Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 340564, 11 pages
http://dx.doi.org/10.1155/2013/340564

*Research Article*

# Reductions and New Exact Solutions of ZK, Gardner KP, and Modified KP Equations via Generalized Double Reduction Theorem

**R. Naz,[1] Z. Ali,[2] and I. Naeem[3]**

[1] *Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore 53200, Pakistan*
[2] *Department of Sciences and Humanities, National University of Computer and Emerging Sciences, Lahore Campus, Lahore 54000, Pakistan*
[3] *Department of Mathematics, School of Science and Engineering, Lahore University of Management Sciences, Opposite Sector U, DHA, Lahore Cantt 54792, Pakistan*

Correspondence should be addressed to I. Naeem; imran.naeem@lums.edu.pk

Received 26 May 2013; Accepted 16 July 2013

Academic Editor: Teoman Özer



We study here the Lie symmetries, conservation laws, reductions, and new exact solutions of (2 + 1) dimensional Zakharov-Kuznetsov (ZK), Gardner Kadomtsev-Petviashvili (GKP), and Modified Kadomtsev-Petviashvili (MKP) equations. The multiplier approach yields three conservation laws for ZK equation. We find the Lie symmetries associated with the conserved vectors, and three different cases arise. The generalized double reduction theorem is then applied to reduce the third-order ZK equation to a second-order ordinary differential equation (ODE) and implicit solutions are established. We use the Sine-Cosine method for the reduced second-order ODE to obtain new explicit solutions of ZK equation. The Lie symmetries, conservation laws, reductions, and exact solutions via generalized double reduction theorem are computed for the GKP and MKP equations. Moreover, for the GKP equation, some new explicit solutions are constructed by applying the first integral method to the reduced equations.

## 1. Introduction

The association of conservation laws with Noether symmetries [1], Lie-Bäcklund symmetries [2], and nonlocal symmetries [3, 4] has been of great interest during the last few decades. This association results in double reduction of a partial differential equation (PDE). For variational partial differential equations (PDEs), the double reduction was achieved by association of a Noether symmetry with a conserved vector [5, 6]. Sjöberg [7, 8] developed a double reduction formula for a nonvariational PDE of order $q$ with two independent and $m$ dependent variables to reduce it to an ODE of order $(q-1)$ provided that the PDE admits a nontrivial conserved vector associated with at least one symmetry. Recently, Bokhari et al. [9] generalized the double reduction theory for the case of several independent variables. According to the generalized double reduction theory, a nonlinear system of $q$th-order PDEs with $n$ independent and $m$ dependent variables can be reduced to a nonlinear system of $(q-1)$th-order ODEs. In every reduction, at least one symmetry should be associated with a nontrivial conserved vector; otherwise, reduction is not possible. Naz et al. [10] utilized the double reduction theory to find some exact solutions of a class of nonlinear regularized long wave equations.

Different methods are developed for the construction of conservation laws compared by Naz et al. [11], and see also references therein. We will use the multiplier approach. The conservation law in characteristic form [12] can be expressed as $D_iT^i = \Lambda^\alpha E_\alpha$, and one can compute the characteristics (multipliers) by taking the variational derivative of $D_iT^i = Q^\alpha E_\alpha$ for the arbitrary functions not only for solutions of system of partial differential equations [6]. It was successfully applied to construct the conservation laws (see, e.g., [11, 13]).

In this paper, we consider $(2+1)$ dimensional ZK [14, 15], GKP [16], and MKP [17] equations. The conservation laws are computed by the multiplier approach. The symmetry conservation law relation is used to determine symmetries associated with the conserved vectors. Reductions and new exact solutions are found by the generalized double reduction theory for ZK, GKP, and MKP equations. We utilize the Sine-Cosine method [18–20] and first integral method [21] to compute new explicit solutions for the reduced conserved forms of ZK and GKP equations. To the best of our knowledge, the exact solutions derived here are new and not reported in the literature.

The detail outline of the paper is as follows. In Section 2, basic definitions, important relations, and the fundamental theorem of generalized double reduction theory are presented. The Lie symmetries, conservation laws, reduced forms, and new exact solutions via generalized double reduction theorem for ZK equation are constructed in Section 3. In Sections 4 and 5, Lie symmetries, conservation laws, reductions, and new exact solutions of GKP and MKP equations are studied. Concluding remarks are summarized in Section 6.

## 2. Fundamental Operators

The following definitions are adopted from the literature [7–9, 11, 22].

Consider the following $q$th-order system of PDEs:

$$E^{\nu} = \left(x, u, u_{(1)}, u_{(2)}, \dots, u_{(q)}\right), \quad \nu = 1, 2, 3, \dots, m, \tag{1}$$

where $x = (x^1, x^2, x^3, \dots, x^n)$ are the independent variables, and $u = (u^1, u^2, u^3, \dots, u^m)$ are the dependent variables.

*Definition 1.* A Lie-Bäcklund or generalized operator is defined by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha} \frac{\partial}{\partial u_i^{\alpha}} + \sum_{s \geq 1} \zeta^{\alpha}_{i_1 \cdots i_s} \frac{\partial}{\partial u^{\alpha}_{i_1 \cdots i_s}}, \tag{2}$$

and the additional coefficients $\zeta^{\alpha}_{i_1 \cdots i_s}$ can be found from

$$\begin{aligned} \zeta_i^{\alpha} &= D_i \left(W^{\alpha}\right) + \xi^j u_{ij}^{\alpha}, \\ \zeta^{\alpha}_{i_1 \cdots i_s} &= D_{i_1} \cdots D_{i_s} \left(W^{\alpha}\right) + \xi^j u^{\alpha}_{j i_1 \cdots i_s}, \quad s > 1, \end{aligned} \tag{3}$$

in which $W^{\alpha}$ is the Lie characteristic function described by

$$W^{\alpha} = \eta^{\alpha} - \xi^j u_j^{\alpha}. \tag{4}$$

*Definition 2.* The Euler operator is defined by

$$\frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} - D_i \frac{\partial}{\partial u_i^{\alpha}} + D_i D_j \frac{\partial}{\partial u_{ij}^{\alpha}} - \cdots, \tag{5}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \cdots, \quad i = 1, 2, \dots, n \tag{6}$$

is the total derivative operator with respect to $x^i$.

*Definition 3.* A conserved vector $T = (T^1, T^2, \dots, T^n)$, $T^i \in \mathscr{A}$, $i = 1, 2, \dots, n$ satisfies $D_i T^i|_{(1)} = 0$ for all solutions of (1) is called a local conservation law. Here $\mathscr{A}$ denotes the space of all differential functions.

*Definition 4.* A Lie-Bäcklund operator $X$ given in (2) is associated with the conserved vector $T$ of (1) if it satisfies the following relation:

$$\mathrm{X}\left(T^i\right) + T^i D_j \left(\xi^j\right) - T^j D_j \left(\xi^i\right) = 0, \quad i = 1, 2, \dots, n. \tag{7}$$

Equation (7) is known as the symmetry conservation laws relationship [22].

New conservation laws can be derived from existing conservation laws and the symmetries by using the following theorem adopted from [22, 23].

**Theorem 5.** *Suppose $X$ is any Lie-Bäcklund operator of* (1) *and $T^i$, $i = 1, 2, 3, \dots, n$ comprise the components of a conserved vector of* (1) *then*

$$\widetilde{T}^i = X\left(T^i\right) + T^i D_j \left(\xi^j\right) - T^j D_j \left(\xi^i\right), \quad i = 1, 2, 3, \dots, n \tag{8}$$

*yields the components of a conserved vector of* (1), *and thus*

$$D_i \widetilde{T}^i\Big|_{(1)} = 0. \tag{9}$$

**Theorem 6** (see [9]). *Suppose $D_i T^i = 0$ is a conservation law of the PDE system* (1). *Then under a contact transformation, there exist functions $\widetilde{T}^i$ such that $J D_i T^i = \widetilde{D}_i \widetilde{T}^i$ where $\widetilde{T}^i$ is given by*

$$\begin{gathered} \begin{pmatrix} \widetilde{T}^1 \\ \widetilde{T}^2 \\ \cdot \\ \cdot \\ \cdot \\ \widetilde{T}^n \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^1 \\ T^2 \\ \cdot \\ \cdot \\ \cdot \\ T^n \end{pmatrix}, \\ J \begin{pmatrix} T^1 \\ T^2 \\ \cdot \\ \cdot \\ \cdot \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \widetilde{T}^1 \\ \widetilde{T}^2 \\ \cdot \\ \cdot \\ \cdot \\ \widetilde{T}^n \end{pmatrix}. \end{gathered} \tag{10}$$

*In* (10)*,* $A$*,* $A^{-1}$*, and* $J$ *can be determined from*

$$A = \begin{pmatrix} \widetilde{D}_1 x_1 & \widetilde{D}_1 x_2 & \cdots & \widetilde{D}_1 x_n \\ \widetilde{D}_2 x_1 & \widetilde{D}_2 x_2 & \cdots & \widetilde{D}_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{D}_n x_1 & \widetilde{D}_n x_2 & \cdots & \widetilde{D}_n x_n \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} D_1 \widetilde{x}_1 & D_1 \widetilde{x}_2 & \cdots & D_1 \widetilde{x}_n \\ D_2 \widetilde{x}_1 & D_2 \widetilde{x}_2 & \cdots & D_2 \widetilde{x}_n \\ \vdots & \vdots & \ddots & \vdots \\ D_n \widetilde{x}_1 & D_n \widetilde{x}_2 & \cdots & D_n \widetilde{x}_n \end{pmatrix}, \tag{11}$$

*and* $J = \det(A)$*.*

The following is the fundamental theorem on double reduction theory [9].

**Theorem 7.** *Suppose* $D_i T^i = 0$ *is a conservation law of the PDE system* (1)*. Then under a similarity transformation of a symmetry* $X$ *of the form* (2) *for the PDE, there exist functions* $\widetilde{T}^i$ *such that* $X$ *is still symmetry for the PDE* $\widetilde{D}_i \widetilde{T}^i = 0$ *and*

$$\begin{pmatrix} X\widetilde{T}^1 \\ X\widetilde{T}^2 \\ \vdots \\ X\widetilde{T}^n \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} \left[T^1, X\right] \\ \left[T^2, X\right] \\ \vdots \\ \left[T^n, X\right] \end{pmatrix}, \tag{12}$$

*where*

$$A = \begin{pmatrix} \widetilde{D}_1 x_1 & \widetilde{D}_1 x_2 & \cdots & \widetilde{D}_1 x_n \\ \widetilde{D}_2 x_1 & \widetilde{D}_2 x_2 & \cdots & \widetilde{D}_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{D}_n x_1 & \widetilde{D}_n x_2 & \cdots & \widetilde{D}_n x_n \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} D_1 \widetilde{x}_1 & D_1 \widetilde{x}_2 & \cdots & D_1 \widetilde{x}_n \\ D_2 \widetilde{x}_1 & D_2 \widetilde{x}_2 & \cdots & D_2 \widetilde{x}_n \\ \vdots & \vdots & \ddots & \vdots \\ D_n \widetilde{x}_1 & D_n \widetilde{x}_2 & \cdots & D_n \widetilde{x}_n \end{pmatrix}, \qquad J = \det(A). \tag{13}$$

**Corollary 8** (the necessary and sufficient condition for reduced conserved form [9]). *The conserved form* $D_i T^i = 0$ *of the PDE system* (1) *can be reduced under a similarity transformation of a symmetry* $X$ *to a reduced conserved form* $\widetilde{D}_i \widetilde{T}^i = 0$ *if and only if* $X$ *is associated with the conservation law* $T$*, that is,* $[T, X]|_{(1)} = 0$*.*

**Corollary 9** (see [9]). *A nonlinear system of* $q$*th-order PDEs with* $n$ *independent and* $m$ *dependent variables which admits a nontrivial conserved form that has at least one associated symmetry in every reduction from the* $n$ *reductions (the first step of double reduction) can be reduced to a* $(q-1)$*th-order nonlinear system of ODEs.*

## 3. Lie Symmetries, Conservation Laws, Reductions, and New Exact Solutions of Zakharov-Kuznetsov Equation

The $(2+1)$ dimensional Zakharov-Kuznetsov (ZK) equation [14, 15] representing the model for nonlinear Rossby waves is

$$u_t + au_x + buu_x + cu_{xxx} + du_{xyy} = 0, \tag{14}$$

where $a, b, c$, and $d$ are arbitrary constants. First we will derive the Lie symmetries of (14). The Lie point symmetry generator

$$X = \xi^1(t,x,y,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u}, \tag{15}$$

of (14), is derived by solving

$$X^{[3]}\left[u_t + au_x + buu_x + cu_{xxx} + du_{xyy}\right]\Big|_{(14)} = 0, \tag{16}$$

in which $X^{[3]}$ is the third prolongation and can be computed from (2). Equation (16), after expansion and separation, yields the following overdetermined system of partial differential equations for the unknown coefficients $\xi^1$, $\xi^2$, $\xi^3$, and $\eta$:

$$\begin{gathered} \xi^1_u = 0, \qquad \xi^1_x = 0, \qquad \xi^1_y = 0, \\ \xi^2_u = 0, \qquad \xi^2_y = 0, \qquad \xi^2_{xx} = 0, \\ \xi^3_u = 0, \qquad \xi^3_t = 0, \qquad \xi^3_x = 0, \\ \eta_{uu} = 0, \qquad \eta_{xu} = 0, \qquad \xi^3_y - \xi^2_x = 0, \\ \xi^3_{yy} - 2\eta_{yu} = 0, \qquad \xi^1_t - 3\xi^2_x = 0, \\ \eta_t + (a+bu)\eta_x + c\eta_{xxx} + d\eta_{xyy} = 0, \\ \xi^2_t - 2(a+bu)\xi^2_x - b\eta - d\eta_{yyu} = 0. \end{gathered} \tag{17}$$

The solution of system (17) gives the following Lie symmetries:

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial y}, \\ X_4 = \frac{\partial}{\partial u} + bt\frac{\partial}{\partial x}, \\ X_5 = 3t\frac{\partial}{\partial t} + (x+2at)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}. \end{gathered} \tag{18}$$

The conservation laws for (14) will be derived by the multiplier approach. Consider the multipliers of the form

$\Lambda = \Lambda(t, x, u)$ for (14). The determining equation for the multipliers is

$$\frac{\delta}{\delta u}\left[\Lambda\left(u_t + au_x + buu_x + cu_{xxx} + du_{xyy}\right)\right] = 0, \tag{19}$$

where $\delta/\delta u$ is the standard Euler operator and can be computed from (5). Expanding and then separating (19) with respect to different combinations of derivatives of $u$ yields the following overdetermined system for the multipliers:

$$\begin{gathered} \Lambda_{xx} = 0, \qquad \Lambda_{xy} = 0, \qquad \Lambda_{ux} = 0, \\ \Lambda_{uy} = 0, \qquad \Lambda_{uu} = 0, \\ \Lambda_t = -\Lambda_x (bu + a), \qquad \Lambda_{u_t} = 0. \end{gathered} \tag{20}$$

The solution of system (20) can be expressed as

$$\Lambda = f(y) + (x - at - btu) c_1 + c_2 u, \tag{21}$$

where $c_1$ and $c_2$ are arbitrary constants, and $f(y)$ is an arbitrary function of $y$. Multipliers $\Lambda$ for (14) satisfy

$$\begin{aligned} &\Lambda\left(u_t + au_x + buu_x + cu_{xxx} + du_{xyy}\right) \\ &\quad = D_t T^t + D_x T^x + D_y T^y, \end{aligned} \tag{22}$$

for all functions $u(t, x, y)$. From (21) and (22), we obtain three conserved vectors, and they are given in Table 1. If we consider multipliers of form $\Lambda = \Lambda(t, x, u, u_t, u_x, u_y)$, we get the same conserved vectors.

Now, we apply the double reduction theorem based on Lie symmetries and conservation laws to find the reductions and exact solutions. Let $T_1 = (T_1^t, T_1^x, T_1^y)$, $T_2 = (T_2^t, T_2^x, T_2^y)$, and $T_3 = (T_3^t, T_3^x, T_3^y)$. Equation (7) for variables $t$, $x$, and $y$ yields

$$\begin{aligned} &X\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} - \begin{pmatrix} D_t\xi^1 & D_x\xi^1 & D_y\xi^1 \\ D_t\xi^2 & D_x\xi^2 & D_y\xi^2 \\ D_t\xi^3 & D_x\xi^3 & D_y\xi^3 \end{pmatrix}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} \\ &\quad + \left(D_t\xi^1 + D_x\xi^2 + D_y\xi^3\right)\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = 0. \end{aligned} \tag{23}$$

Equation (23) is used to find symmetries associated with the conserved vectors presented in Table 1. The symmetries $X_1$, $X_2$, and $X_3$ are associated with the conserved vector $T_2$. The symmetries $X_1$, $X_2$, $X_3$, and $X_5$ are associated with the conserved vector $T_3$ only when $f(y) = 1$.

*3.1. Reduction via $T_3$ Using Combination of Symmetries $X_1$, $X_2$, $X_3$.* The conserved vector $T_3$ for $f(y) = 1$ yields

$$T_3^t = u, \qquad T_3^x = \frac{u^2 b}{2} + au + cu_{xx}, \qquad T_3^y = du_{xy}. \tag{24}$$

The combination of symmetries $X_1$, $X_2$, and $X_3$

$$X = \frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} \tag{25}$$

can be used to obtain a reduced conserved form. The generator, $X$, has a canonical form $X = \partial/\partial q$ when

$$\frac{dt}{1} = \frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dv}{0}, \tag{26}$$

and thus the canonical variables are

$$\begin{gathered} r = y - \beta t, \qquad s = x - \alpha t, \qquad q = t, \\ v(r, s) = u(t, x, y). \end{gathered} \tag{27}$$

The formula (10) for the reduced conserved form in terms of variables $(t, x, y)$ and $(r, s, q)$ can be expressed as

$$\begin{pmatrix} T_3^r \\ T_3^s \\ T_3^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_3^t \\ T_3^x \\ T_3^y \end{pmatrix}, \tag{28}$$

where $A^{-1}$ from (11) is given by

$$A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix}, \qquad J = \det(A). \tag{29}$$

Equations (28) and (29) for the conserved vector (24) results in

$$\begin{gathered} T_3^r = \beta v - dv_{sr}, \\ T_3^s = \alpha v - av - b\frac{v^2}{2} - cv_{ss}, \\ T_3^q = -v, \end{gathered} \tag{30}$$

and reduced conserved form is

$$D_r T_3^r + D_s T_3^s = 0. \tag{31}$$

The generalized double reduction theorem reduced the third-order ZK equation (14) from the third-order PDE in terms of three independent variables $(t, x, y)$ to a system of two second-order PDES with two independent variables $(r, s)$. It can be further reduced to an ODE if the reduced form admits symmetries, and at least one symmetry is associated with a nontrivial conserved vector. The reduced conserved form (31) admits the following two symmetries:

$$\widetilde{X}_1 = \frac{\partial}{\partial r}, \qquad \widetilde{X}_2 = \frac{\partial}{\partial s}. \tag{32}$$

Since $\widetilde{X}_1$ and $\widetilde{X}_2$ *satisfy* the symmetry conservation law relation

$$\begin{aligned} &X\begin{pmatrix} T^r \\ T^s \end{pmatrix} - \begin{pmatrix} D_r\xi^r & D_s\xi^r \\ D_r\xi^s & D_s\xi^s \end{pmatrix}\begin{pmatrix} T^r \\ T^s \end{pmatrix} \\ &\quad + \left(D_r\xi^r + D_s\xi^s\right)\begin{pmatrix} T^r \\ T^s \end{pmatrix} = 0, \end{aligned} \tag{33}$$

therefore $\widetilde{X}_1$, and $\widetilde{X}_2$ are the associated symmetries, and it is possible to find second reduction. A reduced conserved form can be obtained using

$$Y = \frac{\partial}{\partial r} + \gamma\frac{\partial}{\partial s}. \tag{34}$$

Table 1: Multipliers and conserved vectors for (14).

| Multipliers | Conserved vector |
|---|---|
| $\Lambda_1 = x - at - btu$ | $T_1^t = xu - uat - \dfrac{u^2bt}{2}$ <br> $T_1^x = -\dfrac{u^3b^2t}{3} + \dfrac{u^2bx}{2} - abtu^2 - bctuu_{xx} + \dfrac{bctu_x^2}{2} - \dfrac{bdtuu_{yy}}{2} + aux - a^2tu + cxu_{xx} - actu_{xx} - cu_x + dxu_{yy} - adtu_{yy}$ <br> $T_1^y = -\dfrac{bdtuu_{xy}}{2} + \dfrac{bdtu_yu_x}{2} - du_y$ |
| $\Lambda_2 = u$ | $T_2^t = \dfrac{u^2}{2}$ <br> $T_2^x = \dfrac{u^3b}{3} + \dfrac{u^2a}{2} + cuu_{xx} - \dfrac{cu_x^2}{2} + \dfrac{duu_{yy}}{2}$ <br> $T_2^y = \dfrac{duu_{xy}}{2} - \dfrac{du_xu_y}{2}$ |
| $\Lambda_3 = f(y)$ | $T_3^t = f(y)u$ <br> $T_3^x = f(y)\left[\dfrac{bu^2}{2} + au + cu_{xx}\right]$ <br> $T_3^y = f(y)du_{xy}$ |

The canonical form of generator $Y$ is $Y = \partial/\partial m$ with the similarity variables

$$n = \gamma r - s, \qquad m = r, \qquad w(n) = v(r,s). \tag{35}$$

In this case, the formula (10) for the reduced conserved form results in

$$\begin{pmatrix} T_3^n \\ T_3^m \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix}, \tag{36}$$

with

$$A^{-1} = \begin{pmatrix} D_rn & D_rm \\ D_sn & D_sm \end{pmatrix}. \tag{37}$$

Equations (36) and (37) for the conserved vector (30) take the following form:

$$\begin{pmatrix} T_3^n \\ T_3^m \end{pmatrix} = \begin{pmatrix} \gamma & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta v - dv_{sr} \\ \alpha v - av - b\dfrac{v^2}{2} - cv_{ss} \end{pmatrix}. \tag{38}$$

Equation (38) expressed in terms of variable $n$ becomes

$$\begin{gathered} T_3^n = (\gamma\beta - \alpha + a)\,w + \frac{b}{2}w^2 + \left(\gamma^2 d + c\right) w_{nn}, \\ T_3^m = \beta w + \gamma d w_{nn}, \end{gathered} \tag{39}$$

and the reduced conserved form is

$$D_nT_3^n = 0. \tag{40}$$

Equation (40) gives $T_3^n = k_1$, and (39) can be written as

$$(\gamma\beta - \alpha + a)\,w + \frac{b}{2}w^2 + \left(\gamma^2 d + c\right) w_{nn} = k_1. \tag{41}$$

The symmetries $X_1$, $X_2$, and $X_3$ are associated with the conserved vector $T_3$ only when $f(y) = 1$. For this case, the generalized double reduction theorem is applied twice to ZK equation (14) and it is reduced to second-order ODE (41). Next, we find implicit and explicit solutions of reduced form (41) and these constitute the exact solutions of ZK equation (14).

The implicit solution of (41) using Maple is

$$\pm\int \frac{\sqrt{3}\left(\gamma^2 d + c\right)}{\sqrt{\left(\gamma^2 d + c\right)\left(3\alpha w^2 - 3aw^2 - bw^3 - 3\gamma\beta w^2 + 3\gamma^2 dc_1 + 3cc_1 + 6kw\right)}}dw - n - c_2 = 0. \tag{42}$$

Now, we compute the explicit solutions of (41) by utilizing the Sine-Cosine method [18–20]. The solution of (41) can be expressed in the form

$$w(n) = \nu \cos^{\kappa}(\omega n), \tag{43}$$

or

$$w(n) = \nu \sin^{\kappa}(\omega n), \tag{44}$$

where $\nu$, $\kappa \neq 0$ and $\omega$ are parameters need to be determined.

Substituting the values of $w$ from (43) and setting $k_1 = 0$ in (41) yields

$$A\nu\cos^{\kappa}(\omega n) + \frac{b}{2}\nu^2\cos^{2\kappa}(\omega n) + B\nu\kappa^2\omega^2\cos^{\kappa-2}(\omega n) - B\nu\kappa^2\omega^2\cos^{\kappa}(\omega n) - B\nu\kappa\omega^2\cos^{\kappa-2}(\omega n) = 0, \tag{45}$$

where $A = \gamma\beta - \alpha + a$ and $B = \gamma^2 d + c$. Equation (45) is satisfied if

$$\begin{gathered}\kappa - 2 = 2\kappa,\\ \gamma\beta - \alpha + a - 4\mu^2\gamma^2 d - 4\mu^2 c = 0,\\ \frac{1}{2}b\lambda + 6\mu^2\gamma^2 d + 6\mu^2 c = 0.\end{gathered} \tag{46}$$

Ultimately, the solution of algebraic system (46) yields the solution of (41) and is given by

$$\begin{gathered}w(n) = -3\frac{\gamma\beta + a - \alpha}{b}\sec^2\left(\sqrt{\frac{\gamma\beta + a - \alpha}{4c + 4d\gamma^2}}n\right),\\ n = \gamma y - x + t(\alpha - \beta\gamma), \quad w = u.\end{gathered} \tag{47}$$

Similarly, using the Sine function (44) one can easily obtain the solution of (41) as

$$w(n) = -3\frac{\gamma\beta + a - \alpha}{b}\csc^2\left(\sqrt{\frac{\gamma\beta + a - \alpha}{4c + 4d\gamma^2}}n\right), \tag{48}$$

where $n = \gamma y - x + t(\alpha - \gamma\beta)$, $w = u$. The solutions (47) and (48) can be finally expressed in terms of original variables as

$$u(t, x, y) = -3\frac{\gamma\beta + a - \alpha}{b} \times \sec^2\left(\sqrt{\frac{\gamma\beta + a - \alpha}{4c + 4d\gamma^2}}\left[\gamma y - x + t(\alpha - \gamma\beta)\right]\right), \tag{49}$$

$$u(t, x, y) = -3\frac{\gamma\beta + a - \alpha}{b} \times \csc^2\left(\sqrt{\frac{\gamma\beta + a - \alpha}{4c + 4d\gamma^2}}\left[\gamma y - x + t(\alpha - \gamma\beta)\right]\right), \tag{50}$$

and these constitute the exact solutions of ZK equation (14).

*3.2. Reduction via $T_2$ Using Combination of Symmetries $X_1$, $X_2$, and $X_3$.* The symmetries $X_1$, $X_2$, and $X_3$ are associated with the conserved vector $T_2$, and a reduced conserved form can be obtained by the combination $X$ given in (25). For the conserved vector $T_2$ in terms of canonical variables (27), we have

$$\begin{gathered}T_2^r = \beta\frac{v^2}{2} - \frac{dvv_{rs}}{2} + \frac{dv_r v_s}{2},\\ T_2^s = \alpha\frac{v^2}{2} - \frac{bv^3}{3} - \frac{av^2}{2} - cvv_{ss} + \frac{cv_s^2}{2} - \frac{dvv_{rr}}{2},\\ T_2^q = -\frac{v^2}{2},\end{gathered} \tag{51}$$

and reduced conserved form is $D_r T_2^r + D_s T_2^s = 0$. The reduced conserved form admits the symmetries (32) and these symmetries satisfy the symmetry conservation law relation (33) for the conserved vectors $T_2^r$ and $T_2^s$. The combination of symmetries yields the generator $Y$ and the similarity variables (35). Equations (36) and (37) for the conserved vector (51) gives

$$\begin{aligned}T_2^n &= \frac{(\beta\gamma + a - \alpha)}{2}w^2 + (\gamma^2 d + c)ww_{nn} + \frac{bw^3}{3} - \frac{(\gamma^2 d + c)}{2}w_n^2,\\ T_2^m &= -\beta\frac{w^2}{2} - \frac{d\gamma ww_{nn}}{2} + \frac{d\gamma w_n^2}{2}.\end{aligned} \tag{52}$$

The reduced conserved form is

$$D_n T_2^n = 0, \tag{53}$$

and this yields

$$\frac{(\beta\gamma + a - \alpha)}{2}w^2 + (\gamma^2 d + c)ww_{nn} + \frac{bw^3}{3} - \frac{(\gamma^2 d + c)}{2}w_n^2 = k_2, \tag{54}$$

where $k_2$ is a constant. Using Maple, the solution of (54) is

$$\begin{gathered}\pm\int\frac{\sqrt{3}(d\gamma^2 + c)}{\sqrt{(d\gamma^2 + c)(3\alpha w^2 - 3aw^2 - 3\gamma\beta w^2 - bw^3 + 3cc_3 w + 3c_3 d\gamma^2 w - 6k_2)}}dw - n - c_4 = 0,\\ n = \gamma y - x + t(\alpha - \gamma\beta), \quad w = u,\end{gathered} \tag{55}$$

whereas (54) gives the same solution by employing Sine-Cosine method as we have obtained in the previous case. The symmetries $X_1$, $X_2$, and $X_3$ are associated with the conserved vector $T_2$, and in this case generalized double reduction theorem gives one implicit solution (55) for the ZK equation (14). It is interesting to notice that generalized double reduction theorem yields two different reduced forms (41) and (54) for traveling wave solutions (49) and (50), whereas in [14, 15] only one reduced form (41) was obtained. One can use the simple reduced form to construct exact or approximate solutions.

*3.3. Reduction via $T_3$ Using Symmetry $X_5$.* The generator, $X_5$, has canonical form $X = \partial/\partial q$ if

$$\frac{dt}{3t} = \frac{dx}{x+2at} = \frac{dy}{y} = \frac{du}{-2u} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dv}{0}, \tag{56}$$

and thus we have

$$q = \frac{1}{3}\ln t, \qquad r = \frac{y}{t^{1/3}}, \qquad s = \frac{(x-at)}{t^{1/3}}, \qquad v(r,s) = u(t,x,y)\,t^{2/3}. \tag{57}$$

Equations (28) and (29) for the conserved vector $T_3$ in terms of canonical variables (57) result in

$$T_3^r = rv - 3dv_{rs}, \qquad T_3^s = sv - \frac{3}{2}bv^2 - 3cv_{ss}, \tag{58}$$

and reduced conserved form is $D_r T_3^r + D_s T_3^s = 0$. The conserved form (58) cannot be further reduced because it does not admit any symmetry, however one can perform the numerical simulation or any other approximate method to construct the approximate solutions.

The generalized double reduction theorem gives two different reduced forms (41) and (54) for traveling wave solutions. The Sine-Cosine method for each of the reduced forms gives the explicit solutions (49) and (50) for the ZK equation. Also we find two implicit solutions (42) and (55) by Maple. The transformations (57) are obtained due to double reduction theorem, and these transformations are different from the traveling wave transformations. These transformations provide the reduced form (58), and numerical method can be applied to obtain approximate solutions for the ZK equation (14). The exact solutions for ZK equation obtained here are different from the class of exact solutions computed by Exp-function method [14] and by transformation of elliptic equation [15].

## 4. Lie Symmetries, Conservation Laws, and Exact Solutions of Gardner KP Equation

The Gardner KP equation [16] is

$$u_{tx} + 6uu_{xx} + 6u_x^2 + 6u^2u_{xx} + 12uu_x^2 + u_{xxxx} + u_{yy} = 0. \tag{59}$$

The Lie symmetry generator determining equation for Gardner KP equation (59) is

$$X^{[4]}\left[u_{tx} + 6uu_{xx} + 6u_x^2 + 6u^2u_{xx} + 12uu_x^2 + u_{xxxx} + u_{yy}\right]\Big|_{(59)} = 0, \tag{60}$$

where $X^{[4]}$ is the fourth prolongation. Solving (60), after expansion, we have

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial y},$$
$$X_4 = y\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial y}, \tag{61}$$
$$X_5 = -6t\frac{\partial}{\partial t} + (-2x+6t)\frac{\partial}{\partial x} - 4y\frac{\partial}{\partial y} + (2u+1)\frac{\partial}{\partial u},$$

as the Lie symmetry generators for the Gardner KP equation (59). Consider the multipliers of the form $\Lambda = \Lambda(t,x,u)$ for (59). The determining equation for the multipliers is

$$\frac{\delta}{\delta u}\left[\Lambda\left(u_{tx} + 6uu_{xx} + 6u_x^2 + 6u^2u_{xx} + 12uu_x^2 + u_{xxxx} + u_{yy}\right)\right] = 0. \tag{62}$$

Equation (62) finally presents

$$\Lambda_u = 0, \qquad \Lambda_{xx} = 0, \qquad \Lambda_{xyy} = 0, \qquad \Lambda_{yyyy} = 0, \qquad \Lambda_{tx} + \Lambda_{yy} = 0, \tag{63}$$

and this results in four multipliers. The multipliers and conserved vectors are presented in Table 2.

The symmetry conservation law relation (23) is not satisfied for the conserved vectors $T_1$, $T_2$, and $T_3$. The symmetries $X_1$, $X_2$, and $X_3$ are associated with the conserved vector $T_4$ if $f(t) = 1$. Thus we can get a reduced conserved form by the combination of $X$ given in (25). Equations (28) and (29) for the conserved vector $T_4$ in terms of canonical variables (27) yield the following three components of $\widetilde{T}_4$:

$$T_4^r = \beta v_s - v_r,$$
$$T_4^s = \alpha v_s - v_{sss} - 6v^2 v_s - 6vv_s, \tag{64}$$
$$T_4^q = -v_s,$$

and reduced conserved form is

$$D_r T_4^r + D_s T_4^s = 0. \tag{65}$$

The reduced conservation law admits the following symmetries:

$$\widetilde{X}_1 = \frac{\partial}{\partial r}, \qquad \widetilde{X}_2 = \frac{\partial}{\partial s}. \tag{66}$$

The symmetries $\widetilde{X}_1$ and $\widetilde{X}_2$ satisfy the symmetry conservation law relation (33) for the conserved vectors $T_4^r$ and $T_4^s$.

Table 2: Multipliers and conserved vectors for (59).

| Multipliers | Conserved vector |
|---|---|
| $\Lambda_1 = f(t)xy - \frac{1}{6}f_t y^3$ | $T_1^t = \frac{1}{6}yu_x\left(6xf(t) - f_t y^2\right)$ <br> $T_1^x = -\frac{1}{6}yf(t)\left(-36xu^2u_x + 12u^3 - 36xuu_x + 18u^2 - 6xu_{xxx} + 6u_{xx}\right)$ $-\frac{1}{6}yf_t\left(6y^2u^2u_x + 6y^2uu_x + y^2u_{xxx} + 6ux\right) + \frac{1}{6}f_{tt}y^3u$ <br> $T_1^y = f(t)\left(xyu_y - ux\right) + f_t\left(\frac{1}{2}y^2u - \frac{1}{6}y^3u_y\right)$ |
| $\Lambda_2 = f(t)x - \frac{1}{2}f_t y^2$ | $T_2^t = \frac{1}{2}u_x\left(2xf(t) - f_t y^2\right)$ <br> $T_2^x = f(t)\left(6xu^2u_x - 2u^3 + 6xuu_x - 3u^2 + xu_{xxx} - u_{xx}\right)$ $-f_t\left(3y^2u^2u_x + 3y^2uu_x + \frac{1}{2}y^2u_{xxx} + xu\right) + \frac{1}{2}uf_{tt}y^2$ <br> $T_2^y = f(t)xu_y + f_t\left(yu - \frac{1}{2}y^2u_y\right)$ |
| $\Lambda_3 = f(t)y$ | $T_3^t = f(t)yu_x$ <br> $T_3^x = yf(t)\left(6u^2u_x + u_{xxx} + 6uu_x\right) - uyf_t$ <br> $T_3^y = f(t)\left(-u + yu_y\right)$ |
| $\Lambda_4 = f(t)$ | $T_4^t = f(t)u_x$ <br> $T_4^x = \left(6u^2u_x + 6uu_x + u_{xxx}\right)f(t) - f_tu$ <br> $T_4^y = f(t)u_y$ |

Taking the combination of these symmetries yields the same generator $Y$ as given in (34), and the canonical form $Y = \partial/\partial m$ can be obtained from similarity variables (35). Using formula (36), one has the following two components of $\tilde{\tilde{T}}_4$:

$$T_4^n = \left(\alpha - \beta\gamma - \gamma^2\right)w_n - 6w^2w_n - 6ww_n - w_{nnn},$$
$$T_4^m = -\left(\beta + \gamma\right)w_n. \tag{67}$$

The reduced conserved form satisfies $D_nT^n = 0$, and we have

$$\left(\alpha - \beta\gamma - \gamma^2\right)w_n - 6w^2w_n - 6ww_n - w_{nnn} = k_3. \tag{68}$$

The integration of (68) provides

$$\left(\alpha - \beta\gamma - \gamma^2\right)w - 2w^3 - 3w^2 - w_{nn} = k_3n + k_4. \tag{69}$$

Next, we find implicit and explicit solutions of reduced form (69) and these constitute the exact solutions of GKP equation (59).

Equation (69) gives the following solution if $k_3 = 0$:

$$\pm\int\frac{1}{\sqrt{\left(\alpha - \beta\gamma - \gamma^2\right)w^2 - 2w^3 - w^4 - 2k_4w + c_5}}dw - n - c_6 = 0, \tag{70}$$

where $n = (\alpha - \beta\gamma)t + \gamma y - x$, $w = u$ and $c_5$, $c_6$ and $k_4$ are constants.

For explicit solution, we apply the first integral method to the reduced form (69). We substitute $w = X$ and $w' = Y$ with $k_3 = k_4 = 0$ which converts (69) into the following system of ODEs:

$$X' = Y,$$
$$Y' = AX - 3X^2 - 2X^3, \tag{71}$$

where $A = \alpha - \beta\gamma - \gamma^2$. Next, we apply the division theorem to seek the first integral to (71). Assume that $X = X(n)$ and $Y = Y(n)$ are the nontrivial solutions to (71) and

$$p\left(X(n), Y(n)\right) = \sum a_i\left(X(n)\right)Y^i = 0, \quad i = 1, 2, \dots, m \tag{72}$$

is an irreducible polynomial in $c|X, Y|$ such that

$$p\left(X(n), Y(n)\right) = \sum a_i\left(X(n)\right)Y(n)^i = 0, \quad i = 1, 2, \dots, m, \tag{73}$$

where $a_i(X)$, $(i = 1, 2, \dots, m)$ are polynomials of $X$ and all relatively prime in $c|X, Y|$, $a_m(X) \neq 0$. Equation (73) is also called the first integral to (71). Suppose that $m = 1$ in (73). By division theorem, there exist polynomials $H(X, Y) = g(X) + h(X)Y$ in $c|X, Y|$ such that

$$\frac{dp}{dn} = \frac{\partial p}{\partial X}\frac{\partial X}{\partial n} + \frac{\partial p}{\partial Y}\frac{\partial Y}{\partial n} = \left(g(X) + h(X)Y\right)\left(a_0(X) + a_1(X)Y\right), \tag{74}$$

or

$$\left(a_0'(X) + a_1'(X)(Y)\right)Y + a_1(X)Y' = \left(g(X) + h(X)Y\right)\left(a_0(X) + a_1(X)Y\right). \tag{75}$$

Substituting $Y'$ from (71) in (75) and then separating with respect to powers of $Y$, we obtain

$$a_1'(X) = h(X)a_1(X),$$
$$a_0'(X) = g(X)a_1(X) + h(X)a_0(X), \tag{76}$$
$$a_1(X)\left(AX - 3X^2 - 2X^3\right) = g(X)a_0(X).$$

Solving the system (76) for $a_0$ and $a_1$ and then substituting it into (73) yields an ODE which finally gives

$$u(t,x,y) = \frac{1}{c_5\ e^{\iota(\gamma y-\gamma\beta t-x+(\gamma\beta+\gamma^2-1)t)} - 1},$$
$$u(t,x,y) = \frac{1}{c_6\ e^{-\iota(\gamma y-\gamma\beta t-x+(\gamma\beta+\gamma^2-1)t)} - 1}. \tag{77}$$

The explicit solutions (77) form exact solution of GKP equation (59).

The generalized double reduction theorem is applied twice to the GKP equation (59), and it is reduced to an integrable third-order ODE (68). On integration, the third order ODE (68) is further reduced to second-order ODE (69). Using Maple equation (69) yields one implicit solution (70) for the GKP equation (59). Also two explicit solutions (77) for the GKP equation are obtained utilizing the first integral method to the reduced second-order ODE (69). The exact solutions derived here are different from class of multiple-soliton solutions obtained by Hirota's bilinear method [16].

## 5. Lie Symmetries, Conservation Laws, and Exact Solutions of Modified KP Equation

The MKP equation [17] describing the soliton propagation in multitemperature electrons plasmas is

$$u_{tx} + auu_{xx} + au_x^2 + 2duu_x^2 + du^2u_{xx} + bu_{xxxx} + c\left(u_{xx} + u_{yy}\right) = 0, \tag{78}$$

where $a$, $b$, $c$, and $d$ are plasma parameters. For (78) the multipliers of the form $\Lambda = \Lambda(t,x,y,u)$ are considered. The multipliers and conserved vectors are given in Table 3. The MKP equation (78) has the following Lie symmetry generators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x},$$
$$X_3 = \frac{\partial}{\partial y}, \qquad X_4 = y\frac{\partial}{\partial x} - 2ct\frac{\partial}{\partial y},$$
$$X_5 = -6dt\frac{\partial}{\partial t} + \left(-2dx - 4tcd + a^2t\right)\frac{\partial}{\partial x} - 4dy\frac{\partial}{\partial y} + (2du + a)\frac{\partial}{\partial u}. \tag{79}$$

Only the Lie symmetries $X_1$, $X_2$, and $X_3$ are associated with the conserved vector $T_4$ when $f(t) = 1$. By using their combination as we have done in Section 3, and with the aid of $T_4$, we obtain

$$\left(-\beta\gamma - c\gamma^2 + \alpha - c\right)w - \frac{a}{2}w^2 - bw_{nn} - \frac{d}{3}w^3 = k_5 n + k_6. \tag{80}$$

A particular solution of (80) can be found for the case $k_5 = 0$ and is given by

$$\pm\int\frac{6b}{\sqrt{-6b\left(6\left(\beta\gamma + c\gamma^2 - \alpha + c\right)w^2 + 2aw^3 + dw^4 - 12k_6w - 6bc_5\right)}}dw - n - c_6 = 0, \tag{81}$$

where $n = \gamma y - \gamma\beta t - x + \alpha t$, $w = u$.

A class of solitary wave solutions were reported in [17] using (80), whereas the above solution is not reported there.

## 6. Conclusions

The generalized double reduction theorem provides a powerful tool in constructing reduced forms and exact solutions. It enables a systematic way to find not only the transformations providing traveling wave solutions but also other types of transformations. These transformations reduce a nonlinear system of $q$th-order PDEs with $n$ independent and $m$ dependent variables to a nonlinear system of $(q-1)$th-order ODEs provided that in every reduction at least one symmetry is associated with a nontrivial conserved vector. The reduced ODE can be solved either analytically or numerically to derive exact or approximate solutions. It is interesting that the transformations yielding traveling wave solutions can give sometimes more than one reduced form, and one can use the simple one to find exact solution.

The Lie symmetries, conservation laws, reduced forms and new exact solutions of (2 + 1) dimensional ZK, GKP, and MKP equations were derived. First of all ZK equation was considered, and the Lie symmetries and conservation laws were constructed. Multiplier approach yielded three conserved vectors. The symmetry conservation laws relationship was used to determine symmetries associated with the conserved vectors. Three symmetries were associated with the conserved vector $T_3$ if $f(y) = 1$. The generalized double reduction theorem was applied twice to ZK equation to convert it to a second-order ordinary differential equation (41). Thus third-order (2 + 1) dimensional ZK equation was

TABLE 3: Multipliers and conserved vectors for (78).

| Multipliers | Conserved vector |
|---|---|
| $\Lambda_1 = \frac{1}{6c}\left[-f_t y^3 + 6cf(t)xy\right]$ | $T_1^t = \frac{1}{6c}\left[-f_t y^3 + 6cf(t)xy\right]u_x$ <br> $T_1^x = -\frac{1}{6c}yf(t)\left(-6dcxu^2u_x + 2dcu^3 - 6acxuu_x + 3cau^2 + 6bcu_{xx} + 6c^2u - 6c^2xu_x - 6bcxu_{xxx}\right) + \frac{1}{6c}f_{tt}y^3u - \frac{1}{6c}yf_t\left(dy^2u^2u_x + ay^2uu_x + 6cxu + cy^2u_x + by^2u_{xxx}\right)$ <br> $T_1^y = f(t)\left(-cxu + cxyu_y\right) + f_t\left(\frac{1}{2}y^2u - \frac{1}{6}y^3u_y\right)$ |
| $\Lambda_2 = \frac{1}{2c}\left[-f_t y^2 + 2cxf(t)\right]$ | $T_2^t = -\frac{1}{2c}u_x\left(f_t y^2 - 2f(t)cx\right)$ <br> $T_2^x = -\frac{1}{6c}f(t)\left(-6dcxu^2u_x + 2dcu^3 - 6acxuu_x + 3cau^2 + 6bcu_{xx} + 6c^2u - 6c^2xu_x - 6bcxu_{xxx}\right) + \frac{1}{2c}f_{tt}y^2u - \frac{1}{6c}f_t\left(6cxu + 3cy^2u_x + 3by^2u_{xxx} + 3dy^2u^2u_x + 3ay^2uu_x\right)$ <br> $T_2^y = f_t\left(yu - \frac{1}{2}y^2u_y\right) + f(t)cxu_y$ |
| $\Lambda_3 = yf(t)$ | $T_3^t = f(t)yu_x$ <br> $T_3^x = yf(t)\left(du^2u_x + auu_x + cu_x + bu_{xxx}\right) - yf_tu$ <br> $T_3^y = -f(t)cu + f(t)cyu_y$ |
| $\Lambda_4 = f(t)$ | $T_4^t = f(t)u_x$ <br> $T_4^x = f(t)\left[du^2u_x + auu_x + cu_x + bu_{xxx}\right] - f_tu$ <br> $T_4^y = f(t)cu_y$ |

reduced to a second-order ordinary differential equation in terms of canonical variables. Furthermore, one implicit solution was found for (41) which constituted the exact solution of ZK equation. The Sine-Cosine method was applied to the reduced second-order ODE (41), and two explicit solutions were computed for ZK equation. Likewise, symmetries $X_1$, $X_2$, and $X_3$ were associated with the conserved vector $T_2$ and reduced conserved form of ZK equation was derived. One implicit solution was constructed using Maple, whereas same explicit solutions were obtained as in the previous case. The symmetry $X_5$ was associated with the conserved vector $T_3$, and ZK equation was reduced to second-order system (58). It was not possible to further reduce system (58) because it does not admit any symmetry associated with it, however one can apply approximate methods or numerical techniques to compute the approximate solutions.

The Lie symmetries and conservation laws for GKP equation were established. The GKP equation was reduced to a third-order ODE (68), and on integration it was further reduced to a second-order ODE (69). An implicit solution for (70) was found for GKP equation. Two explicit solutions of GKP equation were derived utilizing the first integral method. For MKP equation, we derived the Lie symmetries, conservation laws, reduced form, and one implicit solution.

The solutions found here are new and not found in literature. Due to the lack of experimental basis, the derived solutions cannot be interpreted physically but in applied mathematics these will play a vital role for numerical simulations.

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