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Research Article

Extremal Solutions and Relaxation Problems for Fractional Differential Inclusions

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We present the existence of extremal solution and relaxation problem for fractional differential inclusion with initial conditions.

1. Introduction

Differential equations with fractional order have recently proved to be valuable tools in the modeling of many physical phenomena [1–9]. There has also been a significant theoretical development in fractional differential equations in recent years; see the monographs of Kilbas et al. [10], Miller and Ross [11], Podlubny [12], and Samko et al. [13] and the papers of Kilbas and Trujillo [14], Nahušev [15], Podlubny et al. [16], and Yu and Gao [17].

Recently, some basic theory for initial value problems for fractional differential equations and inclusions involving the Riemann-Liouville differential operator was discussed, for example, by Lakshmikantham [18] and Chalco-Cano et al. [19].

Applied problems requiring definitions of fractional derivatives are those that are physically interpretable for initial conditions containing $y(0)$, $y'(0)$, and so forth. The same requirements are true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both Riemann-Liouville and Caputo types, see Podlubny [12].

Fractional calculus has a long history. We refer the reader to [20].

Recently fractional functional differential equations and inclusions and impulsive fractional differential equations and inclusions with standard Riemann-Liouville and Caputo derivatives with differences conditions were studied by Abbas et al. [21, 22], Benchohra et al. [23], Henderson and Ouahab [24, 25], Jiao and Zhou [26], and Ouahab [27–29] and in the references therein.

In this paper, we will be concerned with the existence of solutions, Filippov's theorem, and the relaxation theorem of abstract fractional differential inclusions. More precisely, we will consider the following problem:

$$\begin{gathered} {}^{c}D^{\alpha} y(t) \in F(t, y(t)), \quad \text{a.e. } t \in J := [0, b], \\ y(0) = y_0, \qquad y'(0) = y_1, \end{gathered} \tag{1}$$

$$\begin{gathered} {}^{c}D^{\alpha} y(t) \in \operatorname{ext} F(t, y(t)), \quad \text{a.e. } t \in J := [0, b], \\ y(0) = y_0, \qquad y'(0) = y_1, \end{gathered} \tag{2}$$

where ${}^{c}D^{\alpha}$ is the Caputo fractional derivatives, $\alpha \in (1, 2]$, $F : J \times \mathbb{R}^N \to \mathscr{P}(\mathbb{R}^N)$ is a multifunction, and $\operatorname{ext} F(t, y)$ represents the set of extreme points of $F(t, y)$. ($\mathscr{P}(\mathbb{R}^N)$ is the family of all nonempty subsets of $\mathbb{R}^N$.

During the last couple of years, the existence of extremal solutions and relaxation problem for ordinary differential inclusions was studied by many authors, for example, see [30–34] and the references therein.

The paper is organized as follows. We first collect some background material and basic results from multivalued analysis and give some results on fractional calculus in Sections 2 and 3, respectively. Then, we will be concerned with the existence of solution for extremal problem. This is the aim of Section 4. In Section 5, we prove the relaxation problem.

2. Preliminaries

The reader is assumed to be familiar with the theory of multivalued analysis and differential inclusions in Banach spaces, as presented in Aubin et al. [35, 36], Hu and Papageorgiou [37], Kisielewicz [38], and Tolstonogov [32].

Let $(X, \|\cdot\|)$ be a real Banach space, $[0,b]$ an interval in R, and $C([0,b],X)$ the Banach space of all continuous functions from J into X with the norm

$$\|y\|_{\infty} = \sup\{\|y(t)\| : 0 \le t \le b\}. \tag{3}$$

A measurable function $y : [0,b] \to X$ is Bochner integrable if $\|y\|$ is Lebesgue integrable. In what follows, $L^1([0,b],X)$ denotes the Banach space of functions $y : [0,b] \to X$, which are Bochner integrable with norm

$$\|y\|_1 = \int_0^b \|y(t)\|\, dt. \tag{4}$$

Denote by $L^1_w([0,b],X)$ the space of equivalence classes of Bochner integrable function $y : [0,b] \to X$ with the norm

$$\|y\|_w = \sup_{t\in[0,t]} \left\| \int_0^t y(s)\, ds \right\|. \tag{5}$$

The norm $\|\cdot\|_w$ is weaker than the usual norm $\|\cdot\|_1$, and for a broad class of subsets of $L^1([0,b],X)$, the topology defined by the weak norm coincides with the usual weak topology (see [37, Proposition 4.14, page 195]). Denote by

$$\begin{gathered} \mathscr{P}(X) = \{Y \subset X : Y \neq \emptyset\}, \\ \mathscr{P}_{\mathrm{cl}}(X) = \{Y \in \mathscr{P}(X) : Y \text{ closed}\}, \\ \mathscr{P}_b(X) = \{Y \in \mathscr{P}(X) : Y \text{ bounded}\}, \\ \mathscr{P}_{\mathrm{cv}}(X) = \{Y \in \mathscr{P}(X) : Y \text{ convex}\}, \\ \mathscr{P}_{\mathrm{cp}}(X) = \{Y \in \mathscr{P}(X) : Y \text{ compact}\}. \end{gathered} \tag{6}$$

A multivalued map $G : X \to \mathscr{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X (i.e., $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

Definition 1. A multifunction $F : X \to \mathscr{P}(Y)$ is said to be upper semicontinuous at the point $x_0 \in X$, if, for every open $W \subseteq Y$ such that $F(x_0) \subset W$, there exists a neighborhood $V(x_0)$ of x_0 such that $F(V(x_0)) \subset W$.

A multifunction is called *upper semicontinuous* (*u.s.c.* for short) on X if for each $x \in X$ it is u.s.c. at x.

Definition 2. A multifunction $F : X \to \mathscr{P}(Y)$ is said to be lower continuous at the point $x_0 \in X$, if, for every open $W \subseteq Y$ such that $F(x_0) \cap W \neq \emptyset$, there exists a neighborhood $V(x_0)$ of x_0 with property that $F(x) \cap W \neq \emptyset$ for all $x \in V(x_0)$.

A multifunction is called *lower semicontinuous* (*l.s.c.* for short) provided that it is lower semicontinuous at every point $x \in X$.

Lemma 3 (see [39, Lemma 3.2]). *Let $F : [0,b] \to \mathscr{P}(Y)$ be a measurable multivalued map and $u : [a,b] \to Y$ a measurable function. Then for any measurable $v : [a,b] \to (0,+\infty)$, there exists a measurable selection f_v of F such that for a.e. $t \in [a,b]$,*

$$\|u(t) - f_v(t)\| \le d(u(t), F(t)) + v(t). \tag{7}$$

First, consider the Hausdorff pseudometric

$$H_d : \mathscr{P}(E) \times \mathscr{P}(E) \longrightarrow \mathbb{R}^+ \cup \{\infty\}, \tag{8}$$

defined by

$$H_d(A,B) = \max\left\{ \sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b) \right\}, \tag{9}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. $(\mathscr{P}_{b,\mathrm{cl}}(E), H_d)$ is a metric space and $(\mathscr{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space.

Definition 4. A multifunction $F : Y \to \mathscr{P}(X)$ is called Hausdorff lower semicontinuous at the point $y_0 \in Y$, if for any $\epsilon > 0$ there exists a neighbourhood $U(y_0)$ of the point y_0 such that

$$F(y_0) \subset F(y) + \epsilon B(0,1), \quad \text{for every } y \in U(y_0), \tag{10}$$

where $B(0,1)$ is the unite ball in X.

Definition 5. A multifunction $F : Y \to \mathscr{P}(X)$ is called Hausdorff upper semicontinuous at the point $y_0 \in Y$, if for any $\epsilon > 0$ there exists a neighbourhood $U(y_0)$ of the point y_0 such that

$$F(y) \subset F(y_0) + \epsilon B(0,1), \quad \text{for every } y \in U(y_0). \tag{11}$$

F is called continuous, if it is Hausdorff lower and upper semicontinuous.

Definition 6. Let X be a Banach space; a subset $A \subset L^1([0,b],X)$ is decomposable if, for all $u, v \in A$ and for every Lebesgue measurable set $I \subset J$, one has

$$u\chi_I + v\chi_{[0,b]\setminus I} \in A, \tag{12}$$

where χ_A stands for the characteristic function of the set A. We denote by $\mathrm{Dco}(L^1([0,b],X))$ the family of decomposable sets.

Let $F : [0,b] \times X \to \mathcal{P}(X)$ be a multivalued map with nonempty closed values. Assign to F the multivalued operator $\mathcal{F} : C([0,b],X) \to \mathcal{P}(L^1([0,b],X))$ defined by

$$\mathcal{F}(y) = \left\{ v \in L^1([0,b],X) : v(t) \in F(t, y(t)), \right. \\ \left. \text{a.e. } t \in [0,b] \right\}. \tag{13}$$

The operator $\mathcal{F}$ is called the Nemyts'kiĭ operator associated to F.

Definition 7. Let $F : [0,b] \times X \to \mathcal{P}(X)$ be a multivalued map with nonempty compact values. We say that F is of lower semicontinuous type (l.s.c. type) if its associated Nemyts'kiĭ operator $\mathcal{F}$ is lower semicontinuous and has nonempty closed and decomposable values.

Next, we state a classical selection theorem due to Bressan and Colombo.

Lemma 8 (see [40]). *Let X be a separable metric space and let E be a Banach space. Then every l.s.c. multivalued operator $N : X \to \mathcal{P}_{cl}(L^1([0,b],E))$ with closed decomposable values has a continuous selection; that is, there exists a continuous single-valued function $f : X \to L^1([0,b],E)$ such that $f(x) \in N(x)$ for every $x \in X$.*

Let us introduce the following hypothesis.

$(\overline{\mathcal{H}}_1)$ $F : [0,b]\times X \to \mathcal{P}(X)$ is a nonempty compact valued multivalued map such that

(a) the mapping $(t,y) \mapsto F(t,y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
(b) the mapping $y \mapsto F(t,y)$ is lower semicontinuous for a.e. $t \in [0,b]$.

Lemma 9 (see, e.g., [41]). *Let $F : J \times X \to \mathcal{P}_{cp}(E)$ be an integrably bounded multivalued map satisfying $(\overline{\mathcal{H}}_1)$. Then F is of lower semicontinuous type.*

Define

$$F(K) = \left\{ f \in L^1([0,b],X) : f(t) \in K \text{ a.e. } t \in [0,b] \right\}, \\ K \subset X, \tag{14}$$

where X is a Banach space.

Lemma 10 (see [37]). *Let $K \subset X$ be a weakly compact subset of X. Then $F(K)$ is relatively weakly compact subset of $L^1([0,b],X)$. Moreover if K is convex, then $F(K)$ is weakly compact in $L^1([0,b],X)$.*

Definition 11. A multifunction $F : [0,b] \times Y \to \mathcal{P}_{wcpcv}(X)$ possesses the Scorza-Dragoni property (S-D property) if for each $\epsilon > 0$, there exists a closed set $J_\epsilon \subset [0,b]$ whose Lebesgue measure $\mu(J_\epsilon) \le \epsilon$ and such that $F : [0,b] \setminus J_\epsilon \times Y \to X$ is continuous with respect to the metric $d_X(\cdot,\cdot)$.

Remark 12. It is well known that if the map $F : [0,b] \times Y \to \mathcal{P}_{wcpcv}(X)$ is continuous with respect to y for almost every $t \in [0,b]$ and is measurable with respect to t for every $y \in Y$, then it possesses the S-D property.

In what follows, we present some definitions and properties of extreme points.

Definition 13. Let A be a nonempty subset of a real or complex linear vector space. An extreme point of a convex set A is a point $x \in A$ with the property that if $x = \lambda y + (1-\lambda)z$ with $y, z \in A$ and $\lambda \in [0,1]$, then $y = x$ and/or $z = x$. $\operatorname{ext}(A)$ will denote the set of extreme points of A.

In other words, an extreme point is a point that is not an interior point of any line segment lying entirely in A.

Lemma 14 (see [42]). *A nonempty compact set in a locally convex linear topological space has extremal points.*

Let $\{x'_n\}_{n\in\mathbb{N}}$ be a denumerable, dense (in $\sigma(X',X)$ topology) subset of the set $\{x \in X : \|x\| \le 1\}$. For any $A \in \mathcal{P}_{cpcv}(X)$ and x'_n define the function

$$d^n(A,u) = \max\left\{ \left\langle y - z, x'_n \right\rangle : y, z \in A,\ u = \frac{y+z}{2} \right\}. \tag{15}$$

Lemma 15 (see [33]). *$u \in \operatorname{ext}(A)$ if and only if $d^n(A,u) = 0$ for all $n \ge 1$.*

In accordance with Krein-Milman and Trojansky theorem [43], the set $\operatorname{ext}(S_F)$ is nonempty and $\overline{\operatorname{co}}(\operatorname{ext}(S_F)) = S_F$.

Lemma 16 (see [33]). *Let $F : [0,b] \to \mathcal{P}_{wcpcv}(X)$ be a measurable, integrably bounded map. Then*

$$\overline{\operatorname{ext}}(S_F) \subseteq S_F, \tag{16}$$

where $\overline{\operatorname{ext}}(S_F)$ is the closure of set $\operatorname{ext}(S_F)$ in the topology of the space $L^1([0,b],X)$.

Theorem 17 (see [33]). *Let $F : [0,b] \times Y \to \mathcal{P}_{wcpcv}(X)$ be a multivalued map that has the S-D property and let it be integrable bounded on compacts from Y. Consider a compact subset $K \subset C([0,b],X)$ and define the multivalued map $G : K \to L^1([0,b],X)$, by*

$$G(y(\cdot)) \\ = \left\{ f \in L^1([0,b],X) : f(t) \in F(t, y(t)) \text{ a.e. on } [0,b] \right\}, \\ y \in K. \tag{17}$$

Then for every K compact in $C([0,b],X)$, $\epsilon > 0$ and any continuous selection $f : K \to L^1([0,b],X)$, there exists a continuous selector $g : K \to L^1([0,b],X)$ of the map $\operatorname{ext}(G) : K \to L^1([0,b],X)$ such that for all $y \in C([0,b],X)$ one has

$$\sup_{t\in[0,b]} \left\| \int_0^t \left((fy)(s) - (gy)(s) \right) ds \right\| \le \epsilon. \tag{18}$$

For a background of extreme point of $F(t, y(t))$ see Dunford-Schwartz [42, Chapter 5, Section 8] and Florenzano and Le Van [44, Chapter 3].

3. Fractional Calculus

According to the Riemann-Liouville approach to fractional calculus, the notation of fractional integral of order α ($\alpha > 0$) is a natural consequence of the well known formula (usually attributed to Cauchy) that reduces the calculation of the n-fold primitive of a function $f(t)$ to a single integral of convolution type. In our notation the Cauchy formula reads

$$I^n f(t) := \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)\,ds, \quad t > 0,\ n \in \mathbb{N}. \tag{19}$$

Definition 18 (see [13, 45]). The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b],\mathbb{R})$ is defined by

$$I_{a^+}^{\alpha} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds, \tag{20}$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $\phi_\alpha(t) = t^{(\alpha-1)}/\Gamma(\alpha)$ for $t > 0$, and we write $\phi_\alpha(t) = 0$ for $t \leq 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function and Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\,dt, \quad \alpha > 0. \tag{21}$$

For consistency, $I^0 = \mathrm{Id}$ (identity operator), that is, $I^0 f(t) = f(t)$. Furthermore, by $I^\alpha f(0^+)$ we mean the limit (if it exists) of $I^\alpha f(t)$ for $t \to 0^+$; this limit may be infinite.

After the notion of fractional integral, that of fractional derivative of order α ($\alpha > 0$) becomes a natural requirement and one is attempted to substitute α with $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integral and preserve the well known properties of the ordinary derivative of integer order. Denoting by D^n, with $n \in \mathbb{N}$, the operator of the derivative of order n, we first note that

$$D^n I^n = \mathrm{Id}, \quad I^n D^n \neq \mathrm{Id}, \quad n \in \mathbb{N}, \tag{22}$$

that is, D^n is the left inverse (and not the right inverse) to the corresponding integral operator J^n. We can easily prove that

$$I^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a^+)\frac{(t-a)^k}{k!}, \quad t > 0. \tag{23}$$

As a consequence, we expect that D^α is defined as the left inverse to I^α. For this purpose, introducing the positive integer n such that $n-1 < \alpha \leq n$, one defines the fractional derivative of order $\alpha > 0$.

Definition 19. For a function f given on interval $[a,b]$, the αth Riemann-Liouville fractional-order derivative of f is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{-\alpha+n-1} f(s)\,ds, \tag{24}$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

Defining for consistency, $D^0 = I^0 = \mathrm{Id}$, then we easily recognize that

$$D^\alpha I^\alpha = \mathrm{Id}, \quad \alpha \geq 0, \tag{25}$$

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \qquad \alpha > 0, \quad \gamma \in (-1,0)\cup(0,+\infty), \quad t > 0. \tag{26}$$

Of course, properties (25) and (26) are a natural generalization of those known when the order is a positive integer.

Note the remarkable fact that the fractional derivative $D^\alpha f$ is not zero for the constant function $f(t) = 1$, if $\alpha \notin \mathbb{N}$. In fact, (26) with $\gamma = 0$ illustrates that

$$D^\alpha 1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha > 0,\ t > 0. \tag{27}$$

It is clear that $D^\alpha 1 = 0$, for $\alpha \in \mathbb{N}$, due to the poles of the gamma function at the points $0, -1, -2, \ldots$.

We now observe an alternative definition of fractional derivative, originally introduced by Caputo [46, 47] in the late sixties and adopted by Caputo and Mainardi [48] in the framework of the theory of Linear Viscoelasticity (see a review in [4]).

Definition 20. Let $f \in AC^n([a,b])$. The Caputo fractional-order derivative of f is defined by

$$\left({}^cD^\alpha f\right)(t) := \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1} f^n(s)\,ds. \tag{28}$$

This definition is of course more restrictive than Riemann-Liouville definition, in that it requires the absolute integrability of the derivative of order m. Whenever we use the operator D_*^α we (tacitly) assume that this condition is met. We easily recognize that in general

$$D^\alpha f(t) := D^m I^{m-\alpha} f(t) \neq J^{m-\alpha} D^m f(t) := D_*^\alpha f(t), \tag{29}$$

unless the function $f(t)$, along with its first $n-1$ derivatives, vanishes at $t = a^+$. In fact, assuming that the passage of the m-derivative under the integral is legitimate, we recognize that, for $m-1 < \alpha < m$ and $t > 0$,

$$D^\alpha f(t) = {}^cD^\alpha f(t) + \sum_{k=0}^{m-1}\frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+), \tag{30}$$

and therefore, recalling the fractional derivative of the power function (26), one has

$$D^\alpha\left(f(t) - \sum_{k=0}^{m-1}\frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+)\right) = D_*^\alpha f(t). \tag{31}$$

The alternative definition, that is, Definition 20, for the fractional derivative thus incorporates the initial values of the function and of lower order. The subtraction of the Taylor polynomial of degree $n-1$ at $t=a^+$ from $f(t)$ means a sort of regularization of the fractional derivative. In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero:

$$^cD^\alpha 1 = 0, \quad \alpha > 0. \tag{32}$$

We now explore the most relevant differences between the two fractional derivatives given in Definitions 19 and 20. From Riemann-Liouville fractional derivatives, we have

$$D^\alpha (t-a)^{\alpha-j} = 0, \quad \text{for } j = 1, 2, \ldots, [\alpha]+1. \tag{33}$$

From (32) and (33) we thus recognize the following statements about functions which, for $t > 0$, admit the same fractional derivative of order α, with $n-1 < \alpha \le n$, $n \in \mathbb{N}$:

$$D^\alpha f(t) = D^\alpha g(t) \Longleftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j (t-a)^{\alpha-j},$$
$$^cD^\alpha f(t) = {}^cD^\alpha g(t) \Longleftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j (t-a)^{n-j}. \tag{34}$$

In these formulas, the coefficients c_j are arbitrary constants. For proving all main results we present the following auxiliary lemmas.

Lemma 21 (see [10]). *Let $\alpha > 0$ and let $y \in L^\infty(a,b)$ or $C([a,b])$. Then*

$$\left({}^cD^\alpha I^\alpha y\right)(t) = y(t). \tag{35}$$

Lemma 22 (see [10]). *Let $\alpha > 0$ and $n = [\alpha]+1$. If $y \in AC^n[a,b]$ or $y \in C^n[a,b]$, then*

$$\left(I^\alpha {}^cD^\alpha y\right)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k. \tag{36}$$

For further readings and details on fractional calculus, we refer to the books and papers by Kilbas [10], Podlubny [12], Samko [13], and Caputo [46–48].

4. Existence Result

Definition 23. A function $y \in C([0,b], \mathbb{R}^N)$ is called mild solution of problem (1) if there exist $f \in L^1(J, \mathbb{R}^N)$ such that

$$y(t) = y_0 + t y_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{1-\alpha} f(s)\,ds, \quad t \in [0,b], \tag{37}$$

where $f \in S_{F,y} = \{v \in L^1([0,b], \mathbb{R}^N) : f(t) \in F(t, y(t))$ a.e. on $[0,b]\}$.

We will impose the following conditions on F.

($\mathcal{H}_1$) The function $F : J \times \mathbb{R}^N \to \mathcal{P}_{cpcv}(\mathbb{R}^N)$ such that

(a) for all $x \in \mathbb{R}^N$, the map $t \mapsto F(t,x)$ is measurable,

(b) for every $t \in [0,b]$, the multivalued map $x \to F(t,x)$ is H_d continuous

($\mathcal{H}_2$) There exist $p \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\|F(t,x)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t,x)\} \le p(t)\,\psi(\|x\|),$$
$$\text{for a.e. } t \in [0,b] \text{ and each } x \in \mathbb{R}^N, \tag{38}$$

with

$$\int_0^b p(s)\,ds < \int_{\|y_0\|+b\|y_1\|}^{\infty} \frac{du}{\psi(u)}. \tag{39}$$

Theorem 24. *Assume that the conditions ($\mathcal{H}_1$)-($\mathcal{H}_2$) and then the problem (2) have at least one solution.*

Proof. From ($\mathcal{H}_2$) there exists $M > 0$ such that $\|y\|_\infty \le M$ for each $y \in S_c$.

Let

$$F_1(t,y) = \begin{cases} F(t,y) & \text{if } \|y\| \le M, \\ F\left(t, \dfrac{My}{\|y\|}\right) & \text{if } \|y\| \ge M. \end{cases} \tag{40}$$

We consider

$$^cD^\alpha y(t) \in F_1(t, y(t)), \quad \text{a.e. } t \in [0,b],$$
$$y(0) = y_0, \qquad y'(0) = y_1. \tag{41}$$

It is clear that all the solutions of (41) are solutions of (2).

Set

$$V = \left\{ f \in L^1\left([0,b], \mathbb{R}^N\right) : \|f(t)\| \le \psi_*(t) \right\},$$
$$\psi_*(t) = p(t)\,\psi(M). \tag{42}$$

It is clear that V is weakly compact in $L^1([0,b], \mathbb{R}^N)$. Remark that for every $f \in V$, there exists a unique solution $L(f)$ of the following problem:

$$^cD^\alpha y(t) = f(t), \quad \text{a.e. } t \in [0,b],$$
$$y(t) = y_0, \qquad y'(0) = y_1; \tag{43}$$

this solution is defined by

$$L(f)(t) = y_0 + t y_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$
$$\text{a.e. } t \in [0,b]. \tag{44}$$

We claim that L is continuous. Indeed, let $f_n \to f$ converge in $L^1([0,b], \mathbb{R}^N)$, as $n \to \infty$, set $y_n = L(f_n)$, $n \in \mathbb{N}$. It is clear that $\{y_n : n \in \mathbb{N}\}$ is relatively compact in $C([0,b],\mathbb{R}^N)$ and y_n converge to $y \in C([0,b],\mathbb{R}^N)$. Let

$$z(t) = y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t \in [0,b]. \tag{45}$$

Then

$$\|y_n - z\|_\infty \le \frac{b^\alpha}{\Gamma(\alpha)} \int_0^b \|f_n(s) - f(s)\|\, ds \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{46}$$

Hence $K = L(V)$ is compact and convex subset of $C([0,b],\mathbb{R}^N)$. Let $S_F : K \to \mathcal{P}_{\mathrm{clcv}}(L^1([0,b],\mathbb{R}^N))$ be the multivalued Nemitsky operator defined by

$$S_{F_1}(y) = \{f \in L^1([0,b],\mathbb{R}^N) : f(t) \in F_1(t, y(t)), \text{ a.e. } t \in [0,b]\} := S_{F_1,y}. \tag{47}$$

It is clear that $F_1(\cdot,\cdot)$ is H_d continuous and $F_1(\cdot,\cdot) \in \mathcal{P}_{wk\mathrm{cpcv}}(\mathbb{R}^N)$ and is integrably bounded, then by Theorem 17 (see also Theorem 6.5 in [32] or Theorem 1.1 in [34]), we can find a continuous function $g : K \to L^1_w([0,b],\mathbb{R}^N)$ such that

$$g(x) \in \operatorname{ext} S_{F_1}(y) \quad \forall y \in K. \tag{48}$$

From Benamara [49] we know that

$$\operatorname{ext} S_{F_1}(y) = S_{\operatorname{ext} F_1(\cdot, y(\cdot))} \quad \forall y \in K. \tag{49}$$

Setting $N = L \circ g$ and letting $y \in K$, then

$$g(y) \in F_1(\cdot, y(\cdot)) \Longrightarrow g(y) \in V \Longrightarrow N(y) = L(g(y)) \in K. \tag{50}$$

Now, we prove that N is continuous. Indeed, let $y_n \in K$ converge to y in $C([0,b],\mathbb{R}^N)$.

Then

$$g(y_n) \text{ converge weakly to } g(y) \quad \text{as } n \longrightarrow \infty. \tag{51}$$

Since $N(y_n) = L(g(y_n)) \in K$ and $g(y_n)(\cdot) \in F(t, y_n(t))$, then

$$g(y_n)(\cdot) \in F(\cdot, \overline{B}_M) \in \mathcal{P}_{\mathrm{cp}}(\mathbb{R}^N). \tag{52}$$

From Lemma 10, $g(y_n)$ converge weakly to y in $L^1([0,b],\mathbb{R}^N)$ as $n \to \infty$. By the definition of N, we have

$$\begin{aligned} N(y_n) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(y_n)(s)\, ds, \quad t \in [0,b], \\ N(y) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(y)(s)\, ds, \quad t \in [0,b]. \end{aligned} \tag{53}$$

Since $\{N(y_n) : n \in \mathbb{N}\} \subset K$, then there exists subsequence of $N(y_n)$ converge in $C([0,b],\mathbb{R}^N)$. Then

$$N(y_n)(t) \longrightarrow N(y)(t), \quad \forall t \in [0,b], \text{ as } n \longrightarrow \infty. \tag{54}$$

This proves that N is continuous. Hence by Schauder's fixed point there exists $y \in K$ such that $y = N(y)$. □

5. The Relaxed Problem

In this section, we examine whether the solutions of the extremal problem are dense in those of the convexified one. Such a result is important in optimal control theory whether the relaxed optimal state can be approximated by original states; the relaxed problems are generally much simpler to build. For the problem for first-order differential inclusions, we refer, for example, to [35, Theorem 2, page 124] or [36, Theorem 10.4.4, page 402]. For the relaxation of extremal problems we see the following recent references [30, 50].

Now we present our main result of this section.

Theorem 25. *Let $F : [0,b] \times \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ be a multifunction satisfying the following hypotheses.*

($\mathcal{H}_3$) *The function $F : [0,b] \times \mathbb{R}^N \to \mathcal{P}_{cpcv}(\mathbb{R}^N)$ such that, for all $x \in \mathbb{R}^N$, the map*

$$t \longmapsto F(t,x) \tag{55}$$

is measurable.

($\mathcal{H}_4$) *There exists $p \in L^1(J,\mathbb{R}^+)$ such that*

$$\begin{aligned} &H_d(F(t,x), F(t,y)) \le p(t)\|x - y\|, \\ &\text{for a.e. } t \in [0,b] \text{ and each } x, y \in \mathbb{R}^N, \\ &H_d(F(t,0), 0) \le p(t) \quad \text{for a.e. } t \in [0,b]. \end{aligned} \tag{56}$$

Then $\overline{S_e} = S_c$.

Proof. By Coviz and Nadlar fixed point theorem, we can easily prove that $S_c \ne \emptyset$, and since F has compact and convex valued, then S_c is compact in $C([0,b],\mathbb{R}^N)$. For more information we see [25, 27–29, 51, 52].

Let $y \in S_c$; then there exists $f \in S_{F,y}$ such that

$$y(t) = y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad \text{a.e. } t \in [0,b]. \tag{57}$$

Let K be a compact and convex set in $C([0,b],\mathbb{R}^N)$ such that $S_c \subset K$. Given that $y_* \in K$ and $\epsilon > 0$, we define the following multifunction $U_\epsilon : [0,b] \to \mathcal{P}(\mathbb{R}^N)$ by

$$U_\epsilon(t) = \{u \in \mathbb{R}^N : \|f(t) - u\| < d(f(t), F(t, y(t))) + \epsilon, \ u \in F(t, y_*(t))\}. \tag{58}$$

The multivalued map $t \to F(t,\cdot)$ is measurable and $x \to F(\cdot,x)$ is H_d continuous. In addition, if $F(\cdot,\cdot)$ has compact values, then $F(\cdot,\cdot)$ is graph measurable, and the mapping $t \to F(t,y(t))$ is a measurable multivalued map for fixed $y \in C([0,b],\mathbb{R}^N)$. Then by Lemma 3, there exists a measurable selection $v_1(t) \in F(t,y(t))$ a.e. $t \in [0,b]$ such that

$$\|f(t) - v_1(t)\| < d(f(t), F(t,y(t))) + \epsilon; \tag{59}$$

this implies that $U_\epsilon(\cdot) \neq \emptyset$. We consider $G_\epsilon : K \to \mathscr{P}(L^1(J,\mathbb{R}^N)$ defined by

$$G_\epsilon(y) = \{f_* \in \mathscr{F}(y) : \|f(t) - f_*(t)\| < \epsilon + d(f_*(t), F(t,y(t)))\}. \tag{60}$$

Since the measurable multifunction F is integrable bounded, Lemma 9 implies that the Nemyts'kiĭ operator $\mathscr{F}$ has decomposable values. Hence $y \to \overline{G_\epsilon(y)}$ is l.s.c. with decomposable values. By Lemma 8, there exists a continuous selection $f_\epsilon : C([0,b],\mathbb{R}^N) \to L^1(J,\mathbb{R}^N)$ such that

$$f_\epsilon(y) \in \overline{G_\epsilon(y)} \quad \forall y \in C([0,b],\mathbb{R}^N). \tag{61}$$

From Theorem 17, there exists function $g_\epsilon : K \to L_w([0,b],\mathbb{R}^N)$ such that

$$\begin{gathered} g_\epsilon(y) \in \operatorname{ext} S_F(y) = S_{\operatorname{ext} F(\cdot,y(\cdot))} \quad \forall y \in K, \\ \| g_\epsilon(y) - f_\epsilon(y)\|_w \le \epsilon, \quad \forall y \in K. \end{gathered} \tag{62}$$

From $(\mathscr{H}_3)$ we can prove that there exists $M > 0$ such that

$$\|y\|_\infty \le M \quad \forall y \in S_c. \tag{63}$$

Consider the sequence $\epsilon_n \to 0$, as $n \to \infty$, and set $g_n = g_{\epsilon_n}$, $f_n = f_{\epsilon_n}$. Set

$$\begin{gathered} V = \{f \in L^1([0,b],\mathbb{R}^N) : \|f(t)\| \le \psi(t) \text{ a.e. } t \in [0,b]\}, \\ \psi(t) = (1+M)p(t). \end{gathered} \tag{64}$$

Let $L : V \to C([0,b],\mathbb{R}^N)$ be the map such that each $f \in V$ assigns the unique solution of the problem

$$\begin{gathered} {}^cD^\alpha y(t) = f(t), \quad \text{a.e. } t \in [0,b], \\ y(0) = y_0, \qquad y'(0) = y_1. \end{gathered} \tag{65}$$

As in Theorem 24, we can prove that $L(V)$ is compact in $C([0,b],\mathbb{R}^N)$ and the operator $N_n = L \circ g_n : K \to K$ is compact; then by Schauder's fixed point there exists $\tilde{y}_n \in K$ such that $\tilde{y}_n \in S_e$ and

$$\tilde{y}_n(t) = y_0 + ty_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_n(y_n)(s)\,ds, \quad \text{a.e. } t \in [0,b],\ n \in \mathbb{N}. \tag{66}$$

Hence

$$\begin{aligned} &\|y(t) - \tilde{y}_n(t)\| \\ &\le \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}[g_n(\tilde{y}_n)(s) - f(s)]\,ds\right\| \\ &\le \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}[g_n(\tilde{y}_n)(s) - f_n(\tilde{y}_n)(s)]\,ds\right\| \\ &\quad + \frac{b^\alpha}{\Gamma(\alpha)}\int_0^t \|f_n(\tilde{y}_n)(s) - f(s)\|\,ds \\ &\le \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}[g_n(\tilde{y}_n)(s) - f_n(\tilde{y}_n)(s)]\,ds\right\| \\ &\quad + \frac{b^\alpha}{\Gamma(\alpha)}\int_0^t (\epsilon_n + d(f(s), f_n(\tilde{y}_n)(s)))\,ds \\ &\le \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}[g_n(\tilde{y}_n)(s) - f_n(\tilde{y}_n)(s)]\,ds\right\| \\ &\quad + \frac{b^\alpha}{\Gamma(\alpha)}\int_0^t (\epsilon_n + H_d(F(s,y(s)), F(s,\tilde{y}_n(s))))\,ds \\ &\le \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\epsilon_n + \frac{b^{\alpha+1}}{\Gamma(\alpha)}\epsilon_n + \int_0^t p(s)\|y(s) - \tilde{y}_n(s)\|. \end{aligned} \tag{67}$$

Let $\tilde{y}(\cdot)$ be a limit point of the sequence $\tilde{y}_n(\cdot)$. Then, it follows that from the above inequality, one has

$$\|y(t) - \tilde{y}(t)\| \le \int_0^t p(s)\|y(s) - \tilde{y}(s)\|\,ds, \tag{68}$$

which implies $y(\cdot) = \tilde{y}(\cdot)$. Consequently, $y \in S_c$ is a unique limit point of $\tilde{y}_n(\cdot) \in S_e$. □

Example 26. Let $F : J \times \mathbb{R}^N \to \mathscr{P}_{\mathrm{cpcv}}(\mathbb{R}^N)$ with

$$F(t,y) = \overline{B}(f_1(t,y), f_2(t,y)), \tag{69}$$

where $f_1, f_2 : J \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory functions and bounded.

Then (2) is solvable.

Example 27. If, in addition to the conditions on F of Example 26, f_1 and f_2 are Lipschitz functions, then $\overline{S_e} = S_c$.

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