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Research Article

On Abstract Economies and Their Applications

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We establish a new equilibrium existence theorem of generalized abstract economies with general preference correspondences. As an application, we derive an existence theorem of generalized quasi-variational inequalities in the general setting of *l.c.*-spaces without any linear structure.

1. Introduction and Preliminary

Let I be any (finite or infinite) set of agents. A *generalized abstract economy* is defined as a family of order quintuples $\mathbf{\Omega} = (X_\alpha, A_\alpha, B_\alpha, F_\alpha, P_\alpha)_{\alpha\in I}$ with $X := \prod_{\alpha\in I} X_\alpha$ such that for each $\alpha \in I$, X_α is a topological space, $A_\alpha, B_\alpha : X \to 2^{X_\alpha}$ are constraint correspondences, $F_\alpha : X \to 2^{X_\alpha}$ is a fuzzy constraint correspondence, and $P_\alpha : X \times X \to 2^{X_\alpha}$ is a preference correspondence. In a real market, any preference of a real agent would be unstable by the fuzziness of consumers' behavior or market situations. Thus, it is reasonable to introduce fuzzy constraint correspondences in defining an abstract economy. An *equilibrium point* of $\mathbf{\Omega}$ is a point $(\widehat{x}, \widehat{y}) \in X \times X$ such that for each $\alpha \in I$, $\widehat{x}_\alpha \in \operatorname{cl} B_\alpha(\widehat{x})$, $\widehat{y}_\alpha \in F_\alpha(\widehat{x})$, and $A_\alpha(\widehat{x}) \cap P_\alpha(\widehat{x}, \widehat{y}) = \emptyset$, where $\widehat{x}_\alpha$ and $\widehat{y}_\alpha$ denote the projections of $\widehat{x}$ and $\widehat{y}$ from X to X_α, respectively.

In case $F_\alpha(x) = X_\alpha$ for each $x \in X$ and P_α is independent of the second variable, that is, $P_\alpha : X \to 2^{X_\alpha}$, the above generalized abstract economy reduces to the standard abstract economy $\mathbf{\Omega_s} := (X_\alpha, A_\alpha, B_\alpha, P_\alpha)_{\alpha\in I}$, in which an equilibrium point of $\mathbf{\Omega_s}$ is a point $\widehat{x} \in X$ such that for each $\alpha \in I$, $\widehat{x}_\alpha \in \operatorname{cl} B_\alpha(\widehat{x})$ and $A_\alpha(\widehat{x}) \cap P_\alpha(\widehat{x}) = \emptyset$. When $A_\alpha = B_\alpha$ and each X_α is a topological vector space, the standard abstract economy $\mathbf{\Omega_s}$ coincides with the classical definition of Shafer and Sonnenschein [1]. For more details on abstract economies, see, for example, [2–14] and the references therein.

Throughout this paper, all topological spaces are assumed to be Hausdorff. In order to establish our main results, we first give some basic notations. For a nonempty set C of a topological space X, we denote the set of all subsets of C by 2^C, the set of all nonempty finite subsets of C by $\langle X\rangle$, the interior of C by $\operatorname{int} C$, and the closure of C by $\operatorname{cl} C$.

Let $\{\Gamma_D\}$ be a family of some nonempty contractible subsets of a topological space X indexed by $D \in \langle X\rangle$ such that $\Gamma_D \subset \Gamma_{D'}$ whenever $D \subset D'$. The pair $(X, \{\Gamma_D\})$ is called an *H-space*. Given an H-space $(X, \{\Gamma_D\})$, a nonempty subset C of X is said to be *H-convex* if $\Gamma_D \subseteq C$ for all $D \in \langle C\rangle$. For a nonempty subset C of X, we define the *H-convex hull* of C as

$$H\text{-co}C := \bigcap \{K \mid K \text{ is } H\text{-convex in } X \text{ and } C \subseteq K\}. \quad (1)$$

It is known that if $x \in H$-co$C$, then there exists a finit subset D of C such that $x \in H$-co$D$. Moreover, for any $D \in \langle X\rangle$, H-co$D$ is called a *polytope*. We will say that X is an *H-space with precompact polytopes* if any polytope of X is precompact. For example, a locally convex topological vector space X is an H-space with precompact polytopes, by setting $\Gamma_D = \operatorname{co} D$ for all $D \in \langle X\rangle$.

An H-space $(X, \{\Gamma_D\})$ is called an *l.c.-space* if X is a uniform space whose topology is induced by its uniformity $\mathscr{U}$, and there is a base $\mathscr{B}$ consisting of symmetric entourages in $\mathscr{U}$ such that for each $V \in \mathscr{B}$, the set $V(E) := \{y \in X \mid (x, y) \in V$ for some $x \in E\}$ is H-convex whenever E is H-convex. We will use the notation $(X, \mathscr{U}, \mathscr{B})$ to stand for an *l.c.*-space. For details of uniform spaces, we refer to [15]. In a recent paper [16], we introduce a new *measure of precompactness* of a subset A in an *l.c.*-space $(X, \mathscr{U}, \mathscr{B})$ by

$$Q(A) := \{V \in \mathscr{B} \mid A \subseteq \mathrm{cl}(V(K)) \text{ for some precompact set } K \text{ of } X\}. \tag{2}$$

Let $(X_\alpha, \mathscr{U}_\alpha, \mathscr{B}_\alpha)_{\alpha\in I}$ be a family of *l.c.*-spaces with precompact polytopes, where I is a finite or infinite index set and $X = \prod_{\alpha\in I} X_\alpha$. For each $\alpha \in I$, let π_α be the projection of X onto X_α and Q_α a measure of precompactness in X_α. We say that a set-valued mapping $T_\alpha : X \to 2^{X_\alpha}$ is Q_α-*condensing* if $Q_\alpha(\pi_\alpha(C)) \subsetneq Q_\alpha(T_\alpha(C))$ for every C satisfying $\pi_\alpha(C)$ is a nonprecompact subset of X_α. It is clear that for any set-valued mapping $T : X \to 2^Y$ and any measure Q in Y, T is Q-condensing whenever Y is compact.

Let X be a topological space, let Y be an H-space, and let $S, T : X \to 2^Y$ be two set-valued mappings.

(1) T is said to be *upper semicontinuous* (*u.s.c.*) if for each $x \in X$ and each open subset V of Y with $T(x) \subseteq V$, there exists a neighborhood N_x of x such that $T(z) \subseteq V$ for all $z \in N_x$.

(2) T is said to be *transfer open valued* on X if for each $x \in X$, for each $y \in T(x)$, there exists some $x' \in X$ such that $y \in \mathrm{int}\, T(x')$.

(3) T is said to be *transfer open inverse valued* in Y if T^{-1} is transfer open valued on Y, where $T^{-1} : Y \to 2^X$ is defined by

$$T^{-1}(y) := \{x \in X \mid y \in T(x)\} \quad \forall y \in Y. \tag{3}$$

(4) The set-valued mappings $S \cap T : X \to 2^Y$ and $\mathrm{cl}\, T : X \to 2^Y$ are defined by

$$(S \cap T)(x) := S(x) \cap T(x), \qquad \mathrm{cl}\, T(x) := \mathrm{cl}(T(x)), \quad \forall x \in X. \tag{4}$$

Further, we denote by $\mathbb{H}(X, Y)$ the class of all *u.s.c.* set-valued mappings $T : X \to 2^Y$ with nonempty closed H-convex values.

2. Main Results

The following fundamental theorems will play an important role in proving our main theorem.

Theorem A (see [16]). *Let $(X_\alpha, \mathscr{U}_\alpha, \mathscr{B}_\alpha)_{\alpha\in I}$ be a family of l.c.-spaces with precompact polytopes, $X := \prod_{\alpha\in I} X_\alpha$, and let $T_\alpha : X \to 2^{X_\alpha}$ be Q_α-condensing. Then there exists a nonempty compact H-convex subset $K := \prod_{\alpha\in I} K_\alpha$ of X such that $T_\alpha(K) \subseteq K_\alpha$.*

Theorem B (see [16]). *Let $(X_\alpha, \mathscr{U}_\alpha, \mathscr{B}_\alpha)_{\alpha\in I}$ be a family of l.c.-spaces with precompact polytopes and $X := \prod_{\alpha\in I} X_\alpha$. If $T_\alpha : X \to 2^{X_\alpha}$ is an u.s.c. Q_α-condensing mapping with closed H-convex values for each $\alpha \in I$, then $T := \prod_{\alpha\in I} T_\alpha$ has a fixed point.*

Next, we list and establish some essential lemmas as follows.

Lemma 1 (see [12]). *If X is an l.c.-space and E is an H-convex subset of X, then $\mathrm{cl}\, E$ is also H-convex.*

Lemma 2 (see [12]). *Let X be a topological space and let $(Y, \{\Gamma_D\})$ be a compact l.c.-space. If $T : X \to 2^Y$ is an u.s.c. set-valued mapping, then the mapping $x \mapsto \mathrm{cl}[H\text{-co}T(x)]$ is also u.s.c. with compact H-convex values.*

Lemma 3 (see [7]). *Let X and Y be topological spaces and let $S : X \to 2^Y$ be a transfer open valued mapping. Then $\bigcup_{x\in X} S(x) = Y \setminus (\cap_{x\in X} \mathrm{cl}(Y \setminus S(x)))$ and hence $\bigcup_{x\in X} S(x)$ is open in Y.*

Lemma 4. *Let X be paracompact, $(Y, \{\Gamma_D\})$ an H-space, and $S, T : X \to 2^Y$ be two set-valued mappings such that*

(1) *$S(x) \neq \emptyset$ and $H\text{-co}S(x) \subseteq T(x)$ for each $x \in X$,*

(2) *S is transfer open inverse valued in Y.*

Then T has a continuous selection; that is, there exists a continuous function $f : X \to Y$ such that $f(x) \in T(x)$ for each $x \in X$.

Proof. Since for each $x \in X$, $S(x) \neq \emptyset$, it follows that $x \in S^{-1}(y)$ for some $y \in Y$. Since S is transfer open inverse valued in Y, there exists some $y_x \in Y$ such that $x \in \mathrm{int}\, S^{-1}(y_x)$. This yields that $\{\mathrm{int}\, S^{-1}(y) \mid y \in Y\}$ forms an open cover of X. Since X is paracompact, there exists a locally finite open cover $\{U_y \mid y \in Y\}$ such that $U_y \subseteq \mathrm{int}\, S^{-1}(y)$ for each $y \in Y$. By [17, Theorem 3.1], there exists a continuous function $f : X \to Y$ such that $f(x) \in \Gamma_{\{y\in Y \mid x\in U_y\}}$ for all $x \in X$. Note that for any $x \in X$, there exist finitely many $y \in Y$ such that $x \in U_y \subseteq \mathrm{int}\, S^{-1}(y) \subseteq S^{-1}(y)$. This implies $y \in S(x) \subseteq H\text{-co}S(x)$, and hence $\{y \in Y \mid x \in U_y\} \in \langle H\text{-co}S(x)\rangle$. It follows that for each $x \in X$, we get

$$f(x) \in \Gamma_{\{y\in Y \mid x\in U_y\}} \subseteq H\text{-co}S(x) \subseteq T(x). \tag{5}$$

Thus, the proof is complete. □

We remark that Lemma 4 extends [7, Theorem 2] from topological vector spaces to general H-spaces. When $S = T$ and S has open lower sections, Lemma 4 reduces to [18, Theorem 3.1].

Lemma 5. *Let X be a compact H-space, and let $P : X \times X \to 2^X$ be a set-valued mapping such that for each $x \in X$, $P^{-1}(x)$ is open: then so is $(H\text{-co}P)^{-1}(x)$.*

Proof. For each $z_0 \in X$, we fix an $(x_0, y_0) \in (H\text{-co}P)^{-1}(z_0)$. Since $z_0 \in H\text{-co}P(x_0, y_0)$, there is a finite set $\{z_1, z_2, \ldots, z_n\}$ in $P(x_0, y_0)$ such that $z_0 \in H\text{-co}\{z_1, z_2, \ldots, z_n\}$. Since each $P^{-1}(z_i)$ is open, it follows that the set $U := \bigcap_{i=1}^n P^{-1}(z_i)$ is also open and $(x_0, y_0) \in U$. To complete the proof, we will show that $U \subseteq (H\text{-co}P)^{-1}(z_0)$. For any $(x, y) \in U$, we have

$(x, y) \in P^{-1}(z_i)$ for all $i = 1, 2, \ldots, n$. Accordingly, $z_i \in P(x, y)$ for all $i = 1, 2, \ldots, n$. Hence,

$$z_0 \in H\text{-co}\{z_1, z_2, \ldots, z_n\} \subseteq H\text{-co}P(x, y). \tag{6}$$

That is, $(x, y) \in (H\text{-co}P)^{-1}(z_0)$. Consequently, $U \subseteq (H\text{-co}P)^{-1}(z_0)$. □

Theorem 6. *Let $\mathbf{\Omega} = (X_\alpha, A_\alpha, B_\alpha, F_\alpha, P_\alpha)_{\alpha \in I}$ be a generalized abstract economy, where I is a set of agents and $X = \prod_{\alpha \in I} X_\alpha$ such that for each $\alpha \in I$,*

(1) *X_α is an l.c.-space with precompact polytopes,*

(2) *$A_\alpha(x) \subseteq \mathrm{cl}\, B_\alpha(x)$ for each $x \in X$,*

(3) *both $\mathrm{cl}\, B_\alpha$ and F_α are Q_α-condensing mappings in $\mathbb{H}(X, X_\alpha)$,*

(4) *$x_\alpha \notin \mathrm{cl}(H\text{-co}P_\alpha(x, y))$ for each $x, y \in X$,*

(5) *$A_\alpha \cap (H\text{-co}P_\alpha)$ is transfer open inverse valued in X_α,*

(6) *$W_\alpha := \{(x, y) \in X \times X \mid A_\alpha(x) \cap (H\text{-co}P_\alpha(x, y)) \neq \emptyset\}$ is paracompact.*

Then $\mathbf{\Omega}$ has an equilibrium point $(\hat{x}, \hat{y}) \in X \times X$.

Proof. For each $\alpha \in I$, we define $\phi_\alpha : X \times X \to 2^{X_\alpha}$ by

$$\phi_\alpha(x, y) := A_\alpha(x) \cap (H\text{-co}P_\alpha(x, y)), \quad \forall (x, y) \in X \times X. \tag{7}$$

Assume that $W_\alpha \neq \emptyset$. Then for each $(x, y) \in W_\alpha$, we have some $y_\alpha \in \phi_\alpha(x, y)$. Equivalently, $(x, y) \in \phi_\alpha^{-1}(y_\alpha)$. It follows that $W_\alpha = \bigcup_{y_\alpha \in X_\alpha} \phi_\alpha^{-1}(y_\alpha)$. Since each ϕ_α is transfer open inverse valued in X_α by (5), it follows from Lemma 3 that W_α is open in $X \times X$.

For $z_\alpha \in X_\alpha$, if $(x, y) \in \phi_\alpha^{-1}(z_\alpha)$, by using (5), we have some $z'_\alpha \in X_\alpha$ such that $(x, y) \in \mathrm{int}\, \phi_\alpha^{-1}(z'_\alpha) \subseteq W_\alpha$. Thus, the restriction $\phi_\alpha|_{W_\alpha} : W_\alpha \to 2^{X_\alpha}$ is transfer open inverse valued in X_α. Moreover, by (3), each $\phi_\alpha|_{W_\alpha}(x, y)$ is nonempty and H-convex. Therefore, by Lemma 4, there exists a continuous function $f_\alpha : W_\alpha \to X_\alpha$ such that $f_\alpha(x, y) \in \phi_\alpha|_{W_\alpha}(x, y)$ for each $(x, y) \in W_\alpha$.

Since $\mathrm{cl}\, B_\alpha$ and F_α are Q_α-condensing, applying Theorem A, we have two nonempty compact H-convex subsets $K := \prod_{\alpha \in I} K_\alpha$ and $K' := \prod_{\alpha \in I} K'_\alpha$ of X such that $\mathrm{cl}\, B_\alpha(K) \subseteq K_\alpha$ and $F_\alpha(K') \subseteq K'_\alpha$. Using these notations, we define a set-valued mapping $S_\alpha : K \times K' \to 2^{K_\alpha \times K'_\alpha}$ by

$$S_\alpha(x, y) = \begin{cases} \mathrm{cl}(H\text{-co}f_\alpha(x, y)) \times F_\alpha(x), \\ \quad \text{if } (x, y) \in (K \times K') \cap W_\alpha, \\ \mathrm{cl}\, B_\alpha(x) \times F_\alpha(x), \\ \quad \text{if } (x, y) \in (K \times K') \setminus W_\alpha. \end{cases} \tag{8}$$

We will show that $S_\alpha \in \mathbb{H}(K \times K', K_\alpha \times K'_\alpha)$. Let V_α be an open subset of $K_\alpha \times K'_\alpha$. Since $\mathrm{cl}[H\text{-co}f_\alpha(x, y)] \subseteq \mathrm{cl}[H\text{-co}\phi_\alpha(x, y)] \subseteq \mathrm{cl}\, B_\alpha(x)$ for each $(x, y) \in W_\alpha$, we have

$$\begin{aligned} U_\alpha &= \{(x, y) \in K \times K' \mid S_\alpha(x, y) \subseteq V_\alpha\} \\ &= \{(x, y) \in (K \times K') \cap W_\alpha \mid \mathrm{cl}[H\text{-co}f_\alpha(x, y)] \\ &\quad \times F_\alpha(x) \subseteq V_\alpha\} \\ &\cup \{(x, y) \in (K \times K') \setminus W_\alpha \mid \mathrm{cl}\, B_\alpha(x) \times F_\alpha(x) \subseteq V_\alpha\} \\ &= \{(x, y) \in (K \times K') \cap W_\alpha \mid \mathrm{cl}[H\text{-co}f_\alpha(x, y)] \\ &\quad \times F_\alpha(x) \subseteq V_\alpha\} \\ &\cup \{(x, y) \in K \times K' \mid \mathrm{cl}\, B_\alpha(x) \times F_\alpha(x) \subseteq V_\alpha\}. \end{aligned} \tag{9}$$

It follows from Lemma 2 and the upper semicontinuity of $\mathrm{cl}\, B_\alpha \times F_\alpha$ that U_α is open in $K \times K'$. Hence, S_α is *u.s.c.* Further, by (3) and Lemma 1, each $S_\alpha(x, y)$ is nonempty, closed, and H-convex. Therefore, $S_\alpha \in \mathbb{H}(K \times K', K_\alpha \times K'_\alpha)$.

Next, we define a set-valued mapping $T_\alpha : K \times K' \to 2^{K_\alpha \times K'_\alpha}$ by

$$T_\alpha(x, y) = \begin{cases} S_\alpha(x, y), & \text{if } (x, y) \in (K \times K') \cap W_\alpha, \\ \mathrm{cl}\, B_\alpha(x) \times F_\alpha(x), & \text{if } (x, y) \in (K \times K') \setminus W_\alpha. \end{cases} \tag{10}$$

Since $K_\alpha \times K'_\alpha$ is compact, each T_α is Q_α-condensing in $\mathbb{H}(K \times K', K_\alpha \times K'_\alpha)$. Hence, by Theorem B, the set-valued mapping $\prod_{\alpha \in I} T_\alpha$ has a fixed point $(\hat{x}, \hat{y}) \in K \times K'$; that is, $(\hat{x}_\alpha, \hat{y}_\alpha) \in T_\alpha(\hat{x}, \hat{y})$ for each $\alpha \in I$. If $(\hat{x}, \hat{y}) \in W_\alpha$, then

$$\begin{aligned} (\hat{x}_\alpha, \hat{y}_\alpha) &\in \mathrm{cl}(S_\alpha(\hat{x}, \hat{y})) \\ &\subseteq \mathrm{cl}(B_\alpha(x) \cap (H\text{-co}P_\alpha(\hat{x}, \hat{y}))) \times F_\alpha(\hat{x}). \end{aligned} \tag{11}$$

Thus, $\hat{x}_\alpha \in \mathrm{cl}(H\text{-co}P_\alpha(\hat{x}, \hat{y}))$, which contradicts with (4). Therefore, $(\hat{x}, \hat{y}) \notin W_\alpha$ and hence $\hat{x}_\alpha \in \mathrm{cl}\, B_\alpha(\hat{x})$, $\hat{y}_\alpha \in F_\alpha(\hat{x})$, and $A_\alpha(\hat{x}) \cap P_\alpha(\hat{x}, \hat{y}) = \emptyset$ for each $\alpha \in I$. That is, $(\hat{x}, \hat{y})$ is an equilibrium of $\mathbf{\Omega}$. □

Remark that condition (4) of Theorem 6 can be replaced by a milder condition $x_\alpha \notin \mathrm{cl}(B_\alpha(x) \cap H\text{-co}P_\alpha(x, y))$ for each $(x, y) \in W_\alpha$. Further, when each *l.c.*-space $(X_\alpha, \Gamma^\alpha)$ satisfies $\Gamma^\alpha_{\{x_\alpha\}} = \{x_\alpha\}$, condition (4) can be modified by $x_\alpha \notin H\text{-co}P_\alpha(x, y)$ without affecting the conclusion.

Corollary 7. *Let $\mathbf{\Omega} = (X_\alpha, A_\alpha, B_\alpha, F_\alpha, P_\alpha)_{\alpha \in I}$ be a generalized abstract economy, where I is a set of agents and $X = \prod_{\alpha \in I} X_\alpha$ such that for each $\alpha \in I$,*

(1) *$(X_\alpha, \Gamma^\alpha)$ is an l.c.-space with precompact polytopes, and $\Gamma^\alpha_{\{x_\alpha\}} = \{x_\alpha\}$ for each $x_\alpha \in X_\alpha$,*

(2) $A_\alpha(x) \subseteq \operatorname{cl} B_\alpha(x)$ *for each* $x \in X$,

(3) *both* $\operatorname{cl} B_\alpha$ *and* F_α *are* Q_α*-condensing mappings in* $\mathbb{H}(X, X_\alpha)$,

(4) $x_\alpha \notin H\text{-co}P_\alpha(x, y)$ *for each* $x, y \in X$,

(5) $A_\alpha \cap (H\text{-co}P_\alpha)$ *is transfer open inverse valued in* X_α,

(6) $W_\alpha := \{(x, y) \in X \times X \mid A_\alpha(x) \cap (H\text{-co}P_\alpha(x, y)) \neq \emptyset\}$ *is paracompact.*

Then $\boldsymbol{\Omega}$ *has an equilibrium point* $(\hat{x}, \hat{y}) \in X \times X$.

Proof. According to the proof of Theorem 6 and by virtue of the condition $\Gamma^\alpha_{\{x_\alpha\}} = \{x_\alpha\}$ for each $x_\alpha \in X_\alpha$, we obtain $\operatorname{cl}(H\text{-co} f_\alpha(x, y)) = f_\alpha(x, y)$. It follows that the set-valued mapping S_α can be defined by

$$S_\alpha(x, y) = \begin{cases} f_\alpha(x, y) \times F_\alpha(x), & \text{if } (x, y) \in (K \times K') \cap W_\alpha, \\ \operatorname{cl} B_\alpha(x) \times F_\alpha(x), & \text{if } (x, y) \in (K \times K') \setminus W_\alpha. \end{cases} \tag{12}$$

Thus, by an analogue proof to Theorem 6, we may conclude that $\boldsymbol{\Omega}$ has an equilibrium point. □

Following the proof of Theorem 6 by taking $\phi_\alpha(x, y) := \operatorname{cl} B_\alpha(x) \cap (H\text{-co}P_\alpha(x, y))$, we may obtain a new version of equilibrium existence theorem as follows.

Corollary 8. *Let* $\boldsymbol{\Omega} = (X_\alpha, A_\alpha, B_\alpha, F_\alpha, P_\alpha)_{\alpha \in I}$ *be a generalized abstract economy, where* I *is a set of agents and* $X = \prod_{\alpha \in I} X_\alpha$ *such that for each* $\alpha \in I$,

(1) X_α *is an l.c.-space with precompact polytopes,*

(2) $A_\alpha(x) \subseteq \operatorname{cl} B_\alpha(x)$ *for each* $x \in X$,

(3) *both* $\operatorname{cl} B_\alpha$ *and* F_α *are* Q_α*-condensing mappings in* $\mathbb{H}(X, X_\alpha)$,

(4) $x_\alpha \notin \operatorname{cl}(H\text{-co}P_\alpha(x, y))$ *for each* $x, y \in X$,

(5) $\operatorname{cl} B_\alpha \cap (H\text{-co}P_\alpha)$ *is transfer open inverse valued in* X_α,

(6) $W_\alpha := \{(x, y) \in X \times X \mid \operatorname{cl} B_\alpha(x) \cap (H\text{-co}P_\alpha(x, y)) \neq \emptyset\}$ *is paracompact.*

Then $\boldsymbol{\Omega}$ *has an equilibrium point* $(\hat{x}, \hat{y}) \in X \times X$.

Notice that Theorem 6 generalizes [7, Kim-Tan, Theorem 2], in which they deal with the case of locally convex topological vector spaces under some compactness conditions, and it also improves [19, Wu-Yuan, Theorem 3] in the setting of locally H-convex spaces. We also note that if X is metrizable, the set W_α is also metrizable and hence is paracompact. Therefore, the assumption (6) of Theorem 6 is automatically satisfied. Furthermore, if each X_α is compact, then both $\operatorname{cl} B_\alpha$ and F_α are obviously Q_α-condensing. Thus, we have an immediate consequence, which is a generalization of [7, Kim-Tan, Corollary 1] to H-spaces.

Corollary 9. *Let* $\boldsymbol{\Omega} = (X_\alpha, A_\alpha, B_\alpha, F_\alpha, P_\alpha)_{\alpha \in I}$ *be a generalized abstract economy, where* I *is a set of agents such that for each* $\alpha \in I$,

(1) $(X_\alpha, \Gamma^\alpha)$ *is a metrizable compact l.c.-space, and* $\Gamma^\alpha_{\{x_\alpha\}} = \{x_\alpha\}$ *for each* $x_\alpha \in X_\alpha$,

(2) $A_\alpha(x) \subseteq \operatorname{cl} B_\alpha(x)$ *for each* $x \in X$,

(3) $\operatorname{cl} B_\alpha \in \mathbb{H}(X, X_\alpha)$, *and* $F_\alpha \in \mathbb{H}(X, X_\alpha)$,

(4) $x_\alpha \notin H\text{-co}P_\alpha(x, y)$ *for each* $x, y \in X$,

(5) $A_\alpha \cap (H\text{-co}P_\alpha)$ *is transfer open inverse valued in* X_α.

Then $\boldsymbol{\Omega}$ *has an equilibrium point* $(\hat{x}, \hat{y}) \in X \times X$.

We note that our main results focus on the setting of general *l.c.*-spaces without any linear structure; further, the correspondences are not necessarily lower semicontinuous and do not require the usual open lower section assumption, such as the earlier works [3, Theorem 4], [13, Theorem 3 and its Corollary], [19, Theorems 1 and 3], and [18, Theorem 6.1]. In fact, we can give a simple example applicable for Corollary 9, while previous results do not.

Example 10. Consider the set I of agents is singleton. Let $X = [0, 1]$ and the correspondences $A, B, F : X \to 2^X$ be defined by $A(x) = B(x) = [0, 1]$, and $F(x) = \{x\}$ for each $x \in X$. The preference correspondence $P : X \times X \to 2^X$ is defined as follows:

$$P(x_1, x_2) = \begin{cases} \left(\dfrac{x_1 + x_2}{2}, 1\right], & \text{if } x_1 < x_2,\ x_1, x_2 \in Q, \\ \left(\dfrac{x_1 + 2x_2}{3}, 1\right], & \text{if } x_1 < x_2,\ x_1 \notin Q, \\ & \text{or } x_2 \notin Q, \\ \emptyset, & \text{if } x_1 = x_2, \\ \{0\}, & \text{if } x_1 > x_2. \end{cases} \tag{13}$$

Then $A \cap (H\text{-co}P) = A \cap P$ is transfer open inverse valued in X. Indeed, $(A \cap P)^{-1}(0) = P^{-1}(0) = \{(x_1, x_2) \mid x_1 > x_2\}$ is open in $X \times X$, and for any $t \in (0, 1]$ and $(x_1, x_2) \in (A \cap P)^{-1}(t) = P^{-1}(t)$, we always have $(x_1, x_2) \in \operatorname{int} P^{-1}(1) = \operatorname{int}(A \cap P)^{-1}(1)$. However, the lower section $(A \cap P)^{-1}(1/2)$ is not open. Indeed, let $a_n = (1/2) - (1/n\sqrt{2})$ and let $b_n = 9/10$; then $(a_n, b_n) \in X \times X \setminus (A \cap P)^{-1}(1/2)$ and (a_n, b_n) converges to $(1/2, 9/10)$, which does not belong to $X \times X \setminus (A \cap P)^{-1}(1/2)$. This means that the set $X \times X \setminus (A \cap P)^{-1}(1/2)$ is not closed, and hence $(A \cap P)^{-1}(1/2)$ is not open. Further, for each $x_1, x_2 \in X$, $x_1 \notin P(x_1, x_2) = H\text{-co}P(x_1, x_2)$. Thus, all hypotheses of Corollary 9 are satisfied so that the generalized abstract economy $\boldsymbol{\Omega}$ has an equilibrium point in $X \times X$. In fact, all the equilibria of $\boldsymbol{\Omega}$ are the points (a, a), where $a \in [0, 1]$.

Let X and Y be two topological spaces. Given three set-valued mappings $T : X \to 2^Y$, $F : X \to 2^X$, $A : X \to 2^X$, and a function $\phi : X \times X \times Y \to \mathbb{R}$, a *generalized quasi-variational inequality* is defined as follows:

$$(\mathbf{GQVI}) \begin{cases} \text{Find } (\hat{x}, \hat{w}, \hat{y}) \in X \times X \times Y \\ \quad \text{such that } \hat{x} \in \mathrm{cl}A(\hat{x}), \ \hat{w} \in F(\hat{x}), \\ \quad \hat{y} \in T(\hat{x}), \\ \phi(z, \hat{x}, \hat{y}) \geq 0, \\ \quad \forall z \in A(\hat{x}) \cap \left(F^{-1}(\hat{w})\right)^C. \end{cases} \tag{14}$$

In particular, if $F(x) = \{x\}$ for each $x \in X$, then $(F^{-1}(w))^C = \{w\}^C = X \setminus \{w\}$. Therefore, the (**GQVI**) reduces to the usual quasi-variational inequality as follows:

$$(\mathbf{QVI}) \begin{cases} \text{Find } (\hat{x}, \hat{y}) \in X \times Y \\ \quad \text{such that } \hat{x} \in \mathrm{cl}\, A(\hat{x}), \quad \hat{y} \in T(\hat{x}), \\ \phi(z, \hat{x}, \hat{y}) \geq 0, \quad \forall z \in A(\hat{x}) \setminus \{\hat{x}\}. \end{cases} \tag{15}$$

Theorem 11. *Let (X, Γ) be an l.c.-space with precompact polytopes, $\Gamma_{\{x\}} = \{x\}$ for each $x \in X$, and let Y be a topological space. The set-valued mappings $T : X \to 2^Y$ and $F, A : X \to 2^X$ satisfy $T \in H(X, Y)$, $F \in H(X, X)$, and $\mathrm{cl}\, A \in H(X, X)$, and $A^{-1}(x)$ is open for all $x \in X$. Suppose that $\phi : X \times X \times Y \to \mathbb{R}$ is a function such that*

(1) *$\phi_\alpha(x, x, y) \geq 0$ for all $x \in X$ and $y \in T(x)$,*

(2) *for each fixed $z \in X$, the mapping $(x, y) \mapsto \phi(z, x, y)$ is lower semicontinuous,*

(3) *for each fixed $(x, y) \in X \times Y$, the mapping $z \mapsto \phi(z, x, y)$ is H-quasiconvex in the following sense that for any finite set D in X,*

$$\phi(z, x, y) \leq \max_{u \in D} \phi(u, x, y), \quad \forall z \in H\text{-co}D. \tag{16}$$

*Then there is a solution to (**GQVI**).*

Proof. Define a set-valued mapping $P : X \times X \to 2^X$ by

$$P(x, w) := \left\{z \in X \mid \inf_{y \in T(x)} \phi(z, x, y) < 0\right\} \cap \left(F^{-1}(w)\right)^C, \quad \forall (x, w) \in X \times X. \tag{17}$$

By [20, Proposition 23, page 121], for each fixed $z \in X$, the mapping $x \mapsto \inf_{y \in T(x)} \phi(z, x, y)$ is lower semicontinuous. Thus, the set $\{x \in X \mid \inf_{y \in T(x)} \phi(z, x, y) > 0\}$ is open for each $z \in X$. It follows that

$$P^{-1}(z) = \left(\left\{x \in X \mid \inf_{y \in T(x)} \phi(z, x, y) > 0\right\} \times X\right) \cap \left(X \times (T(z))^C\right) \tag{18}$$

is open. By Lemma 5, $(H\text{-co}P)^{-1}(z)$ is also open. Next, we show that $x \notin H\text{-co}P(x, w)$ for all $x, w \in X$. Assume that there are x_0 and w_0 satisfying $x_0 \in H\text{-co}P(x_0, w_0)$. Then there is a finite subset D of $P(x_0, w_0)$ such that $x_0 \in H\text{-co}D$. For each fixed $y \in T(x_0)$, since the mapping $z \mapsto \phi(z, x_0, y)$ is H-quasiconvex, it follows that

$$0 \leq \inf_{y \in T(x_0)} \phi(x_0, x_0, y) \leq \inf_{y \in T(x_0)} \max_{z \in D} \phi(z, x_0, y). \tag{19}$$

By Kneser's minimax theorem [21], together with $z \in P(x_0, w_0)$ for all $z \in D$, we have

$$\inf_{y \in T(x_0)} \max_{z \in D} \phi(z, x_0, y) = \max_{z \in D} \inf_{y \in T(x_0)} \phi(z, x_0, y) < 0. \tag{20}$$

This is a contradiction. Thus, all hypotheses of Corollary 7 are satisfied. Therefore, there exist $\hat{x}, \hat{w} \in X$ such that $\hat{x} \in \mathrm{cl}\, A(\hat{x})$, $\hat{w} \in F(\hat{x})$, and $A(\hat{x}) \cap P(\hat{x}, \hat{w}) = \emptyset$, It follows that

$$\inf_{y \in T(\hat{x})} \phi(z, \hat{x}, y) \geq 0, \quad \forall z \in A(\hat{x}) \cap \left(F^{-1}(\hat{w})\right)^C. \tag{21}$$

Since $T(\hat{x})$ is compact, there is $\hat{y} \in T(\hat{x})$ such that $\phi(z, \hat{x}, \hat{y}) \geq 0$ for all $z \in A(\hat{x}) \cap (F^{-1}(\hat{w}))^C$. That is, $(\hat{x}, \hat{w}, \hat{y})$ is a solution to (**GQVI**). □

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