denjoy in 1912 and also by Perron in 1914, it was difficult to handle using their definitions. But with the Riemann-type definition introduced more recently by Henstock [12] in 1963 and also independently by Kurzweil

[13], the definition is now simple, and furthermore the proof involving the integral also turns out to be easy. For more detailed results about the Henstock integral, we refer to [14]. Recently, Wu and Gong [15, 16] have combined the fuzzy set theory and nonabsolute integration theory, and they discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva [1] integration. In order to complete the theory of fuzzy calculus and to meet the solving need of transferring a fuzzy differential equation into a fuzzy integral equation, Gong and Shao [17, 18] defined the strong fuzzy Henstock integrals and discussed some of their properties and the controlled convergence theorem.

In this paper, according to the idea of [19] and using the concept of generalized differentiability [20], we will prove other controlled convergence theorems for the strong fuzzy Henstock integrals, which will be of foundational significance for studying the existence and uniqueness of solutions to the fuzzy discontinuous systems. As we know, we inevitably use the controlled convergence theorems for solving the numerical solutions of differential equations. As the main outcomes, we will deal with the Cauchy problem of discontinuous fuzzy systems as follows:

$$\begin{aligned} x'(t) &= \tilde{f}(t,x), \\ x(\tau) &= \xi \in E^n, \end{aligned} \tag{4}$$

where $\tilde{f} : U \to E^n$ is a strong fuzzy Henstock integrable function and

$$U = \{(t,x) : |t-\tau| \le a, \quad x \in E^n, \quad D(x,\xi) \le b\}. \tag{5}$$

To make our analysis possible, we will first recall some basic results of fuzzy numbers and give some definitions of absolutely continuous fuzzy-number-valued function. In addition, we present the concept of generalized differentiability. In Section 3, we present the concept of strong fuzzy Henstock integrals, and we prove a controlled convergence theorem for the strong fuzzy Henstock integrals. In Section 4, we deal with the Cauchy problem of discontinuous fuzzy systems. And in Section 5, we present some concluding remarks.

2. Preliminaries

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n, and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric [21] as follows:

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|b-a\|\right\}. \tag{6}$$

Denote that $E^n = \{u : R^n \to [0,1] \mid u$ satisfies (1)–(4) below$\}$ is a fuzzy number space, where

(1) u is normal; that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(2) u is fuzzy convex; that is, $u(\lambda x + (1-\lambda)y) \ge \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \le \lambda \le 1$;

(3) u is upper semicontinuous;

(4) $[u]^0 = \mathrm{cl}\{x \in R^n \mid u(x) > 0\}$ is compact.

For $0 < \alpha \le 1$, denote that $[u]^\alpha = \{x \in R^n \mid u(x) \ge \alpha\}$. Then from the above (1)–(4), it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \le \alpha < 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows [21]:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [ku]^\alpha = k[u]^\alpha, \tag{7}$$

where $u, v \in E^n$ and $0 \le \alpha \le 1$.

Define $D : E^n \times E^n \to [0,\infty)$

$$D(u,v) = \sup\{d_H([u]^\alpha,[v]^\alpha) : \alpha \in [0,1]\}, \tag{8}$$

where d is the Hausdorff metric defined in $P_k(R^n)$. Then it is easy to see that D is a metric in E^n. Using the results [22], we know that

(1) (E^n, D) is a complete metric space,

(2) $D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^n$,

(3) $D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for all $u, v, w \in E^n$ and $\lambda \in R$.

Let $x, y \in E^n$. If there exist $z \in E^n$ such that $x = y + z$, then z is called the H-difference of x and y and is denoted by $x -_H y$. As mentioned above which always is called the condition (H). It is well known that the H-derivative for fuzzy-number-functions was initially introduced by Puri et al. [5, 23] and it is based on the condition (H) of sets. We note that this definition is fairly strong, because the family of fuzzy-number-valued functions H-differentiable is very restrictive. For example, the fuzzy-number-valued function $\tilde{f} : [a,b] \to R_{\mathcal{F}}$ defined by $\tilde{f}(x) = C \cdot g(x)$, where C is a fuzzy number, $\cdot$ is the scalar multiplication (in the fuzzy context) and $g : [a,b] \to R^+$, with $g'(t_0) < 0$, is not H-differentiable in t_0 (see [20, 24]). To avoid the above difficulty, in this paper we consider a more general definition of a derivative for fuzzy-number-valued functions enlarging the class of differentiable fuzzy-number-valued functions, which has been introduced in [20].

Definition 2 (see [20]). Let $\tilde{f} : (a,b) \to E^n$ and $x_0 \in (a,b)$. One says that $\tilde{f}$ is differentiable at x_0, if there exists an element $\tilde{f}'(t_0) \in E^n$, such that

(1) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)$, $\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits (in the metric D)

$$\begin{aligned} \lim_{h\to 0}\frac{\tilde{f}(x_0+h) -_H \tilde{f}(x_0)}{h} &= \lim_{h\to 0}\frac{\tilde{f}(x_0) -_H \tilde{f}(x_0-h)}{h} \\ &= \tilde{f}'(x_0) \end{aligned} \tag{9}$$

or

(2) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)$, $\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h \to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h \to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0) \tag{10}$$

or

(3) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)$, $\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h \to 0} \frac{\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)}{h} = \lim_{h \to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0) \tag{11}$$

or

(4) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)$, $\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits

$$\lim_{h \to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h \to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)}{h} = \tilde{f}'(x_0) \tag{12}$$

(h and $-h$ at denominators mean $(1/h)\cdot$ and $-(1/h)\cdot$, resp.).

3. The Convergence Theorem of Strong Fuzzy Henstock Integral

In this section, we define the strong Henstock integrals of fuzzy-number-valued functions in fuzzy number space E^n, and we give some properties and controlled convergence theorem of this integral by using new conditions.

Definition 3 (see [18]). A fuzzy-number-valued function $\tilde{f}$ is said to be termed additive on $[a, b]$ if, for any division $T : a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b$, one has $\tilde{f}([x_i, x_j])$ $(1 \leq i < j \leq n)$ that exists and $\tilde{f}([x_i, x_j]) = \sum_{k=i}^{j-1} \tilde{f}([x_k, x_{k+1}])$ or $\tilde{f}([x_j, x_i])(1 \leq i < j \leq n)$ that exists and $(-1) \cdot \tilde{f}([x_j, x_i]) = (-1) \cdot \sum_{k=i}^{j-1} \tilde{f}([x_{k+1}, x_k])$. For convenience, $\tilde{f}([s, t])$ denotes $\tilde{f}(t) -_H \tilde{f}(s)$.

Definition 4 (see [17, 18]). A fuzzy-number-valued function $\tilde{f}$ is said to be strong Henstock integrable on $[a, b]$ if there exists a piecewise additive fuzzy-number-valued function $\tilde{F}$ on $[a, b]$ such that for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ and for any δ-fine division $P = \{([u, v], \xi)\}$ of $[a, b]$ one has

$$\sum_{i \in K_n} D\left(\tilde{f}(\xi_i)(v_i - u_i), \tilde{F}([u_i, v_i])\right) + \sum_{j \in I_n} D\left(\tilde{f}(\xi_j)(v_j - u_j), (-1) \cdot \tilde{F}\left([u_j, v_{j-1}]\right)\right) < \varepsilon, \tag{13}$$

where $K_n = \{i \in \{1, 2, \ldots, n\}$ such that $\tilde{F}([x_{i-1}, x_i])$ is a fuzzy number and $I_n = \{j \in \{1, 2, \ldots, n\}$ such that $\tilde{F}([x_j, x_{j-1}])$ is a fuzzy number. One writes $\tilde{f} \in \mathrm{SFH}[a, b]$.

Definition 5. A fuzzy-number-valued function $\tilde{F}$ defined on $X \subset [a, b]$ is said to be $AC^*_\delta(X)$ if for every $\varepsilon > 0$ there exists $\eta > 0$ and $\delta(\xi) > 0$ such that for any δ-fine partial division $P = \{([u, v], \xi)\}$ with $\xi \in X_i$ satisfying $\sum_{i=1}^{n} |v - u| < \eta$ one has $\sum D(\tilde{F}[u, v]) < \varepsilon$.

Definition 6. A fuzzy-number-valued function $\tilde{F}$ is said to be ACG^*_δ on $X \subset [a, b]$ if X is the union of a sequence of closed sets $\{X_i\}$ such that on each X_i, $\tilde{F}$ is $AC^*_\delta(X_i)$.

Definition 7. The sequence of fuzzy-number-function $\{\tilde{F}_n\}$ is $U\ ACG^*_\delta$ on $X \subset [a, b]$ if X is the sequence of subsets X_i such that $\{\tilde{F}_n\}$ is $U\ AC^*_\delta$ for each i, independent of n.

Definition 8. Let $\{\tilde{F}_n\}$ be a sequence of fuzzy-number-function defined on $[a, b]$, and let $x \subset [a, b]$ be measurable.

(i) The sequence of fuzzy-number-function $\{\tilde{F}_n\}$ is $\mathscr{P}$-Cauchy on E^n if $\{\tilde{F}_n\}$ converges pointwise on X and if for each $\varepsilon > 0$ there exist $\delta(\xi) > 0$ on X and a positive integer N such that $D(F_m(P), F_n(P)) < \varepsilon$ for all $m, n \geq N$ whenever P is X-subordinate to $\delta(\xi)$.

(ii) The sequence of fuzzy-number-function $\{\tilde{F}_n\}$ is generalized $\mathscr{P}$-Cauchy on X if X can be written as a countable union of measurable sets on each of which $\{\tilde{F}_n\}$ is $\mathscr{P}$-Cauchy.

Theorem 9. *Let the following conditions be satisfied:*

(i) $\tilde{f}_{n,X}(x) \to \tilde{f}_x$ *a.e. on* $[a, b]$ *as* $n \to \infty$ *where each* $\tilde{f}_{n,X}$ *is strong Fuzzy Henstock integrable on* $[a, b]$*;*

(ii) *the primitives* $\tilde{F}_{n,X}$ *of* $\tilde{f}_{n,X}$ *are* $U\ AC^*_\delta$ *with closed set* X *in* $[a, b]$.

Then $\tilde{f}_X(x)$ *is strong fuzzy Henstock integrable on* $[a, b]$ *with the primitive* $\tilde{F}_X(x)$.

Proof. By (ii), for every $\varepsilon > 0$ there exist a $\delta(\xi) > 0$ and $\eta > 0$, such that for any δ-fine partial division P of X satisfying $\sum |v - u| < \eta$ we have $\sum D(\tilde{F}_{n,X}(v, u), \tilde{0}) < \varepsilon$. By Egoroff's theorem [18, Theorem 3.4], there is an open set G with $|G| < \eta$ such that $D(\tilde{f}_n(\xi), \tilde{f}_m(\xi)) < \varepsilon$ for $n, m \geq N$ and $\xi \notin G$. Consider the following, in which P is a δ-fine division of $[x, y]$ and

$P = P_1 \cup P_2$ so that P_1 contains the intervals with the associated points $\xi \notin G$ and P_2 otherwise:

$$\begin{aligned}
& D\left(\tilde{F}_{n,X}(x,y), \tilde{F}_{m,X}(x,y)\right) \\
& \quad = (P_2)\sum D\left(\tilde{F}_{n,X}(u,v), \tilde{F}_{m,X}(u,v)\right) \\
& \quad \leq \sum D\left(\tilde{F}_{n,X}(u,v), \tilde{f}_{n,X}(\xi)(v-u)\right) \\
& \qquad + \sum D\left(\tilde{F}_{m,X}(u,v), \tilde{f}_{m,X}(\xi)(v-u)\right) \\
& \qquad + \sum D\left(\tilde{f}_{m,X}(\xi)(v-u), \tilde{f}_{m,X}(\xi)(v-u)\right) \\
& \qquad + \sum D\left(\tilde{F}_{n,X}(v), \tilde{F}_{n,X}(u)\right) + \sum D\left(\tilde{F}_{m,X}(v), \tilde{F}_{m,X}(u)\right) \\
& \quad < \varepsilon(4+b-a).
\end{aligned} \tag{14}$$

Hence, for any δ-fine partial division P of $[a,b]$ we have

$$\left|\sum D\left(\tilde{F}_{n,X}(u,v), \tilde{F}_{m,X}(u,v)\right)\right| < \varepsilon, \tag{15}$$

for $m, n \geq N$. Therefore the fuzzy sequence $\{\tilde{F}_{n,X}\}$ is generalized $\mathscr{P}$-Cauchy on $[a,b]$. Then, by (i), we have that $\tilde{f}_X$ is strong fuzzy Henstock integrable on $[a,b]$ with primitive $\tilde{F}_X$. □

Definition 10. (a) A sequence $\{\tilde{F}_n\}$ of fuzzy-number-valued function is uniformly AC^{∇} on X whenever to each $\varepsilon > 0$ there exist $\eta > 0$ and $\delta(x) > 0$ such that

(1) $\sup_n D(\sum_{J_k \in P_1} \tilde{F}_n(J_k), \sum_{L_h \in P_2} \tilde{F}_n(L_h)) < \varepsilon$, for each P_1, $P_2 \in \prod(X,\delta)$,

(2) with $|(\cup P_1)\Delta(\cap P_2)| < \eta$.

(b) A sequence $\{\tilde{F}_n\}$ of fuzzy-number-valued function is uniformly ACG^{∇} on $[a,b]$ if $[a,b] = \cup_i X_i$, where X_i are measurable sets and $\{\tilde{F}\}$ is uniformly AC^{∇} on each X_i.

Theorem 11. *If $\{\tilde{F}\}$ is uniformly ACG^{∇}, then $\{\tilde{F}\}$ is uniformly ACG^{*}_{δ}.*

Proof. Let $[a,b] = \cup_i X_i$ be such that $\{\tilde{F}\}$ is uniformly ACG^{∇} on each X_i. So, for each $\varepsilon > 0$ there exist $\eta > 0$ and $\delta(x) > 0$ such that (1) holds in Definition 10 for each $P_1, P_2 \in \prod(X,\delta)$ satisfying condition (2). We take $P = \{([c_k,d_k],x_k)\}_{i=1}^{p}$ with $\sum_k |d_k - c_k| < \eta$ and put $P_1 = \{([c_k,d_k],x_k) : \tilde{F}_n(C_k, d_k \geq 0)\}$ and $P_2 = \{([c_k,d_k],x_k) : \tilde{F}_n(c_k,d_k < 0)\}$. So, we have $|(\cup P_1)\Delta(\cap P_2)| = \sum_{k=1}^{p} |d_k - c_k| < \eta$. Then by condition (1) in Definition 10, we have

$$\begin{aligned}
& \sup_n \sum_{k=1}^{p} D\left(\tilde{F}_n(c_k), \tilde{F}_n(d_k)\right) \\
& \quad = \sup D\left(\sum_{(c_k,d_k)\in P_1} \tilde{F}_n(c_k,d_k), \right. \\
& \qquad \left. \sum_{(c_k,d_k)\in P_2} \tilde{F}_n(c_k,d_k)\right) < \varepsilon.
\end{aligned} \tag{16}$$

Hence, we have that $\{\tilde{F}\}$ is uniformly ACG^{*}_{δ}. □

We get the following theorem by Theorems 9 and 11.

Theorem 12. *Let the following conditions be satisfied:*

(i) *$\tilde{f}_{n,X} \to \tilde{f}_X$ a.e. in $[a,b]$ where each $\tilde{f}_{n,X}$ is strong fuzzy Henstock integrable on $[a,b]$;*

(ii) *the primitives $\tilde{F}_{n,X}$ of $\tilde{f}_{n,X}$ are $UAC^{\nabla}(X)$ with closed set X in $[a,b]$.*

Then $\tilde{f}_X$ is strong fuzzy Henstock integrable on $[a,b]$ with primitive $\tilde{F}_X$.

Next, we give the controlled convergence theorem for the strong fuzzy Henstock integrals by the definition of the $UACG_{\delta}$ for a fuzzy-number-valued function.

Definition 13. Let $\tilde{F} : [a,b] \to E^n$, and let $X \subset [a,b]$. A fuzzy-number-valued function $\tilde{F}$ is AC_{δ} on X if for each $\varepsilon > 0$ there exist $\eta > 0$ and $\delta(x) > 0$ on X such that $\sum_{i=1}^{N} D(\tilde{F}(c_i), \tilde{F}(d_i)) < \varepsilon$ for $\sum_{i=1}^{N}(d_i - c_i) < \eta$. A fuzzy-number-valued function $\tilde{F}$ is ACG_{δ} on $[a,b]$ if $[a,b]$ is the union of a sequence of set $\{X_i\}$ such that the function $\tilde{F}$ is $AC_{\delta}(X_i)$ for each i.

Definition 14 (see [18]). A fuzzy-number-valued function $\tilde{F}$ defined on $X \subset [a,b]$ is said to be $AC^{*}(X)$ if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every finite sequence of nonoverlapping intervals $\{[a_i,b_i]\}$, satisfying $\sum_{i=1}^{n} |b_i - a_i| < \eta$ where $a_i, b_i \in X$ for all i, one has $\sum \omega(\tilde{F},[a_i,b_i]) < \varepsilon$, where ω denotes the oscillation of $\tilde{F}$ over $[a_i,b_i]$; that is, $\omega(\tilde{F},[a_i,b_i]) = \sup\{D(\tilde{F}(y),\tilde{F}(x)); x,y \in [a_i,b_i]\}$. A fuzzy-number-valued function $\tilde{F}$ is said to be ACG^{*} on X if X is the union of a sequence of closed sets $\{X_i\}$ such that, on each X_i, $\tilde{F}$ is $AC^{*}(X_i)$.

Theorem 15. *A fuzzy-number-valued function $\tilde{F}$ is ACG_{δ} if and only if it is ACG^{*} on $[a,b]$.*

Theorem 16 (controlled convergence theorem). *Let the following conditions be satisfied:*

(1) *$\tilde{f}_n(x) \to \tilde{f}(x)$ almost everywhere in $[a,b]$ as $n \to \infty$ where each $\tilde{f}_n$ is strong fuzzy Henstock integrable on $[a,b]$;*

(2) *the primitives* $\widetilde{F}_n(x) = (SFH)\int_a^x \widetilde{f}_n(s)dx$ *of* f_n *are* $UACG_\delta$ *uniformly in* n;

(3) *the sequence* $\{\widetilde{F}_n(x)\}$ *converges uniformly to a continuous function on* $[a,b]$. *Then* $f(x)$ *is strong fuzzy Henstock integrable on* $[a,b]$ *and one has*

$$\lim_{n\to\infty}(SFH)\int_a^b \widetilde{f}_n(x)\,dx = (SFH)\int_a^b \widetilde{f}(x)\,dx. \tag{17}$$

If conditions (1) *and* (2) *are replaced by condition* (4):

(4) $\widetilde{g}(x) \le \widetilde{f}(x) \le \widetilde{h}(x)$ *almost everywhere on* $[a,b]$, *where* $\widetilde{g}(x)$ *and* $\widetilde{h}(x)$ *are strong fuzzy Henstock integrable.*

Proof. By condition (2) there exists a sequence $\{X_i\}$ such that $\widetilde{F}_n \in UAC_\delta$ in X_i a bounded, closed set with bounds a and b, and put $X = X_i$. We note that $\widetilde{F}_n(x) \to \widetilde{F}(x)$; we have $\widetilde{F} \in AC_\delta$ on X, and hence $\widetilde{F} \in ACG_\delta$ on $[a,b]$. By Theorem 15, $\widetilde{F} \in AC^*$ on X and hence $\widetilde{F} \in ACG^*$ on $[a,b]$ and also $\widetilde{F} \in AC$ on X and hence $\widetilde{F} \in ACG$ on $[a,b]$.

Now we prove that $\widetilde{F}'(x) = \widetilde{f}(x)$ a.e. on $[a,b]$. In fact, let $\widetilde{G} : [a,b] \to E^n$ equal $\widetilde{F}_n$ on X, and extend $\widetilde{G}_n$ linearly to the closed interval contiguous to X. Likewise, we define $\widetilde{G}$ from $\widetilde{F}$. We see that $\widetilde{G}_n$ and $\widetilde{G}$ are UAC on $[a,b]$. By condition (3), we have $\widetilde{G}_n \to \widetilde{G}$ on $[a,b]$. Let $[c_k,d_k]$ be the intervals contiguous to X. Then we have $D(\widetilde{G}_n') \le M_k$. We define a fuzzy-number-valued function as follows:

$$\widetilde{G}'(x) = \frac{\widetilde{G}_n(d_k) -_H \widetilde{G}_n(c_k)}{d_k - c_k}, \quad x \in (c_k, d_k). \tag{18}$$

Consequently, $\widetilde{G}_n'(x)$ converges on (c_k,d_k). Hence $\widetilde{G}_n'$ converges on $[a,b]$ a.e. Since $\{\widetilde{G}_n\} \in AC$ on $[a,b]$, then $\widetilde{G}_n'(x) = \widetilde{f}_n \to \widetilde{f}$ on X. Therefore, we have $\widetilde{G}_n'(x) = \widetilde{g}(x) = \widetilde{f}(x) = \widetilde{F}'(x)$ a.e. on X. Thus $\widetilde{F}'(x) = \widetilde{f}(x)$ a.e. on $[a,b]$ by Theorem 15. Therefore, there exists an ACG_δ function on $[a,b]$ such that $\widetilde{F}'(x) = \widetilde{f}(x)$ a.e. on $[a,b]$. Hence $\widetilde{f}$ is strong fuzzy Henstock integrable on $[a,b]$, and we have

$$\lim_{n\to\infty}(\mathrm{SFH})\int_a^b \widetilde{f}_n(x)\,\mathrm{d}x = (\mathrm{SFH})\int_a^b \widetilde{f}(x)\,\mathrm{d}x. \tag{19}$$

□

4. The Generalized Solutions of Discontinuous Fuzzy Differential Equations

In this section, a generalized fuzzy differential equation of form (4) is defined by using strong fuzzy Henstock integral. The main results of this section are existence theorems for the generalized solution to the discontinuous fuzzy differential equation.

Definition 17 (see [11]). Let τ and ξ be fixed, and let a fuzzy-number-valued function $\widetilde{f}(t,x)$ be a Carathéodory function defined on a rectangle $U : |t-\tau| \le a,\ D(x,\xi) \le b$; that is, $\widetilde{f}$ is continuous in x for almost all t and measurable in t for each fixed x.

Theorem 18. *Let a fuzzy-number-valued function* $\widetilde{f}$ *be a function as given in Definition 17; then there exist two strong fuzzy Henstock integrable functions* $\widetilde{h}$ *and* $\widetilde{g}$ *defined on* $|t-\tau| \le a$ *such that* $\widetilde{g}(t) \le \widetilde{f}(t,x) \le \widetilde{h}(t)$ *for all* $(t,x) \in U$.

Proof. Note that $\widetilde{f}$ is a Carathéodory function. Thus, there exist two measurable functions $u(t)$ and $v(t)$ defined on $|t-\tau| \le a$ with values in $D(x,\xi) \le b$ such that $\widetilde{f}(t,u(t)) \le \widetilde{f}(t,x) \le \widetilde{f}(t,v(t))$ for all $(t,x) \in U$. Next, we will show that $\widetilde{f}(t,u(t))$ and $\widetilde{f}(t,v(t))$ are fuzzy Henstock integrable by using controlled convergence Theorem 16. First, there exists a sequence $\{k_n(t)\}$ of step functions defined on $|t-\tau| \le a$ with values in $D(x,\xi) \le b$ such that $k_n(t) \to u(t)$ almost everywhere as $n \to \infty$. Let $\widetilde{F}_n(t) = \int_\tau^t \widetilde{f}(s,k_n(s))\mathrm{d}s$. Then $\{\widetilde{F}_n(t)\}$ is $UACG_\delta$ uniformly in n and equicontinuous. By controlled convergence Theorem 16, $\widetilde{f}(t,u(t))$ is strong fuzzy Henstock integrable. Similarly, $\widetilde{f}(t,v(t))$ is strong fuzzy Henstock integrable. □

Definition 19. A fuzzy-number-valued function $x(t) : I \to E^n$ is said to be a solution of the discontinuous fuzzy differential equation (4) if $x(t)$ satisfies the following conditions:

(i) $x(t)$ is ACG_δ on each compact subinterval of I;

(ii) $(t,x) \in U$ for $t \in I$;

(iii) $x'(t)$ for almost everywhere $t \in I$.

Now we will state the existence theorem for the generalized solution of discontinuous fuzzy differential equation (4).

Theorem 20. *Suppose that* $\widetilde{f}$ *satisfies the condition of Theorem 18; then there exists a generalized solution* Φ *of the discontinuous fuzzy differential equation* (4) *on some interval* $|t-\tau| \le a$ *which satisfies* $\Phi(\tau) = \xi$.

Proof. Given $\widetilde{g}(t) \le \widetilde{f}(t,x) \le \widetilde{h}(t)$ for all x and almost all t with $(t,x) \in U$, we get $\widetilde{0} \le \widetilde{f}(t,x) -_H \widetilde{g}(t) \le \widetilde{h}(t) -_H \widetilde{g}(t)$. Let

$$\widetilde{F}(t,x) = \widetilde{f}\left(t, x + \int_\tau^t \widetilde{g}(s)\,\mathrm{d}s\right) -_H \widetilde{g}(t), \tag{20}$$

or

$$\widetilde{F}(t,x) = \widetilde{f}\left(t, x + (-1)\cdot\int_\tau^t \widetilde{g}(s)\,\mathrm{d}s\right) -_H \widetilde{g}(t). \tag{21}$$

Then $\widetilde{F}$ is a Carathéodory function. Furthermore, $\widetilde{0} \le \widetilde{F}(t,x) \le \widetilde{h}(t) -_H \widetilde{g}(t)$ for all $(t,x) \in U_0$, where $U_0 \subset U$, such that

$$D\left(x + \int_\tau^t \widetilde{g}(s)\,\mathrm{d}s, \xi\right) \le b \quad \forall (t,x) \in U_0. \tag{22}$$

By Carathéodory existence theorem (see Theorem 7 in [11]), there is a fuzzy-number-valued function ψ on some interval $|t-\tau| \le a$ such that $\psi'(t) = \widetilde{F}(t,\psi(t))$ almost everywhere in this interval and $\psi(\tau) = \xi$. Let

$$\phi(t) = \psi(t) + \int_\tau^t \widetilde{g}(s)\,\mathrm{d}s \quad \text{or} \quad \phi(t) = \psi(t) + (-1)\cdot\int_\tau^t \widetilde{g}(s)\,\mathrm{d}s. \tag{23}$$

Then, for almost all t, we have the following.

Case 1. Consider

$$\begin{aligned}\phi'(t) &= \psi'(t) + \widetilde{g}(t) = \widetilde{F}(t,\psi(t)) + \widetilde{g}(t)\\ &= \widetilde{f}\left(t,\psi(t) + \int_\tau^t \widetilde{g}(s)\,\mathrm{d}s\right) -_H \widetilde{g}(t) + \widetilde{g}(t)\\ &= \widetilde{f}(t,\phi(t)),\\ \phi(t) &= \psi(\tau) + \int_\tau^t \widetilde{g}(s)\,\mathrm{d}s = \xi.\end{aligned} \tag{24}$$

Case 2. Consider

$$\begin{aligned}\phi'(t) &= \psi'(t) + \widetilde{g}(t) = \widetilde{F}(t,\psi(t)) + \widetilde{g}(t)\\ &= \widetilde{f}\left(t,\psi(t) + (-1)\cdot\int_\tau^t \widetilde{g}(s)\,\mathrm{d}s\right) -_H \widetilde{g}(t) + \widetilde{g}(t)\\ &= \widetilde{f}(t,\phi(t)),\\ \phi(t) &= \psi(\tau) + (-1)\cdot\int_\tau^t \widetilde{g}(s)\,\mathrm{d}s = \xi.\end{aligned} \tag{25}$$

The proof is complete. □

Example 21. Consider fuzzy differential equation $x' = \widetilde{f}(t,x) = \widetilde{g}(t,x) + \widetilde{h}(t)$, where $D(\widetilde{g}(t,x),\widetilde{0}) \le D(\widetilde{g}_1(t),\widetilde{0})$ for all $|t| \le 1$, $D(x,\widetilde{0}) \le 1$ and $\widetilde{g}_1(t)$ is Kaleva integrable on $|t| \le 1$ and $\widetilde{h}(t) = \widetilde{A}\cdot(d/dt)(t^2\sin t^{-2})$ if $t \neq 0$ and $\widetilde{h}(0) = \widetilde{0}$. Here $\widetilde{A}$ is defined in Example 1. Note that $\widetilde{h}$ is strong fuzzy Henstock integrable but not Kaleva integrable, and

$$\begin{aligned}\widetilde{h}(t) -_H \widetilde{g}_1(t) \le \widetilde{f}(t,x) \le \widetilde{h}(t) + \widetilde{g}_1(t),\\ \text{for } |t| \le 1,\ D\left(x,\widetilde{0}\right) \le 1.\end{aligned} \tag{26}$$

Thus, by Theorem 20, there exists a solution of $x' = \widetilde{f}(t,x)$ with $x(0) = \widetilde{0}$. For instance, if $\widetilde{g}(t,x) = t^2 x$, then

$$\phi(t) = e^{t^3/3}\cdot\int_0^t e^{-s^3/3}\widetilde{h}(s)\,\mathrm{d}s \tag{27}$$

is a solution by using integrating factor.

We get the following existence theorem by Theorems 18 and 20.

Theorem 22. *Let a fuzzy-number-valued function $\widetilde{f}$ be a Carathéodory function defined on a rectangle $U : |t-\tau| \le a, D(x,\xi) \le b$. Let $\widetilde{f}(t,u(t))$ be strong fuzzy Henstock integrable on $|t-\tau| \le a$ for any step function $u(t)$ defined on $|t-\tau| \le a$ with values in $D(x,\xi) \le b$. Denote that $\widetilde{F}_u(t) = \int_\tau^t \widetilde{f}(s,u(s))ds$. If $\{\widetilde{F}_u : u$ is a step function$\}$ is $UACG_\delta$ uniformly in u and equicontinuous on $|t-\tau| \le a$, then there exists a solution ϕ of $x' = \widetilde{f}(t,x)$ on some interval $|t-\tau| \le \beta$ with $\phi(\tau) = \xi$.*

Finally, in this paper, we will show the continuous dependence of a solution on parameters by using Theorems 16, 18, and 22.

Let U_p be a connected set in U. Let $c > 0$ and let μ_0 be fixed:

$$\begin{aligned}I_\mu &= \{\mu; |\mu-\mu_0| < c\},\\ U_\mu &= \left\{(t,x,\mu);(t,x)\in U_p, \mu\in I_\mu\right\}.\end{aligned} \tag{28}$$

Let $\widetilde{f}(t,x,\mu)$ be a fuzzy-number-valued function defined on U_μ such that, for each fixed μ, the function $\widetilde{f}$ is a Carathéodory function defined on U_p for each fixed t and continuous at (x,μ_0) for every x. For $\mu = \mu_0$, let

$$\begin{aligned}x'(t) &= \widetilde{f}(t,x,\mu),\\ x(\tau) &= \xi \in E^n\end{aligned} \tag{29}$$

have a solution ϕ_0 on $[a,b]$, where $\tau \in [a,b]$. Let $\widetilde{f}(t,u(t),\mu)$ be strong fuzzy Henstock integrable on $[a,b]$ for any step function $u(t)$ with $(t,u(t)) \in U_p$ for $t \in [a,b]$.

Definition 23. Denote

$$\widetilde{F}_{\mu,u}(t) = \int_a^t \widetilde{f}(s,u(s),\mu)\,\mathrm{d}s. \tag{30}$$

The family $\{\widetilde{F}_{\mu,u}\}$ is said to be equicontinuous in u and near μ_0 if, for each $t_0 \in [a,b]$, there exists an interval $|\mu-\mu_0| < r_0$ such that the family $\{\widetilde{F}_{\mu,u}(t) : u$ and μ with $|\mu-\mu_0| < r_0\}$ is equicontinuous at t_0.

Theorem 24. *Let $\widetilde{f}$ be a fuzzy-number-valued function as given above. If the primitive $\widetilde{F}_{\mu,u}(t)$ of $\widetilde{f}(t,x,\mu)$ is ACG_δ uniformly in u and μ and equicontinuous in u and near μ_0, then there exists $\delta > 0$ such that, for any fixed μ with $|\mu-\mu_0| < \delta$, a solution ϕ_μ of discontinuous fuzzy differential equation* (29) *exists over $[a,b]$ and as $\mu \to \mu_0$, $\phi_\mu \to \phi_0$ uniformly over $[a,b]$.*

Proof. Firstly, we will consider the case $\tau \in (a,b)$. By Theorem 22 and the equicontinuity of $\widetilde{F}_{\mu,u}(t)$, all solutions ϕ_μ of problem (29) with $|\mu-\mu_0| \le \delta_1$ for some $\delta_1 > 0$ exist over some interval $|t-\tau| \le \alpha$ with $\alpha > 0$. Then $G = \{\phi_\mu : |\mu-\mu_0| \le \delta_1\}$ is equicontinuous on $|t-\tau| \le \alpha$. Because $\phi_\mu(\tau) = \xi$, G is uniformly bounded. Hence, for all sequence of G with $\mu \to \mu_0$, there exists a subsequence which converges uniformly. That is to say, $\phi_{\mu(k)} \to \psi$ uniformly on $|t-\tau| \le \alpha$ as $\mu(k) \to \mu_0$. Since we have

$$\phi_{\mu(k)}(t) = \xi + \int_\tau^t \widetilde{f}\left(s,\phi_{\mu(k)},\mu(k)\right)\mathrm{d}s \tag{31}$$

or

$$\phi_{\mu(k)}(t) = \xi + (-1)\cdot\int_\tau^t \widetilde{f}\left(s,\phi_{\mu(k)},\mu(k)\right)\mathrm{d}s, \tag{32}$$

by Theorem 16 we get

$$\psi(t) = \xi + \int_\tau^t \widetilde{f}(s,\psi(s),\mu_0)\,\mathrm{d}s \tag{33}$$

or

$$\psi(t) = \xi + (-1) \cdot \int_{\tau}^{t} \widetilde{f}(s, \psi(s), \mu_0)\, ds. \tag{34}$$

Thus, ψ is a solution of (29) for $\mu = \mu_0$. Hence, $\psi = \phi_0$ on $|t - \tau| \le \alpha$ because the solution ϕ_0 is unique. Consequently, $\phi_{u(k)} \to \phi_0$ uniformly on $|t-\tau| \le \alpha$ as $\mu(k) \to \mu_0$. By Reductio ad absurdum, $\phi_\mu \to \phi_0$ uniformly there as $\mu \to \mu_0$.

Secondly, we will extend the result over $[a, b]$. We consider the case $[\tau, b]$. Assume that there exists $t_0 \in [\tau, b)$ such that the result is valid over $[\tau, t_0 - h]$ but not $[\tau, t_0 + h]$. Obviously, $t_0 \ge \tau + \alpha$. Let γ, β be such that

$$H = \{(t, x) : |t - t_0| \le \alpha, D(x, \phi_0(t - \gamma) \le \beta)\} \tag{35}$$

is contained in U. Since $\{\widetilde{F}_{\mu,u}(t)\}$ is equicontinuous in u and near μ_0, we may choose $\gamma > 0$ small enough such that

$$D\left(\int_{t_0}^{t} \widetilde{f}(s, u(s), \mu)\, ds, \widetilde{0}\right) < \frac{\beta}{3} \tag{36}$$

for all step functions u whenever $|t - t_0| \le \gamma$ and $|\mu - \mu_0| < \delta_2$. Since $\phi_\mu \to \phi_0$ uniformly on $[\tau, t_0 - h]$, there exists a δ such that

$$D(\phi_\mu(t_0 - \gamma), \phi_0(t_0 - \gamma)) < \frac{\beta}{3} \tag{37}$$

for $|\mu - \mu_0| < \delta$. Hence, we have

$$\begin{aligned}
&D\left(\int_{t_0-\gamma}^{t} \widetilde{f}(s, u(s), \mu)\, ds, \phi_{mu}(t_0 - \gamma) -_H \phi_0(t_0 - \gamma)\right) \\
&\quad\le D\left(\int_{t_0-\gamma}^{t_0} \widetilde{f}(s, u(s), \mu)\, ds, \widetilde{0}\right) \\
&\qquad + D\left(\int_{t_0}^{t} \widetilde{f}(s, u(s), \mu)\, ds, \widetilde{0}\right) \\
&\qquad + D(\phi_{mu}(t_0 - \gamma), \phi_0(t_0 - \gamma)) < 3 \cdot \frac{\beta}{3}.
\end{aligned} \tag{38}$$

Thus, for each u with $|\mu - \mu_0| < \delta$, a solution of (29) with $x(t_0 - \gamma) = \phi_\mu(t_0 - \gamma)$ exists on $|t - t_0| \le \gamma$. Hence, ϕ_μ can be continued to $t_0 + \gamma$. So, in the case $[\tau - \alpha, \tau + \alpha]$, $\phi_\mu \to \phi_0$ uniformly on $[\tau, t_0 + \gamma]$. It leads to a contradiction, similarly, for the cases $t_0 = b$ and $[a, \tau]$. Therefore, the theorem holds over $[a, b]$. □

Example 25. Let

$$\widetilde{H}(t) = \begin{cases} \widetilde{A} \cdot t^2 \sin t^{-2} & t \ne 0, \\ \widetilde{0}, & t = 0, \end{cases} \tag{39}$$

where fuzzy number $\widetilde{A}$ is defined in Example 1. Let $\widetilde{h}(t) = \widetilde{H}'(t)$. We define

$$\widetilde{h}_\mu(t) = \begin{cases} \widetilde{h}(t) & \mu \in [0, 1),\ t \in \dfrac{(-2, 2)}{(-\mu, \mu)}, \\ \widetilde{0}, & \text{otherwise}, \end{cases} \tag{40}$$

and $\widetilde{h}_\mu(t) = \widetilde{h}_{-\mu}(t)$ for $\mu \in (-1, 0]$. Let $\widetilde{f}(t, x, \mu) = t^2 x + \widetilde{h}_\mu(t)$ defined on $(-2, 2) \times (-1, 1)$. Then, we have

$$\begin{aligned}
D\left(\widetilde{F}_{\mu,u}(t_1), \widetilde{F}_{\mu,u}(t_2)\right) &= D\left(\int_{t_1}^{t_2} \widetilde{f}(s, u(s), \mu)\, ds, \widetilde{0}\right) \\
&\le \int_{t_1}^{t_2} s^2 ds + D\left(\int_{t_1}^{t_2} \widetilde{h}_\mu ds, \widetilde{0}\right),
\end{aligned} \tag{41}$$

$$D\left(\int_{t_1}^{t_2} \widetilde{h}_\mu ds, \widetilde{0}\right) \le \int_{t_1}^{t_2} D\left(\widetilde{h}(s)\, ds, \widetilde{0}\right),$$

where $t_1, t_2 \in (0, 1]$ or $t_1, t_2 \in [-1, 0)$, and

$$\begin{aligned}
&D\left(\int_0^t \widetilde{h}_\mu(s)\, ds, \widetilde{0}\right) \\
&\quad = \begin{cases} D\left(\widetilde{A} \cdot t^2 \sin t^{-2}, \widetilde{A} \cdot \mu^2 \sin \mu^{-2}\right), & \mu \ne 0, \\ D\left(\widetilde{A} \cdot t^2 \sin t^{-2}, \widetilde{0}\right), & \mu = 0. \end{cases}
\end{aligned} \tag{42}$$

Note that $\widetilde{h}$ is Kaleva integrable on every subinterval of $[-1, 0)$ and $(0, 1]$. Therefore, we have that $\widetilde{F}_{\mu,u}(t)$ is equicontinuous on $[-1, 1]$ in u and near $\mu_0 = 0$. Furthermore, $\widetilde{F}_{\mu,u}(t)$ is $AC_\delta(X_n)$ uniformly in μ and u, where $X_n = [1/n, 1]$, $n = 1, 2, \ldots$. On the other hand, by using integrating factor, for $t \in [-1, 1]$, we have

$$\phi_\mu(t) = e^{t^2/3} \int_0^t e^{-s^3/3} \widetilde{A} \cdot \widetilde{h}_\mu(s)\, ds, \tag{43}$$

with $\phi_\mu(0) = \widetilde{0}$. Obviously, ϕ_0 is unique. Thus, by Theorem 24, $\phi_\mu \to \phi_0$ uniformly on $[-1, 1]$.

5. Conclusion

In this paper, we give the definition of the $UACG_\delta$ for a fuzzy-number-valued function and the nonabsolute fuzzy integral and its controlled convergence theorem. In addition, we deal with the Cauchy problem and the continuous dependence of a solution on parameters of discontinuous fuzzy differential equations involving the strong fuzzy Henstock integral in fuzzy number space. The function governing the equations is supposed to be discontinuous with respect to some variables and satisfy nonabsolute fuzzy integrability. Our result improves the result given in [1, 11, 19, 20] (where uniform continuity was required), as well as those referred therein.

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References

[1] O. Kaleva, "Fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 301–317, 1987.

[2] Z. Gong and Y. Shao, "Global existence and uniqueness of solutions for fuzzy differential equations under dissipative-type conditions," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2716–2723, 2008.

[3] O. Kaleva, "The Cauchy problem for fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 35, no. 3, pp. 389–396, 1990.

[4] J. J. Nieto, "The Cauchy problem for continuous fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 102, no. 2, pp. 259–262, 1999.

[5] M. L. Puri and D. A. Ralescu, "Differentials of fuzzy functions," *Journal of Mathematical Analysis and Applications*, vol. 91, no. 2, pp. 552–558, 1983.

[6] S. Seikkala, "On the fuzzy initial value problem," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 319–330, 1987.

[7] C. Wu and S. Song, "Existence theorem to the Cauchy problem of fuzzy differential equations under compactness-type conditions," *Information Sciences*, vol. 108, no. 1–4, pp. 123–134, 1998.

[8] C. Wu, S. Song, and Z. Qi, "Existence and uniqueness for a solution on the closed subset to the Cauchy problem of fuzzy diffrential equations," *Journal of Harbin Institute of Technology*, vol. 2, pp. 1–7, 1997.

[9] C. Wu, S. Song, and E. S. Lee, "Approximate solutions, existence, and uniqueness of the Cauchy problem of fuzzy differential equations," *Journal of Mathematical Analysis and Applications*, vol. 202, no. 2, pp. 629–644, 1996.

[10] V. Lakshmikantham and R. N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, vol. 6, Taylor & Francis, London, UK, 2003.

[11] X. Xue and Y. Fu, "Carathéodory solutions of fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 125, no. 2, pp. 239–243, 2002.

[12] R. Henstock, *Theory of Integration*, Butterworths, London, UK, 1963.

[13] J. Kurzweil, "Generalized ordinary differential equations and continuous dependence on a parameter," *Czechoslovak Mathematical Journal*, vol. 7, no. 82, pp. 418–449, 1957.

[14] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, vol. 2, World Scientific Publishing, Teaneck, NJ, USA, 1989.

[15] C. Wu and Z. Gong, "On Henstock integrals of interval-valued functions and fuzzy-valued functions," *Fuzzy Sets and Systems*, vol. 115, no. 3, pp. 377–391, 2000.

[16] C. Wu and Z. Gong, "On Henstock integral of fuzzy-number-valued functions. I," *Fuzzy Sets and Systems*, vol. 120, no. 3, pp. 523–532, 2001.

[17] Z. Gong, "On the problem of characterizing derivatives for the fuzzy-valued functions. II. Almost everywhere differentiability and strong Henstock integral," *Fuzzy Sets and Systems*, vol. 145, no. 3, pp. 381–393, 2004.

[18] Z. Gong and Y. Shao, "The controlled convergence theorems for the strong Henstock integrals of fuzzy-number-valued functions," *Fuzzy Sets and Systems*, vol. 160, no. 11, pp. 1528–1546, 2009.

[19] T. S. Chew and F. Flordeliza, "On $x' = f(t, x)$ and Henstock-Kurzweil integrals," *Differential and Integral Equations*, vol. 4, no. 4, pp. 861–868, 1991.

[20] B. Bede and S. G. Gal, "Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 151, no. 3, pp. 581–599, 2005.

[21] D. Dubois and H. Prade, "Towards fuzzy differential calculus. I. Integration of fuzzy mappings," *Fuzzy Sets and Systems*, vol. 8, no. 1, pp. 1–17, 1982.

[22] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets*, World Scientific Publishing, River Edge, NJ, USA, 1994.

[23] R. Goetschel Jr. and W. Voxman, "Elementary fuzzy calculus," *Fuzzy Sets and Systems*, vol. 18, no. 1, pp. 31–43, 1986.

[24] Y. Chalco-Cano and H. Román-Flores, "On new solutions of fuzzy differential equations," *Chaos, Solitons and Fractals*, vol. 38, no. 1, pp. 112–119, 2008.