*Research Article*

# First Integrals, Integrating Factors, and Invariant Solutions of the Path Equation Based on Noether and $\lambda$-Symmetries

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We analyze Noether and $\lambda$-symmetries of the path equation describing the minimum drag work. First, the partial Lagrangian for the governing equation is constructed, and then the determining equations are obtained based on the partial Lagrangian approach. For specific altitude functions, Noether symmetry classification is carried out and the first integrals, conservation laws and group invariant solutions are obtained and classified. Then, secondly, by using the mathematical relationship with Lie point symmetries we investigate $\lambda$-symmetry properties and the corresponding reduction forms, integrating factors, and first integrals for specific altitude functions of the governing equation. Furthermore, we apply the Jacobi last multiplier method as a different approach to determine the new forms of $\lambda$-symmetries. Finally, we compare the results obtained from different classifications.

## 1. Introduction

In a fluid medium, drag forces are the major sources of energy loss for moving objects. Fuel consumption may have reduced to minimize the drag work. This can be achieved by the selection of optimum path. The drag force depends on the density of fluid, the drag coefficient, the cross-sectional area, and the velocity. These parameters are the combination of the altitude-dependent parameters which can be expressed as a single arbitrary function. If all parameters are assumed to be constants, then the minimum drag work path would be a linear path. But these parameters change during the motion. And all parameters can be defined as the function of altitude [1, 2].

The main purpose of the work is to study Noether and $\lambda$-symmetry classifications of the path equation for the different forms of arbitrary function of the governing equation [3–7]. Based on Noether's theorem, if Noether symmetries of an ordinary differential equation are known, then the conservation laws of this equation can be obtained directly by using Euler-Lagrange equations [8]. However, in order to apply this theorem, a differential equation should have standard Lagrangian. Thus, an important problem in such studies is to determine the standard Lagrangian of

the differential equation. In fact, for many problems in the literature, it may not be possible to determine the Lagrangian function of the equation. To overcome this problem, partial Lagrangian method can be used alternatively and the Noether symmetries and first integrals can be obtained in spite of the fact that the differential equation does not have a standard Lagrangian [9]. Here, we examine the partial Lagrangian of path equation and classify the Noether symmetries and first integrals corresponding to special forms of arbitrary function in the governing equation.

The second type of classification that is called $\lambda$-symmetries is carried out by using the relation with Lie point symmetries as a direct method. For second-order ordinary differential equation, the method of finding $\lambda$-symmetries has been investigated extensively by Muriel and Romero [10, 11]. They have demonstrated that integrating factors and the integrals from $\lambda$-symmetries for a second-order ordinary differential equation can be determined algorithmically [12]. In their studies, for the sake of simplicity, the $\lambda$-symmetry is assumed to be a linear form as $\lambda(x, y) = \lambda_1(x, y)y' + \lambda_2(x, y)$. However, it is possible to show that the $\lambda$-symmetry cannot be chosen generally in this linear form. Therefore, we propose in this study to use the relation between Lie point symmetries and $\lambda$-symmetries for the classification.

The other classification that we discuss in our study is how to obtain $\lambda$-symmetries with the Jacobi last multiplier approach. Recently, Nucci and Levi [13] have shown that $\lambda$-symmetries and corresponding invariant solutions can be algorithmically obtained by using the Jacobi last multiplier. This new approach includes the new determining equation including $\lambda$-function that can be obtained from the divergence of the ordinary differential equation. In the $\lambda$-symmetries approach based on a new form of the prolongation formula, the determining equations are difficult to solve since they include three unknown variables to determine and then the determining equation cannot be reduced to a simpler form. However, by considering the Jacobi last multiplier approach, first we determine the $\lambda$-function, which reduces to two the number of unknown functions, and then the other functions called infinitesimals functions can be calculated easily. Taking into account these ideas we analyze $\lambda$-symmetries of the path equation for different cases of the altitude function.

The outline of this work is as follows. In the next section, we present the necessary preliminaries. In Section 3, Noether symmetries, first integrals, and some invariant solutions of path equation are obtained. In Section 4, firstly we introduce some fundamental information about $\lambda$-symmetries, integration factors, and first integrals, and then $\lambda$-symmetries corresponding to different choice of the arbitrary function are investigated. Also for some cases the reduced forms of path equation are found and the new solutions of path equation are established. Section 5 is devoted to introduce another approach that is called Jacobi last multiplier to investigate the $\lambda$-symmetries. The conclusions and results are discussed in Section 6.

## 2. Preliminaries

Let us assume that $x$ be the independent variable and $y = (y^1, \ldots, y^m)$ be the dependent variable with functions $y^\alpha$. The derivatives of $y^\alpha$ with respect to $x$ are given by

$$y_x^\alpha = y_1^\alpha = D_x(y^\alpha), \quad y_s^\alpha = D_x^s(y^\alpha),$$
$$s \geq 2, \quad \alpha = 1, 2, \ldots, m,$$
(1)

where $D_x$ is the *total derivative operator* [14–18] with respect to $x$, which can be defined as

$$D_x = \frac{\partial}{\partial x} + y_x^\alpha \frac{\partial}{\partial y^\alpha} + y_{xx}^\alpha \frac{\partial}{\partial y_x^\alpha}.$$
(2)

*Definition 1.* For each $\alpha$ we can define the operator

$$\frac{\delta}{\delta y^\alpha} = \frac{\partial}{\partial y^\alpha} + \sum_{s\geq 1}(-D_x)^s \frac{\partial}{\partial y_s^\alpha}, \quad \alpha = 1, 2, \ldots, m,$$
(3)

which is called the *Euler-Lagrange operator*.

*Definition 2.* *Generalized operator* can be formulated as

$$X = \xi \frac{\partial}{\partial x} + \eta^\alpha \frac{\partial}{\partial y^\alpha} + \sum_{s\geq 1}\zeta_s^\alpha \frac{\partial}{\partial y_s^\alpha},$$
(4)

where

$$\zeta_s^\alpha = D_x^s(W^\alpha) + \xi y_{s+1}^\alpha, \quad s \geq 2, \quad \alpha = 1, 2, \ldots, m,$$
(5)

in which $W^\alpha$ is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi y_x^\alpha, \quad \alpha = 1, 2, \ldots, m.$$
(6)

For convenience the generalized operator (4) can be rewritten by using characteristic function such as

$$X = \xi D_x + W^\alpha \frac{\partial}{\partial y^\alpha} + \sum_{s\geq 1}D_x^s(W^\alpha) \frac{\partial}{\partial y_s^\alpha},$$
(7)

and the Noether operator associated with a generalized operator $X$ can be defined

$$N = \xi + W^\alpha \frac{\partial}{\partial y^\alpha} + \sum_{s\geq 1}D_x^s(W^\alpha) \frac{\partial}{\partial y_s^\alpha}.$$
(8)

*Definition 3.* Let us consider an $n$th-order ordinary differential equation system

$$F_\alpha(x, y, y_{(1)}, y_{(2)}, \ldots, y_{(n)}) = 0, \quad \alpha = 1, 2, \ldots, m,$$
(9)

then the *first integral* of this system is a differential function (9) $I \in \mathcal{A}$, the universal space and the *vector space of all differential functions of all finite orders*, which is given by the following formula:

$$D_x I = 0,$$
(10)

and this equality is valid for every solution of (9). The first integral is also referred to as *the local conservation law*.

*Definition 4.* Let (9) be in the following form

$$F_\alpha \equiv F_\alpha^0 + F_\alpha^1 = 0, \quad \alpha = 1, 2, \ldots, m.$$
(11)

and $L = L(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(\alpha)}) \in \mathcal{A}, \alpha \leq k$ and then nonzero functions $f_\alpha^\beta \in \mathcal{A}$ satisfy the relations $\delta L/\delta u^\alpha = f_\alpha^\beta F_\beta^1, F_\beta^1 \neq 0$, in which $L$ is called *partial Lagrangian* of (11). Otherwise, $L$ is a *standard Lagrangian*.

On the other hand the *Euler-Lagrange equations* can be defined as following form

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, 2, \ldots, m,$$
(12)

and similarly the form of *partial Euler-Lagrange equations* is

$$\frac{\delta L}{\delta u^\alpha} = f_\alpha^\beta F_\beta^1.$$
(13)

*Definition 5.* Let $B \in \mathcal{A}$ be a vector that satisfies $B \neq NL + C$, where $C$ is a constant. Then $X_{(\alpha)}$ represents $\alpha$th prolongation of the generalized operator (7), and *partial Noether* operator corresponding to a partial Lagrangian is formulated as

$$X_{(\alpha)}L + LD_x(\xi) = W^\alpha \frac{\delta L}{\delta y^\alpha} + D_x(B),$$
(14)

in which $W = (W^1, \ldots, W^m), W^\alpha \in \mathcal{A}$, is the characteristic of $X$. Also $B(x, y)$ is called the *gauge* function.

*Definition 6.* If $X$ is a partial Noether operator corresponding to partial Lagrangian $L$, then the gauge function $B(x, y)$ exists. Hence, the first integral is given by

$$I = \xi L + (\eta - y'\xi) L_{y'} - B. \tag{15}$$

## 3. Noether Symmetries of Path Equation

The differential equation describing the path of the minimum drag work is given in the form

$$y'' - \frac{f'(y)}{f(y)} - y'^2 \frac{f'(y)}{f(y)} = 0, \tag{16}$$

where $y = y(x)$ is the altitude function. In this section we use partial Lagrangian approach to analyze Noether symmetries. Firstly, we can determine the *Euler-Lagrange* operator (3) for the path equation (16) such as

$$\frac{\delta}{\alpha y^{\alpha}} = \frac{\partial}{\partial y^{\alpha}} - D_x \frac{\partial}{\partial y_x} + D_x^2 \frac{\partial}{\partial y_{xx}}, \tag{17}$$

and the partial Lagrangian $L$ for the path equation (16) is

$$L = \frac{1}{2} y'^2 + \ln f(y). \tag{18}$$

Then the application of (18) to (14) and separation with respect to powers of $y'$ and arranging yield the set of determining equations, the over-system of partial differential equations

$$\frac{1}{2}\xi_y + \xi \frac{f'(y)}{f(y)} = 0, \tag{19}$$

$$\eta_y - \frac{1}{2}\xi_x + \eta \frac{f'(y)}{f(y)} = 0, \tag{20}$$

$$\eta_x + \xi_y \ln f(y) - B_y = 0, \tag{21}$$

$$\xi_x \ln f(y) - B_x + \eta \frac{f'(y)}{f(y)} = 0. \tag{22}$$

To find the infinitesimals $\xi$ and $\eta$, (19)–(22) should be solved together. First, (19) is integrated as

$$\xi = \frac{a(x)}{f(y)^2}, \tag{23}$$

and then substituting (23) into (20) and solving for $\eta$ yield

$$\eta = \frac{1}{f(y)} \left( \frac{a'(x)}{2} \int \frac{dy}{f(y)} + b(x) \right). \tag{24}$$

Differentiating (21)-(22) with respect to $x$ and $y$, respectively, gives

$$B_{yx} = \eta_{xx} + \xi_{yx} \ln f(y),$$

$$B_{xy} = \xi_{xy} \ln f(y) + \xi_x \frac{f'(y)}{f(y)} + \eta_y \frac{f'(y)}{f(y)} + \eta \left( \frac{f'(y)}{f(y)} \right)^2. \tag{25}$$

Using (25) and eliminating $B$, we find that

$$\eta \left( \frac{f'(y)}{f(y)} \right)^2 + (\xi_x - \eta_y) \frac{f'(y)}{f(y)} - \eta_{xx} = 0. \tag{26}$$

If the infinitesimals $\xi$ (23) and $\eta$ (24) are inserted into (26) then one can find the following classification relationship in terms of $f(y)$:

$$\begin{aligned} b(x) \left( 4f(y)^2 - 2f(y) f''(y) \right) \\ + a'(x) \left( -3f'(y) + 2 \int f(y)\, dy \right. \\ \left. -f(y) \int \frac{dy}{f(y)} f''(y) \right) \\ + f(y)^2 \left( 2b''(x) + a'''(x) \int \frac{dy}{f(y)} \right) = 0. \end{aligned} \tag{27}$$

Here several cases should be examined separately for different forms of $f(y)$.

*3.1. $f(y) = k = Constant$.* For this case the solution of (27) gives to the following infinitesimals:

$$\xi = \frac{c_1 + c_2 x + c_3 x^2}{k^2},$$

$$\eta = \frac{c_2 y + 2xyc_3 + 2kc_4 + 2kxc_5}{2k^2}, \tag{28}$$

where $c_i$ are constants $i = 1, \dots, 5$. Integrating (21) with respect to $y$ gives

$$B(x, y) = \frac{y^2 c_3 + 2yc_5 + 2xc_2 \ln k + 2x^2 c_3 \ln k}{2k^2}. \tag{29}$$

The associated infinitesimal generators turn out to be

$$X_1 = \frac{1}{k^2} \frac{\partial}{\partial x},$$

$$X_2 = \frac{x}{k^2} \frac{\partial}{\partial x} + \frac{y}{2k^2} \frac{\partial}{\partial y},$$

$$X_3 = \frac{x^2}{k^2} \frac{\partial}{\partial x} + \frac{xy}{k^2} \frac{\partial}{\partial y}, \tag{30}$$

$$X_4 = \frac{1}{k} \frac{\partial}{\partial y}, \qquad X_5 = \frac{x}{k} \frac{\partial}{\partial y}.$$

Thus, the first integrals by *Definition 6* are given as follows:

$$I_1 = \frac{c_1 \left( 2 \ln k - y'^2 \right)}{2k^2}, \qquad I_2 = \frac{yy' - xy'^2}{2k^2},$$

$$I_3 = \frac{-y^2 + 2xyy' - x^2 y'^2}{2k^2}, \tag{31}$$

$$I_4 = \frac{y'}{k}, \qquad I_5 = \frac{-2ky + 2kxy'}{2k^2}.$$

*3.2. $f(y) = y$.* For the linear case of $f(y)$, we obtain

$$\xi = \frac{c_1}{y^2}, \qquad \eta = 0, \qquad B(x, y) = c_1 \left( \frac{1}{2y^2} + \frac{\ln y}{y^2} \right),$$

$$(32)$$

where $c_1$ is a constant. The partial Noether operator is

$$X_1 = \frac{1}{y^2} \frac{\partial}{\partial x},$$

$$(33)$$

and the first integral is

$$I_1 = -\frac{\left( 1 + y'^2 \right)}{2y^2}.$$

$$(34)$$

*3.3. $f(y) = ke^{\alpha} y$.* The solution of determining equations for the form of $f(y) = ke^{\alpha} y$ gives the following infinitesimals

$$\xi = \frac{e^{-2y\alpha}}{2k^2\alpha} \left( c_1 \sin 2x\alpha - c_2 \cos 2x\alpha + 2c_3\alpha \right),$$

$$\eta = \frac{e^{-2y\alpha}}{2k^2\alpha} \left( 2c_4 e^{\alpha y} k\alpha \cos x\alpha \right.$$
$$\left. - c_1 \cos 2x\alpha + 2c_5 e^{\alpha y} \sin x\alpha - c_2 \sin 2x\alpha \right),$$

$$(35)$$

where $c_i$ are constants $i = 1, \ldots, 5$, and the gauge function is

$$B(x, y) = \frac{e^{-2y\alpha}}{4k^2\alpha} \left( 2c_3\alpha + 4yc_3\alpha^2 - 4e^{y\alpha} c_5 \cos x\alpha + 4c_3\alpha \ln k \right.$$
$$- c_2 \cos 2x\alpha \left( 2\ln k + 2y\alpha - 1 \right)$$
$$\times 4c_4 e^{y\alpha} k\alpha \sin x\alpha - c_1 \sin 2x\alpha$$
$$\left. + 2c_1 y\alpha \sin 2x\alpha + 2c_1 \ln k \sin 2x\alpha \right).$$

$$(36)$$

The associated five-parameter symmetry generators take the form

$$X_1 = \frac{e^{-2y\alpha} \sin 2x\alpha}{2k^2\alpha} \frac{\partial}{\partial x} - \frac{e^{-2y\alpha} \cos 2x\alpha}{2k^2\alpha} \frac{\partial}{\partial x},$$

$$X_2 = -\frac{e^{-2y\alpha} \cos 2x\alpha}{2k^2\alpha} \frac{\partial}{\partial x} - \frac{e^{-2y\alpha} \sin 2x\alpha}{2k^2\alpha} \frac{\partial}{\partial x},$$

$$X_3 = \frac{e^{-2y\alpha}}{k^2\alpha} \frac{\partial}{\partial x}, \qquad X_4 = \frac{e^{-y\alpha} \cos x\alpha}{k} \frac{\partial}{\partial y},$$

$$X_5 = \frac{e^{-y\alpha} \sin x\alpha}{k} \frac{\partial}{\partial y},$$

$$(37)$$

and the corresponding first integrals are

$$I_1 = \frac{e^{-2y\alpha}}{4k^2\alpha} \left( -2\cos 2x\alpha y' - \sin 2\alpha x \left( y'^2 - 1 \right) \right),$$

$$I_2 = \frac{e^{-2y\alpha}}{4k^2\alpha} \left( -2\sin 2x\alpha y' + \cos 2\alpha x \left( y'^2 - 1 \right) \right),$$

$$I_3 = -\frac{e^{-2y\alpha}}{2k^2\alpha} \left( 1 + y'^2 \right), \qquad I_4 = \frac{e^{-y\alpha}}{k} \left( y' \cos x\alpha - \sin x\alpha \right),$$

$$I_5 = \frac{e^{-y\alpha}}{k} \left( \cos \alpha x + y' \sin x\alpha \right).$$

$$(38)$$

*3.4. $f(y) = 1/(my + n)$.* For this case, the infinitesimal functions read

$$\xi = (my + n)^2 \left( c_1 + c_2 x + c_3 x^2 \right),$$

$$\eta = (my + n) \left( -\frac{3}{4}mx^2 c_2 - \frac{1}{2}mx^3 c_3 \right.$$
$$\left. + \frac{1}{4}y \left( 2n + my \right) \left( c_2 + 2xc_3 \right) + c_4 + xc_5 \right),$$

$$(39)$$

where $c_i$ are constants $i = 1, \ldots, 5$, and the gauge function is

$$B(x, y) = \frac{1}{2} \left( \frac{1}{4} (2n + my) \right.$$
$$\times \left( 2nyc_3 + m \left( 4c_1 - 2xc_2 - 2x^2 c_3 + y^2 c_3 \right) + 4c_5 \right)$$
$$+ 2my (2n + my) \left( c_1 + x (c_2 + c_3) \right) \ln \frac{1}{my + n}$$
$$- 2n^2 \left( c_1 + x (c_2 + c_3) \right) \ln (my + n) \right)$$
$$+ \frac{1}{8}x \left( m \left( 2mx^2 c_2 + mx^3 c_3 - 8c_4 - 4xc_5 \right) \right.$$
$$+ 8n_2 (c_2 + xc_3) \ln \frac{1}{my + n}$$
$$\left. + 8n_2 (c_2 + xc_3) \ln (my + n) \right).$$

$$(40)$$

The corresponding Noether symmetry generators are

$$X_1 = (my + n)^2 \frac{\partial}{\partial x},$$

$$X_2 = x(my + n)^2 \frac{\partial}{\partial x} + (my + n)$$
$$\times \left( -\frac{3mx^2}{4} + \frac{1}{2} \left( ny + \frac{my^2}{2} \right) \right) \frac{\partial}{\partial y},$$

$$X_3 = x^2 (my + n)^2 \frac{\partial}{\partial x} + (my + n)$$
$$\times \left( -\frac{mx^3}{4} + x \left( ny + \frac{my^2}{2} \right) \right) \frac{\partial}{\partial y},$$

$$X_4 = (my + n) \frac{\partial}{\partial y}, \qquad X_5 = x (my + n) \frac{\partial}{\partial y}.$$

$$(41)$$

And the conservation laws are

$$I_1 = \frac{1}{8}\left(-8mny - 4m^2y^2 + 8n^2\ln\frac{1}{my+n}\right.$$
$$\left. + 8n^2\ln(my+n) - (4my+n)^2 y'^2\right),$$

$$I_2 = \frac{1}{8}\left(-2m^2x^3 + 4mnxy + 2m^2xy^2 - 6mx^2(my+n)y'\right.$$
$$\left. + 2my^2(my+n)y' - 4x(my+n)^2 y'^2\right),$$

$$I_3 = -\frac{1}{8}\left(-m^2x^4 + 4mnx^2y - 4n^2y^2 + 2m^2x^2y^2\right.$$
$$- 4mny^3 - m^2y^4 - 4mx^3(my+n)y'$$
$$+ 8nxy(my+n)y' + 4mxy^2(my+n)y'$$
$$\left. - 4x^2(my+n)^2 y'^2\right),$$

$$I_4 = mx + (my+n)y',$$

$$I_5 = \frac{1}{8}\left(4mx^2 - 8ny - 4my^2 + 8x(my+n)y'\right).$$
$$\tag{42}$$

3.5. $f(y) = y^n$. For this choice of $f(y)$, we find the infinitesimals

$$\xi = c_1 y^{-2n}, \qquad \eta = 0,$$
$$B(x,y) = \frac{1}{2}c_1 y^{-2n}(1 + 2\ln(y^n)),$$
$$\tag{43}$$

where $c_1$ is constant, and we have the first integral

$$I = -\frac{1}{2}y^{-2n}\left(1 + y'^2\right).$$
$$\tag{44}$$

For convenience all Noether symmetries and first integrals are presented in Table 1.

3.6. Invariant Solutions. Invariant solutions that satisfy the original path equation can be obtained by first integrals according to the relation $D_x I = 0$. We here determine some special cases and investigate the corresponding invariant solutions.

Case 1. (a) For the case of $f(y) = ke^{\alpha y}$, the conservation law is

$$I = -\frac{e^{-2y\alpha}}{2k^2\alpha}\left(1 + y'^2\right);$$
$$\tag{45}$$

by using the relation $D_x I = 0$, then the invariant solution of path equation (16) is

$$y(x) = \frac{1}{\alpha}\ln\left(-\frac{\sqrt{-1 - \tan(\alpha(x+c_1))^2}}{k\sqrt{2c}}\right),$$
$$\tag{46}$$

where $c_1, c$ are constants.

(b) For the same $f(y)$ function, the conservation law is

$$I = \frac{e^{-y\alpha}}{k}\left(\cos x\alpha y' - \sin x\alpha\right),$$
$$\tag{47}$$

and the invariant solution similar to previous one is

$$y(x) = -\frac{1}{\alpha}\ln\left(ck\alpha^3\cos x\alpha\left(-c_1\frac{1}{\alpha^3} - \tan x\alpha\right)\right),$$
$$\tag{48}$$

where $c_1, c$ are constants.

Case 2. Let us consider $f(y) = 1/(my+n)$, then the first integral yields

$$I = -\frac{1}{8}\left(-m^2x^4 + 4mnx^2y - 4n^2y^2 + 2m^2x^2y^2\right.$$
$$- 4mny^3 - m^2y^4 - 4mx^3(my+n)y'$$
$$+ 8nxy(my+n)y'$$
$$\left. + 4mxy^2(my+n)y' - 4x^2(my+n)^2 y'^2\right),$$
$$\tag{49}$$

and the solution of this equation gives

$$y(x) = \frac{-n - \sqrt{-2m\sqrt{-2c} + n^2 - m^2x^2 - 2m^2xc_1}}{m},$$
$$\tag{50}$$

where $c_1, c$ are constants, in which it is obvious that the invariant solution (50) satisfies the original path equation.

## 4. $\lambda$-Symmetries of Path Equation

The relationship between $\lambda$-symmetries, integration factors and first integrals of second-order ordinary differential equation is very important from the mathematical point of view [10–12]. Let us consider first the second-order differential equation of the form

$$y'' = \phi(x, y, y'),$$
$$\tag{51}$$

and let vector field of (51) be in the form of

$$A = \partial_x + y'\partial_y + \phi(x, y, y')\partial_{y'}.$$
$$\tag{52}$$

In terms of $A$, a first integral of (51) is any function in the form of $I(x, y, y')$ providing equality of $A(I) = 0$. An integrating factor of (51) is any function satisfying the following equation:

$$\mu\left[y'' - \phi(x, y, y')\right] = D_x I,$$
$$\tag{53}$$

where $D_x$ is total derivative operator in the form of

$$D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots.$$
$$\tag{54}$$

Thus $\lambda$-symmetries of second-order differential equation (51) can be obtained directly by using Lie symmetries of this same equation. Secondly, let

$$v = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$
$$\tag{55}$$

TABLE 1: Noether symmetry classification table of path equation.

| Function | Infinitesimals and first integrals |
| --- | --- |
| $f(y) = k$ | $\xi = \dfrac{c_1 + c_2 x + c_3 x^2}{k^2},\ \eta = \dfrac{c_2 y + 2xy c_3 + 2k c_4 + 2kx c_5}{2k^2}$ $I_1 = \dfrac{c_1(2\ln k - y'^2)}{2k^2},\ I_2 = \dfrac{yy' - xy'^2}{2k^2},\ I_3 = \dfrac{-y^2 + 2xyy' - x^2 y'^2}{2k^2}$ $I_4 = \dfrac{y'}{k},\ I_5 = \dfrac{-2ky + 2kxy'}{2k^2}$ |
| $f(y) = y$ | $\xi = \dfrac{c_1}{y^2},\ \eta = 0,\ I_1 = -\dfrac{(1 + y'^2)}{2y^2}$ |
| $f(y) = ke^{\alpha y}$ | $\xi = \dfrac{e^{-2y\alpha}}{2k^2\alpha}(c_1 \sin 2x\alpha - c_2 \cos 2x\alpha + 2c_3)$ $\eta = \dfrac{e^{-2y\alpha}}{2k^2\alpha}(2c_4 k\alpha e^{y\alpha}\cos x\alpha - c_1 \cos 2x\alpha + 2c_5 k\alpha e^{y\alpha}\sin x\alpha - c_2 \sin 2x\alpha)$ $I_1 = \dfrac{e^{-2y\alpha}}{4k^2\alpha}(-2\cos 2x\alpha y' - \sin 2x\alpha(y'^2 - 1))$ $I_2 = \dfrac{e^{-2y\alpha}}{4k^2\alpha}(-2\sin 2x\alpha y' + \cos 2x\alpha(y'^2 - 1))$ $I_3 = -\dfrac{e^{-2y\alpha}}{2k^2\alpha}(1 + y'^2),\ I_4 = \dfrac{e^{-y\alpha}}{k}(\cos x\alpha y' - \sin x\alpha),\ I_5 = \dfrac{e^{-y\alpha}}{k}(\cos x\alpha + \sin x\alpha y')$ |
| $f(y) = \dfrac{1}{my + n}$ | $\xi = (my + n)^2(c_1 + c_2 x + c_3 x^2)$ $\eta = (my + n)(-\dfrac{3}{4}mx^2 c_2 - \dfrac{1}{2}mx^3 c_3 + \dfrac{1}{4}y(2n + my)(c_2 + 2x c_3) + c_4 + x c_5)$ $I_1 = \dfrac{1}{8}(-8mny - 4m^2 y^2 + 8n^2 \ln \dfrac{1}{my + n} + 8n^2\ln(my + n) - 4(my + n)^2 y'^2)$ $I_2 = \dfrac{1}{8}(-2m^2 x^3 + 4mnxy + 2m^2 xy^2 - 6mx^2(my + n)y'$ $+ 2my^2(my + n)y' - 4x(my + n)^2 y'^2)$ $I_3 = -\dfrac{1}{8}(-m^2 x^4 + 4mnx^2 y - 4n^2 y^2 + 2m^2 x^2 y^2 - 4mny^3 - m^2 y^4 - 4mx^3(my + n)y'$ $+ 8nxy(my + n)y' + 4mxy^2(my + n)y' - 4x^2(my + n)^2 y'^2)$ $I_4 = mx + (my + n)y,\ I_5 = \dfrac{1}{8}(4mx^2 - 8ny - 4my^2 + 8x(my + n)y')$ |
| $f(y) = y^n$ | $\xi = c_1 y^{-2n},\ \eta = 0,\ I = -\dfrac{1}{2}y^{-2n}(1 + y'^2)$ |

be a Lie point symmetry of (51), and then the characteristic of $v$ is

$$Q = \eta - \xi y', \qquad (56)$$

and for the path equation (16) the total derivative operator can be written as

$$A = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \left(1 + y'^2\right)\frac{f'(y)}{f(y)}\frac{\partial}{\partial y'}; \qquad (57)$$

thus the vector field $\partial_y$ is called $\lambda$-symmetry of (16) if the following equality is satisfied.

$$\lambda = \frac{A(Q)}{Q} \qquad (58)$$

The following four steps can be defined for finding $\lambda$-symmetries and first integrals.

(1) Find a first integral $w(x, y, y')$ of $v^{[\lambda,(1)]}$, that is, a particular solution of the equation

$$w_y + \lambda w_{y'} = 0, \qquad (59)$$

where $v^{[\lambda,(1)]}$ is the first-order $\lambda$-prolongation of the vector field $v$.

(2) The solution of (59) will be in terms of first order derivative of $y$. To write equation of (51) in terms of the reduced equation of $w$, we can obtain the first-order derivative the solution of (59) and we can write (51) equation in terms of $w$.

(3) Let $G$ be an arbitrary constant of the solution of the reduced equation written in terms of $w$. Therefore,

$$\mu = G_w w_{y'} \qquad (60)$$

is an integrating factor of (51).

(4) The solution of $w(x, y, y')$ is the first integral of $v^{[\lambda,(1)]}$.

*4.1. $\lambda$-Symmetries Using Lie Symmetries of Path Equation.* Let us consider an $n$th-order ODE as follows:

$$y^{(n)} = f\left(x, y, y', y'', \ldots, y^{(n-1)}\right). \qquad (61)$$

Thus the invariance criterion of (61) is

$$\text{pr}X\left(y^{(n)} - f\left(x, y, y', y'', \ldots, y^{(n-1)}\right)\right)\Big|_{y^{(n)}=f} = 0. \qquad (62)$$

The expansion of relation (62) gives the determining equation related to path equation, which is the system of partial differential equations. In this system there are three unknowns, namely, $\lambda$, $\xi$, and $\eta$, which are difficult to solve because they are highly nonlinear. In the literature [10–12], for the convenience the $\lambda$ function are chosen generally in the form

$$\lambda\left(x, y, y'\right) = \lambda_1\left(x, y\right) y' + \lambda_2\left(x, y\right). \tag{63}$$

In addition, for solving the remaining determining equations, the infinitesimal functions $\xi$ and $\eta$ are chosen specifically as $\xi = 0$ and $\eta = 1$ [10–12]. Therefore, the number of unknowns in the equation is reduced to find $\lambda_1(x, y)$ and $\lambda_2(x, y)$ functions, and finally, $\lambda$-symmetries can be determined explicitly.

However, for the path equation (16), it is possible to check that $\lambda$-symmetries of this equation cannot be determined by taking the form of $\lambda$ in (63). Thus, we study $\lambda$-symmetries of path equation by using the relation with the Lie point symmetries of the same equation [2, 19]. Here Lie point symmetries of path equation are examined by considering four different cases of function $f(y)$.

*4.1.1. Arbitrary $f(y)$.* For arbitrary $f(y)$ the one-parameter Lie group of transformations is

$$\xi = a, \qquad \eta = 0, \tag{64}$$

and the generator is

$$X = a\frac{\partial}{\partial x}. \tag{65}$$

Applying this generator (56), we obtain the characteristic

$$Q = -ay'. \tag{66}$$

Using (58), the $\lambda$-symmetry is obtained in the following form:

$$\lambda = \frac{A(Q)}{Q} = \frac{\left(1 + y'^2\right) f'(y)}{f(y) y'}. \tag{67}$$

If we substitute $\lambda$-symmetry (67) in (59), then we have

$$w_y + \frac{\left(1 + y'^2\right) f'(y)}{f(y) y'} w_{y'} = 0. \tag{68}$$

It is clear that a solution of (68) is

$$w\left(x, y, y'\right) = \frac{1}{2} \ln\left(\frac{\left(1 + y'^2\right)}{f(y)^2}\right). \tag{69}$$

To write (16) in terms of $\{x, w, w'\}$, we can express the following equality using (69):

$$y' = \sqrt{-1 + e^{2w(x)} f(y(x))^2}. \tag{70}$$

Taking derivative of (70) with respect to $x$ gives

$$y'' = e^{2w(x)} f(y(x))$$
$$\times \left( f'(y(x)) + \frac{f(y(x)) w'(x)}{\sqrt{-1 + e^{2w(x)} f(y(x))^2}} \right), \tag{71}$$

and by using $y'$ and $y''$, (16) becomes

$$w'(x) = 0. \tag{72}$$

It is easy to see that the general solution of this equation is

$$w(x) = G, \quad G \in \mathbb{R}. \tag{73}$$

According to (60), we find the integration factor $\mu$ to be of the form

$$\mu = \frac{y'}{1 + y'^2}. \tag{74}$$

Then the conserved form satisfies the following equality:

$$D_x\left(\frac{1}{2} \ln\left(\frac{1 + y'(x)^2}{f(y(x))^2}\right)\right) = 0, \tag{75}$$

which gives the original path equation. Thus the reduced equation is

$$\frac{1}{2} \ln\left(\frac{1 + y'(x)^2}{f(y(x))^2}\right) - k = 0, \tag{76}$$

where $k$ is a constant, and the solution of (76) is determined for two different cases of arbitrary $f(y)$ function.

(i) For $f(y) = y$,

$$y(x) = \frac{1}{4} e^{-k - e^k x - e^k c_1} \left(4 e^{2e^k} + e^{2e^k c_1}\right), \tag{77}$$

where $c_1$ is a constant, is the solution of original path equation (16).

(ii) For $f(y) = e^y$,

$$y(x) = -k + \ln\left(-\cot\left(x - c_1\right)\right) \sqrt{1 + \tan\left(x - c_1\right)^2} \tag{78}$$

is the other solution of the same equation.

*4.1.2. $f(y) = k = Constant.* For another case $f(y) = k$, the infinitesimal generators are

$$X_1 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \qquad X_2 = y\frac{\partial}{\partial x},$$

$$X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \qquad X_4 = x\frac{\partial}{\partial x},$$

$$X_5 = x\frac{\partial}{\partial x}, \qquad X_6 = y\frac{\partial}{\partial y}, \qquad X_6 = x\frac{\partial}{\partial y}, \tag{79}$$

$$X_7 = \frac{\partial}{\partial y}, \qquad X_8 = \frac{\partial}{\partial x}.$$

Thus, we can calculate $\lambda$-symmetry of path equation using, for example, $X_1$ Lie symmetry generator. For this generator $X_1$ the infinitesimals are

$$\xi = xy, \qquad \eta = y^2. \tag{80}$$

Therefore, the characteristic is written as

$$Q = y^2 - xyy'. \tag{81}$$

By using (58) we obtain the $\lambda$-symmetry

$$\lambda = \frac{y'}{y}. \tag{82}$$

A solution of (59) for this case is

$$w(x, y, y') = \frac{y'}{y}, \tag{83}$$

and we can write $w = y'/y$, then to obtain path equation in terms of $\{x, w, w'\}$ one can have

$$y' = wy, \qquad y'' = w^2 y + yw'. \tag{84}$$

By using these equalities (84) we find the following equation:

$$w' + w^2 = 0, \tag{85}$$

in which the general solution is

$$w(x) = \frac{1}{x - G}, \quad G \in \mathbb{R}. \tag{86}$$

To find the integration factor one can write above equation in terms of $G$ as

$$G = \frac{wx - 1}{w}, \tag{87}$$

and then the integration factor becomes

$$\mu = \frac{1}{w^2 y}. \tag{88}$$

If we substitute $w = y'/y$ in (87), then the reduced equation in terms of $y'$ is

$$\left(x - \frac{y}{y'}\right) - c = 0, \tag{89}$$

and the solution of (89) is

$$y(x) = (x - c)c_3, \tag{90}$$

where $c$ and $c_3$ are constants. It is clear that this solution satisfies the original path equation (16). Also, one can write

$$D_x\left(x - \frac{y(x)}{y'(x)}\right) = 0, \tag{91}$$

which is the first integral of equation that provides the path equation (16).

*4.1.3. $f(y) = 1/(my + n)$.* For this case the eight-parameter symmetry generators are obtained as follows:

$$
\begin{aligned}
X_1 &= \left(\frac{mx^3}{2} + x\left(ny + \frac{my^2}{2}\right)\right)\frac{\partial}{\partial x} \\
&\quad + \left(\frac{m^2 x^4}{4(my+n)}\right. \\
&\qquad \left. + \frac{n^2 y^2 + mny^3 + (m^2 y^4/4)}{my + n}\right)\frac{\partial}{\partial y}, \\
X_2 &= \left(ny + \frac{my^2}{2}\right)\frac{\partial}{\partial x} \\
&\quad + \left(-\frac{m^2 x^3}{4(my+n)} - \frac{3mx\left(ny + (my^2/2)\right)}{2(my+n)}\right)\frac{\partial}{\partial y}, \\
X_3 &= \frac{x^2}{2}\frac{\partial}{\partial x} + \left(-\frac{mx^3}{4(my+n)} + \frac{x\left(ny + (my^2/2)\right)}{2(my+n)}\right)\frac{\partial}{\partial y}, \\
X_4 &= x\frac{\partial}{\partial x} + \left(\frac{y(2n+my)}{my+n}\right)\frac{\partial}{\partial y}, \\
X_5 &= \frac{2ny + m(x^2 + y^2)}{2(my+n)}\frac{\partial}{\partial y}, \\
X_6 &= \frac{x}{my+n}\frac{\partial}{\partial y}, \quad X_7 = \frac{1}{my+n}\frac{\partial}{\partial y}, \quad X_8 = \frac{\partial}{\partial x}.
\end{aligned}
\tag{92}
$$

Now let us consider $X_1$ operator, and then the corresponding infinitesimals $\xi$ and $\eta$ are

$$
\begin{aligned}
\xi &= \frac{mx^3}{2} + x\left(ny + \frac{my^2}{2}\right), \\
\eta &= -\frac{m^2 x^4}{4(my+n)} + \frac{n^2 y^2 + mny^3 + (m^2 y^4/4)}{my + n}.
\end{aligned}
\tag{93}
$$

Using these infinitesimals we find the characteristic

$$
\begin{aligned}
Q = &\left(\left(2ny + m(x^2 + y^2)\right)\right. \\
&\times \left(2n(y - xy') + m(-x^2 + y^2 - 2xyy')\right)\right) \\
&\times \left(4(my + n)\right)^{-1},
\end{aligned}
\tag{94}
$$

and the $\lambda$-symmetry is

$$\lambda = \frac{2n^2 y' + 2mn(x + yy') + m^2(2xy - x^2 y' + y^2 y')}{(my + n)(2ny + m(x^2 + y^2))}. \tag{95}$$

By using (95) the equation (59) becomes

$$w_y + \frac{2n^2 y' + 2mn\left(x + yy'\right) + m^2\left(2xy - x^2 y' + y^2 y'\right)}{\left(my + n\right)\left(2ny + m\left(x^2 + y^2\right)\right)}$$
$$\times w_{y'} = 0. \tag{96}$$

A solution of (96) is

$$w\left(x, y, y'\right) = \frac{mx + ny' + myy'}{mx^2 + 2ny + my^2}. \tag{97}$$

This equation can be written as

$$y' = \frac{-mx + mwx^2 + 2nwy + mwy^2}{my + n}. \tag{98}$$

By differentiation of (98) we have

$$y'' = \frac{-m\left(mx - w\left(mx^2 + 2ny + my^2\right)\right)^2}{\left(my + n\right)^3}$$
$$+ \frac{-m + 2w^2\left(mx^2 + 2ny + my^2\right) + \left(mx^2 + 2ny + my^2\right)w'}{my + n}, \tag{99}$$

and if we substitute (98) and (99) into the path equation, we obtain

$$w' + 2w^2 = 0, \tag{100}$$

and the solution of (100) is

$$w\left(x\right) = \frac{1}{2x - G}, \quad G \in \mathbb{R}. \tag{101}$$

To define $G$, one can write

$$G = \frac{2wx - 1}{w}. \tag{102}$$

Therefore, by using the relation (60) we find the integration factor

$$\mu = \frac{my + n}{w^2\left(mx^2 + 2ny + my^2\right)}. \tag{103}$$

If we rewrite (102) in terms of $y'$ and then we substitute this expression into integration factor, the reduced equation of path equation becomes

$$\left(-my(x)^2 + x\left(mx + 2ny'\left(x\right)\right.\right.$$
$$\left.\left. - 2y\left(x\right)\left(n - mxy'\left(x\right)\right)\right)\right) \tag{104}$$
$$\times\left(mx + \left(my\left(x\right) + n\right)y'\left(x\right)\right)^{-1} - c = 0,$$

where $c$ is a constant. By the solution of (104), we obtain the solution that satisfies the original path equation (16) as

$$y\left(x\right) = \left(-2n + m\sqrt{-\frac{1}{m(c - 2x)^2}}\left(c - 2x\right)\right.$$
$$\left.\times\sqrt{2c^2 - \frac{4n^2}{m} + 4mx\left(x - 2c_3\right) + 4cm\left(c_3 - x\right)}\right)$$
$$\times\left(2m\right)^{-1}, \tag{105}$$

where $c_3$ is a constant, and the corresponding conservation law is

$$D_x\left(\left(-my(x)^2\right.\right.$$
$$+ x\left(mx + 2ny'\left(x\right) - 2y\left(x\right)\left(n - mxy'\left(x\right)\right)\right)\right) \tag{106}$$
$$\left.\times\left(mx + \left(my\left(x\right) + n\right)y'\left(x\right)\right)^{-1}\right)$$
$$= 0.$$

*4.1.4.* $f\left(y\right) = ke^{\alpha y}$. For this case the infinitesimal generators of path equation are

$$X_1 = e^{-\alpha y}\cos\alpha x\frac{\partial}{\partial x} + e^{-\alpha y}\sin\alpha x\frac{\partial}{\partial y},$$

$$X_2 = e^{-\alpha y}\sin\alpha x\frac{\partial}{\partial x} - e^{-\alpha y}\cos\alpha x\frac{\partial}{\partial y},$$

$$X_3 = \cos 2\alpha x\frac{\partial}{\partial x} + \sin 2\alpha x\frac{\partial}{\partial y},$$

$$X_4 = \sin 2\alpha x\frac{\partial}{\partial x} - \cos 2\alpha x\frac{\partial}{\partial y}, \tag{107}$$

$$X_5 = \frac{\partial}{\partial x}, \quad X_6 = \frac{\partial}{\partial y},$$

$$X_7 = e^{\alpha y}\cos\alpha x\frac{\partial}{\partial x}, \quad X_8 = e^{\alpha y}\sin\alpha x\frac{\partial}{\partial x}.$$

If we consider, for example, $X_1$ symmetry generator and then $\xi$ and $\eta$ are

$$\xi = e^{-\alpha y}\cos\alpha x, \quad \eta = e^{\alpha y}\sin\alpha x, \tag{108}$$

then the characteristic by (56) is

$$Q = -e^{-y\alpha}y'\cos x\alpha - e^{-y\alpha}\sin x\alpha. \tag{109}$$

If we apply the operator $A$ (52) to this characteristic (109), we obtain $A(Q) = 0$, and the $\lambda$-symmetry is equal to zero. For $X_2$ symmetry generator we find also $\lambda = 0$ similar to previous one. Hence, we can use another symmetry generator, for example, $X_7$ to obtain $\lambda$-symmetry. For this case,

$$\xi = 0, \quad \eta = e^{y\alpha}\cos x\alpha, \tag{110}$$

are infinitesimals, and the corresponding characteristic is

$$Q = e^{y\alpha} \cos x\alpha. \tag{111}$$

We find the $\lambda$-symmetry from (58) as in the following form:

$$\lambda = \alpha \left( y' - \tan x\alpha \right). \tag{112}$$

By applying (112) to (59) we obtain the solution

$$w(x, y, y') = e^{y\alpha} \left( y' - \tan x\alpha \right). \tag{113}$$

And we write this expression (113) in terms of $\{x, w, w'\}$ as

$$y' = e^{y\alpha} w + \tan x\alpha. \tag{114}$$

By differentiating $y'$ (114) with respect to $y'$ one can write

$$y'' = \alpha\sec(x\alpha)^2 + e^{y\alpha}\alpha w \left( \tan x\alpha + e^{y\alpha} w \right) + e^{y\alpha} w', \tag{115}$$

and by substituting $y'$ and $y''$ to the original path equation we obtain

$$w' - \alpha \tan(x\alpha) w = 0, \tag{116}$$

where the solution of (116) is

$$w(x) = \sec(x\alpha). \tag{117}$$

To define this equality in terms of variable $w$ then $G$ is defined as follows:

$$G = w \cos(x\alpha), \tag{118}$$

so we obtain the integration factor using (60)

$$\mu = e^{-\alpha y} \cos x\alpha. \tag{119}$$

Finally one can write the conservation law

$$D_x \left( e^{-\alpha y(x)} \cos x\alpha \left( y'(x) - \tan x\alpha \right) \right) = 0, \tag{120}$$

which gives the original path equation. And thus we can express the first integral, which is reduced form of the path equation

$$e^{-\alpha y(x)} \cos x\alpha \left( y'(x) - \tan x\alpha \right) - c = 0, \tag{121}$$

where $c$ is a constant. Integrating (121) we obtain the solution that satisfies the original equation

$$y(x) = \frac{-\ln \left( c\alpha^3 \cos(x\alpha) \left( -c_1 - \left( \tan x\alpha / \alpha^3 \right) \right) \right)}{\alpha}, \tag{122}$$

where $c_1$ is a constant.

*4.1.5.* $f(y) = y^n$. If $f(y)$ is assumed in the polynomial form and then Lie symmetry generators are

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial x}. \tag{123}$$

$X_2$, for example, can be used to obtain $\lambda$-symmetry, and for this generator the infinitesimals are

$$\xi = 1, \qquad \eta = 0. \tag{124}$$

By using $\xi$, $\eta$ the characteristic function is written as

$$Q = -y'. \tag{125}$$

By considering (58), the $\lambda$-symmetry becomes

$$\lambda = \frac{n\left(1 + y'^2\right)}{yy'}. \tag{126}$$

The solution of (59) is

$$w(x, y, y') = \frac{1}{2} \ln \left( \frac{\left(1 + y'^2\right)}{y^{2n}} \right). \tag{127}$$

To write (16) in terms of $\{x, w, w'\}$, we can express the following equality:

$$y' = -\sqrt{-1 + e^{2w(x)} y(x)^{2n}}. \tag{128}$$

By taking derivative (128) with respect to $x$, then we have

$$y'' = e^{2w(x)} y(x)^{(2n-1)} \left( n - \frac{y(x) w'(x)}{\sqrt{-1 + e^{2w(x)} y(x)^{2n}}} \right). \tag{129}$$

If we substitute $y'$ and $y''$ into the path equation, then one can find

$$w'(x) = 0, \tag{130}$$

and a solution of this equation (130) is

$$w(x) = G, \quad G \in \mathbb{R}. \tag{131}$$

By using (60) we find the integration factor $\mu$ of the form

$$\mu = \frac{y'}{1 + y'^2}. \tag{132}$$

It is easy to see that the conserved form satisfies the following equality:

$$D_x \left( \frac{1}{2} \ln \left( \frac{1 + y'(x)^2}{y(x)^{2n}} \right) \right) = 0, \tag{133}$$

and this equality gives the original path equation. Thus the reduced form of path equation is

$$\frac{1}{2} \ln \left( \frac{1 + y'(x)^2}{y(x)^{2n}} \right) - k = 0, \tag{134}$$

where $k$ is a constant. And all results are summarized in Table 2.

TABLE 2: Table of $\lambda$-symmetry classification with Lie symmetry of path equation.

| Function | Lie symmetries | Integration factor | $\lambda$-Symmetries |
|---|---|---|---|
| Arbitrary $f(y)$ | $\xi = a,\ \eta = 0$ | $\mu = \dfrac{y'}{1 + y'^2}$ | $\lambda = \dfrac{(1 + y'^2)\,f'(y)}{f(y)\,y'}$ |
| $f(y) = k$ | $\xi = xy,\ \eta = y^2$ | $\mu = \dfrac{1}{w^2 y}$ | $\lambda = \dfrac{y'}{y}$ |
| $f(y) = y^n$ | $\xi = 1,\ \eta = 0$ | $\mu = \dfrac{y'}{1 + y'^2}$ | $\lambda = \dfrac{n(1 + y'^2)}{yy'}$ |
| $f(y) = \dfrac{1}{my + n}$ | $\xi = \dfrac{mx^3}{2} + x(ny + \dfrac{my^2}{2})$ $\eta = -\dfrac{m^2 x^4}{4(my + n)}$ $+ \dfrac{n^2 y^2 + mny^3 + (m^2 y^4/4)}{my + n}$ | $\mu = \dfrac{my + n}{w^2(mx^2 + 2ny + my^2)}$ | $\lambda = \dfrac{2n^2 y' + 2mn(x + yy')}{(my + n)(2ny + m(x^2 + y^2))}$ $+ \dfrac{m^2(2xy - x^2 y_1 + y^2 y')}{(my + n)(2ny + m(x^2 + y^2))}$ |
| $f(y) = k\mathrm{Exp}(\alpha y)$ | $\xi = 0,\ \eta = e^{y\alpha}\cos x\alpha$ | $\mu = e^{-\alpha y}\cos x\alpha$ | $\lambda = \alpha(y' - \tan x\alpha)$ |

## 5. $\lambda$-Symmetries and Jacobi Last Multiplier Approach

*Definition of $\lambda \in C^\infty(M^{(1)})$-Symmetry.* Let $v$ be a vector field on $M$ which is open subset, and has the property of $M \subset X \times Y$. For $k \in \mathbb{N}$, $M^{(k)} \subset X \times Y^{(k)}$ denotes the corresponding $k$-jet space, and their elements are $(x, y^{(k)}) = (x, y, y_1, \ldots, y_k)$, where, for $i = 1, \ldots, k$, $y_i$ denotes the derivative of order $i$ of $y$ with respect to $x$. In addition let $X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ be a vector field defined on $M$, and let $\lambda \in C^\infty(M^{(1)})$ be an arbitrary function. Then the $\lambda$-prolongation of $X$ is

$$\mathrm{pr}X = \xi(x, y)\,\partial_x + \eta(x, y)\,\partial_y$$
$$+ \eta^{(1)}\left(x, y, y', y'', \ldots, y^{(n-1)}\right)\partial_{y'} \qquad (135)$$
$$+ \eta^{(2)}\left(x, y, y', y'', \ldots, y^{(n-1)}\right)\partial_{y''},$$

with

$$\eta^{(n+1)} = \left[(D_x + \lambda)\eta^{(n)} - y'(D_x + \lambda)\xi\right], \qquad (136)$$

where $D_x$ is total derivative operator with respect to $x$ such that

$$D_x = \partial_x + \sum_{k=0}^n y^{(k+1)}\partial_{y^{(k)}}, \qquad y^{(0)} \equiv y, \qquad \eta^{(0)} \equiv \eta. \qquad (137)$$

In this section we analyze $\lambda$-symmetries of path equation by using Jacobi last multiplier as another approach. First (61) can be written by using system of first-order equations, which is equivalent to the expression

$$w'_i = W_i(t, w_1, \ldots, w_n), \qquad (138)$$

and by solving the following differential equation, the Jacobi last multiplier of (138) $M$ is found:

$$\frac{d \log(M)}{dt} + \sum_{i=1}^n \frac{\partial W_i}{\partial w_i} = 0, \qquad (139)$$

where, namely, $M$ is

$$M = \exp\left(-\int \sum_{i=1}^n \frac{\partial W_i}{\partial w_i}\,dt\right). \qquad (140)$$

The nonlocal approach [13, 20] to $\lambda$-symmetries is analyzed to seek $\lambda$-symmetries such that

$$w' = \lambda = \sum_{i=1}^n \frac{\partial W_i}{\partial w_i}. \qquad (141)$$

With this idea $w$ always can be considered to be of the form such as $w = \log(1/M)$. But this relation cannot be considered if the divergence of (138) $\mathrm{Div} \equiv \sum_{i=1}^n (\partial W_i/\partial w_i)$ is equal to zero. So $w$ is chosen like this form because any Jacobi last multiplier is a first integral of (138). In this section we again consider different choices of $f(y)$ for $\lambda$-symmetry classification.

*5.1. $f(y) = k = Constant.* For this case the divergence of the path equation yields

$$\lambda_j = 0. \qquad (142)$$

Substituting $\lambda_j$ into (135) then from the solution of the determining equations (62) we obtain eight-parameter $\lambda$-infinitesimals

$$\xi^{(\lambda)} = y(c_2 + c_1 x) + c_3 x^2 + c_8 x + c_7,$$
$$\eta^{(\lambda)} = y(c_4 + c_3 x) + c_5 x + c_1 y^2 + c_6, \qquad (143)$$

and the generators are

$$X_1^{(\lambda)} = yx\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \qquad X_2^{(\lambda)} = y\frac{\partial}{\partial x},$$
$$X_3^{(\lambda)} = x^2\frac{\partial}{\partial x} + yx\frac{\partial}{\partial x}, \qquad X_4^{(\lambda)} = y\frac{\partial}{\partial y},$$

$$X_5^{(\lambda)} = x\frac{\partial}{\partial y}, \qquad X_6^{(\lambda)} = \frac{\partial}{\partial y},$$

$$X_7^{(\lambda)} = \frac{\partial}{\partial x}, \qquad X_8^{(\lambda)} = x\frac{\partial}{\partial x},$$

$$(144)$$

which corresponds to the classical Lie point symmetries since $\lambda_j$ is equal to zero.

**5.2. $f(y) = y$.** Another special form we consider here is $f(y) = y$. For this case we obtain the divergence of (16) in the form

$$\lambda_j = \frac{2y'}{y}, \qquad (145)$$

and by substituting $\lambda_j$ into the prolongation formula, the $\lambda$-infinitesimals can be found as follows:

$$\xi^{(\lambda)} = \frac{c_1}{y^2}, \qquad \eta^{(\lambda)} = 0, \qquad (146)$$

and the corresponding generator is

$$X_1^{(\lambda)} = \frac{1}{y^2}\frac{\partial}{\partial x}. \qquad (147)$$

which is a new $\lambda$-symmetry.

**5.3. $f(y) = ke^{\alpha y}$.** For this case of $f(y)$ the divergence of (16) gives

$$\lambda_j = 2\alpha y', \qquad (148)$$

and the corresponding $\lambda$-infinitesimals are

$$\xi^{(\lambda)} = e^{-3y\alpha}\left(c_1\cos x\alpha + c_2\sin x\alpha\right.$$
$$\left. + \frac{e^{y\alpha}}{\alpha}\left(2c_4\alpha - c_5\cos 2x\alpha + c_6\sin 2x\alpha\right)\right),$$

$$\eta^{(\lambda)} = e^{-3y\alpha}\left(c_1\sin x\alpha - c_2\cos x\alpha\right.$$
$$+ e^{2y\alpha}\left(c_7\cos x\alpha + c_8\sin x\alpha\right)$$
$$\left. + e^{y\alpha}\left(c_3 - \frac{1}{2\alpha}c_6\cos 2x\alpha + c_5\sin 2x\alpha\right)\right),$$

$$(149)$$

while the corresponding new $\lambda$-symmetries are found to be as follows

$$X_1^{(\lambda)} = e^{-3y\alpha}\cos x\alpha\frac{\partial}{\partial x} + e^{-3y\alpha}\sin x\alpha\frac{\partial}{\partial y},$$

$$X_2^{(\lambda)} = e^{-3y\alpha}\sin x\alpha\frac{\partial}{\partial x} - e^{-3y\alpha}\cos x\alpha\frac{\partial}{\partial y},$$

$$X_3^{(\lambda)} = e^{-2y\alpha}\frac{\partial}{\partial y}, \qquad X_4^{(\lambda)} = e^{-2y\alpha}\frac{\partial}{\partial x},$$

$$X_5^{(\lambda)} = -e^{-2y\alpha}\frac{\cos 2x\alpha}{2\alpha}\frac{\partial}{\partial x} - e^{-2y\alpha}\frac{\sin 2x\alpha}{2\alpha}\frac{\partial}{\partial y},$$

$$X_6^{(\lambda)} = e^{-2y\alpha}\frac{\sin 2x\alpha}{2\alpha}\frac{\partial}{\partial x} - e^{-2y\alpha}\frac{\cos 2x\alpha}{2\alpha}\frac{\partial}{\partial y},$$

$$X_7^{(\lambda)} = e^{-y\alpha}\cos x\alpha\frac{\partial}{\partial y}, \qquad X_8^{(\lambda)} = e^{-y\alpha}\sin x\alpha\frac{\partial}{\partial y}.$$

$$(150)$$

**5.4. $f(y) = 1/(my + n)$.** For this case we find that

$$\lambda_j = -\frac{2my'}{my + n}, \qquad (151)$$

and it is clear that $\lambda$-infinitesimal functions are

$$\xi^{(\lambda)} = (my + n)^2\left(c_1 m^2 x^3 + 3c_2 m^2 x^2 + 2c_3 x^2 + c_6 x + c_5\right)$$
$$+ (my + n)^4\left(c_1 x + c_2\right),$$

$$\eta^{(\lambda)} = \frac{1}{2}(my + n)$$
$$\times\left(2c_7 + \frac{c_1(my + n)^2\left(m^2 y^2 + 2mny - n^2\right)}{m}\right.$$
$$+ 2(my + n)^2\left(a_4 + a_3 x\right)$$
$$- x\left(-2c_8 + mx\left(2c_6 - 2mc_4\right.\right.$$
$$+ 2a_1 n^2 + 2c_3 mx$$
$$\left.\left.\left. + 4c_2 m^2 x + c_1 m^2 x^2\right)\right)\right).$$

$$(152)$$

Therefore, we find new $\lambda$-symmetries as follows:

$$X_1^{(\lambda)} = (my + n)^4 x + m^2(my + n)x^3\frac{\partial}{\partial x} + \frac{1}{2}(my + n)$$
$$\times\left(\frac{(my + n)^2\left(-n^2 + 2mny + m^2 y^2\right)}{m}\right.$$
$$\left. - mx^2\left(2n^2 + m^2 x^2\right)\right)\frac{\partial}{\partial y},$$

$$X_2^{(\lambda)} = \left((my + n)^4 + 3m^2(my + n)^2 x^2\right)\frac{\partial}{\partial x}$$
$$- 2m^3(my + n)x^3\frac{\partial}{\partial y},$$

$$X_3^{(\lambda)} = 2m(my + n)^2 x^2\frac{\partial}{\partial x}$$
$$+ (my + n)\left((my + n)^2 x - m^2 x^3\right)\frac{\partial}{\partial y},$$

$$X_4^{(\lambda)} = (my + n)\left((my + n)^2 + m^2 x^2\right)\frac{\partial}{\partial y},$$

$$X_5^{(\lambda)} = (my + n)^2 \frac{\partial}{\partial x},$$

$$X_6^{(\lambda)} = (my + n)^2 \frac{\partial}{\partial x} - m (my + n) x^2 \frac{\partial}{\partial y},$$

$$X_7^{(\lambda)} = (my + n) \frac{\partial}{\partial x}, \qquad X_8^{(\lambda)} = (my + n) \frac{\partial}{\partial x}.$$

$$(153)$$

### 5.5. $f(y) = y^n$, $n \neq 1/3, 1/2, 1$.

The divergence of the path equation yields

$$\lambda_j = \frac{2ny'}{y}.$$

$$(154)$$

If we substitute (154) $\lambda_j$ into (135), we obtain $\lambda$-infinitesimals

$$\xi^{(\lambda)} = c_1 y^{((\sqrt{(n-1)^2(3n-1)}/\sqrt{3n-1}) + (1/2)(1-5n - \sqrt{(n-1)^2(3n-1)}/\sqrt{3n-1}))},$$

$$\eta^{(\lambda)} = 0$$

$$(155)$$

and the corresponding one-parameter $\lambda$-generator

$$X^{(\lambda)} = y^{((\sqrt{(n-1)^2(3n-1)}/\sqrt{3n-1}) + (1/2)(1-5n - \sqrt{(n-1)^2(3n-1)}/\sqrt{3n-1}))} \frac{\partial}{\partial x}.$$

$$(156)$$

It is clear that we should analyze two specific values for $n$.

*Case 1 ($n = 1/3$).* The divergence of path equation for this value of $n$ is

$$\lambda_j = \frac{2y'}{3y},$$

$$(157)$$

the $\lambda$-infinitesimals can be written as

$$\xi^{(\lambda)} = -\frac{3c_1}{2y^{2/3}}, \qquad \eta^{(\lambda)} = 0,$$

$$(158)$$

and the $\lambda$-generator is

$$X^{(\lambda)} = -\frac{3}{2y^{2/3}} \frac{\partial}{\partial x}.$$

$$(159)$$

*Case 2 ($n = 1/2$).* For another specific value of $n$ the divergence is

$$\lambda_j = \frac{y'}{y},$$

$$(160)$$

the $\lambda$-infinitesimals are found as follows:

$$\xi^{(\lambda)} = \frac{c_1}{y}, \qquad \eta^{(\lambda)} = 0.$$

$$(161)$$

and the $\lambda$-generator is

$$X^{(\lambda)} = \frac{1}{y} \frac{\partial}{\partial x},$$

$$(162)$$

In summary all new $\lambda$-symmetries are presented in Table 3.

### 5.6. Invariant Solutions.

In this section we present some invariant solutions based on Jacobi multiplier approach.

*Case 1.* For the case $f(y) = ke^{\alpha y}$ we can investigate $X_1^\lambda$ to find the invariant solution of path equation. The first prolongation of $X_1^\lambda$ is

$$\mathrm{Pr}X_1^\lambda = e^{-y\alpha} \cos x\alpha \partial_y + e^{-y\alpha} (\alpha y' \cos x\alpha - \alpha \sin x\alpha) \partial_{y'},$$

$$(163)$$

and the Lagrange equations are

$$\frac{dx}{0} = \frac{dy}{e^{-y\alpha} \cos x\alpha} = \frac{dy'}{e^{-y\alpha} (\alpha y' \cos x\alpha - \alpha \sin x\alpha)},$$

$$(164)$$

gives the first order invariants

$$\tilde{x} = x, \qquad \tilde{y} = \frac{y' - \tan x\alpha}{e^{y\alpha}},$$

$$(165)$$

that replaced into path equation generate the first-order equation

$$\frac{d\tilde{y}}{d\tilde{x}} = \tilde{y}\alpha \tan x\alpha,$$

$$(166)$$

the solution of this equation yields

$$\tilde{y} \cos \tilde{x}\alpha = c_1,$$

$$(167)$$

and the first integral is

$$D_x \left( \left( \frac{y' - \tan x\alpha}{e^{y\alpha}} \right) \cos x\alpha \right) = 0;$$

$$(168)$$

this equality gives the original path equation (16). The reduced form of path equation is

$$\left( \frac{y' - \tan x\alpha}{e^{y\alpha}} \right) \cos x\alpha - c = 0,$$

$$(169)$$

in which the solution of (169) is

$$y(x) = -\frac{1}{\alpha} \ln \left( c\alpha^3 \cos x\alpha \left( -c_1 - \frac{\tan x\alpha}{\alpha^3} \right) \right),$$

$$(170)$$

where $c_1$ and $c$ are constants. It is clear that (170) is similar to the solutions (48) and (122). If we apply similar process for the $X_2$ symmetry generator, we obtain first-order invariants for this case as

$$\tilde{x} = x, \qquad \tilde{y} = \frac{y' + \cot x\alpha}{e^{y\alpha}},$$

$$(171)$$

and the first integral is

$$D_x \left( \left( \frac{y' + \cot x\alpha}{e^{y\alpha}} \right) \sin x\alpha \right) = 0;$$

$$(172)$$

another reduced form of path equation (16) is

$$\left( \frac{y' + \cot x\alpha}{e^{y\alpha}} \right) \sin x\alpha - c = 0.$$

$$(173)$$

TABLE 3: $\lambda$-Symmetry classification table of path equation with Jacobi last multiplier.

| Function | $\lambda$-Symmetries with Jacobi last multiplier |
|---|---|
| $f(y) = k$ | $\xi^{(\lambda)} = y(c_2 + c_1 x) + c_3 x^2 + c_8 x + c_7$ |
|  | $\eta^{(\lambda)} = y(c_4 + c_3 x) + c_5 x + c_1 y^2 + c_6$ |
| $f(y) = y$ | $\xi^{(\lambda)} = \dfrac{c_1}{y^2}, \; \eta^{(\lambda)} = 0$ |
| $f(y) = ke^{\alpha y}$ | $\xi^{(\lambda)} = e^{-3y\alpha}(c_1 \cos x\alpha + c_2 \sin x\alpha + \dfrac{e^{y\alpha}}{\alpha}(2c_4\alpha - c_5 \cos 2x\alpha + c_6 \sin 2x\alpha))$ |
|  | $\eta^{(\lambda)} = e^{-3y\alpha}(c_1 \sin x\alpha - c_2 \cos x\alpha + e^{2y\alpha}(c_7 \cos x\alpha + c_8 \sin x\alpha)$ |
|  | $+ e^{y\alpha}(c_3 - \dfrac{1}{2\alpha}c_6 \cos 2x\alpha + c_5 \sin 2x\alpha))$ |
| $f(y) = \dfrac{1}{my + n}$ | $\xi^{(\lambda)} = (my + n)^2(c_1 m^2 x^3 + 3c_2 m^2 x^2 + 2c_3 x^2 + c_6 x + c_5) + (my + n)^4(c_1 x + c_2)$ |
|  | $\eta^{(\lambda)} = \dfrac{1}{2}(my + n)(2c_7 + \dfrac{c_1(my + n)^2(m^2 y^2 + 2mny - n^2)}{m} + 2(my + n)^2 (a_4 + a_3 x))$ |
| $f(y) = y^n, n > 1$ | $\xi^{(\lambda)} = c_1 y^{(\sqrt{(n-1)^2(3n-1)}/\sqrt{3n-1})+(1/2)(1-5n-(\sqrt{(n-1)^2(3n-1)}/\sqrt{3n-1}))}, \; \eta^{(\lambda)} = 0$ |
| $f(y) = y^n, n = 1/3$ | $\xi^{(\lambda)} = -\dfrac{3c_1}{2 y^{2/3}}, \; \eta^{(\lambda)} = 0$ |
| $f(y) = y^n, n = 1/2$ | $\xi^{(\lambda)} = \dfrac{c_1}{y}, \; \eta^{(\lambda)} = 0$ |

The solution of (173) is given by

$$y(x) = -\frac{1}{\alpha} \ln \left( c\alpha^3 \sin x\alpha \left( -c_1 - \frac{\cot x\alpha}{\alpha^3} \right) \right). \quad (174)$$

*Case 2.* As another case $f(y) = 1/(my + n)$, we can analyze $X_4^\lambda$ generator to find the invariant solution of path equation. The first prolongation of $X_4^\lambda$ is written as

$$\text{Pr}X_4^\lambda = (my + n) \left( (my + n)^2 + m^2 x^2 \right) \partial_y$$

$$+ m \left( 2mx (my + n) \right. \quad (175)$$

$$\left. + \left( n^2 + 2mny + m^2 \left( y^2 - x^2 \right) \right) y' \right) \partial_{y'}$$

and the Lagrange equation are

$$\frac{dx}{0} = \frac{dy}{(my + n) \left( (my + n)^2 + m^2 x^2 \right)}$$

$$= \frac{dy'}{m \left( 2mx (my + n) + (n^2 + 2mny + m^2 (y^2 - x^2)) y' \right)}, \quad (176)$$

and the corresponding first-order invariants become

$$\widetilde{x} = x, \qquad \widetilde{y} = \frac{y' (my + n) + mx}{n^2 + m^2 x^2 + 2mny + m^2 y^2}, \quad (177)$$

that replaced into path equation generate the first order equation

$$\frac{d\widetilde{y}}{d\widetilde{x}} + 2m\widetilde{y}^2 = 0. \quad (178)$$

The solution of (178) is

$$\frac{2m\widetilde{x}\widetilde{y} - 1}{\widetilde{y}} = c_1, \quad (179)$$

and the first integral is

$$D_x \left( \frac{m^2 x^2 - n^2 - m^2 y^2 + 2mnxy' + 2my \left( mxy' - n \right)}{y' (my + n) + mx} \right) = 0, \quad (180)$$

which is equal to the original path equation (16). The new reduced form is

$$\frac{m^2 x^2 - n^2 - m^2 y^2 + 2mnxy' + 2my \left( mxy' - n \right)}{y' (my + n) + mx} - c = 0, \quad (181)$$

and the solution of (181) is

$$y(x) = \frac{1}{2m \sqrt{1/m(c - 2mx)^2} (2mx - c)}$$

$$\times \left( 2cn \sqrt{\frac{1}{m(c - 2mx)^2}} - 4mnx \sqrt{\frac{1}{m(c - 2mx)^2}} \right.$$

$$\left. \pm \sqrt{-\frac{2c^2}{m} + 4cx - 4mx^2 + 4mc_1 - 8m^2 xc_1} \right), \quad (182)$$

where $c_1$ and c are constants, and it is clear that (182) is similar to the solution (105).

## 6. Conclusion

The aim of this study is to classify Noether and $\lambda$-symmetries of path equation describing the minimum drag work. The symmetry classification of the equation is analyzed with respect to different choices of altitude-dependent arbitrary function $f(y)$ of the governing equation, which represents a combination of the density, the drag coefficient, the cross sectional area, and the velocity. It is a fact that an ordinary differential equation should have a Lagrangian function to obtain Noether symmetries. In this study we consider the partial Lagrangian approach for obtaining Noether symmetries and constructing a classification in the problem. Thus, new first integrals (conserved forms) are obtained directly by using each Noether symmetry given by symmetry of the equation. With this point of view we find and classify the new forms of first integrals, and then the invariant solutions of path equation are constructed for specific forms of $f(y)$.

In the literature, as a different and a new concept, $\lambda$-symmetries of the second order ordinary differential equations are analyzed by assuming $\lambda$-function in the linear form. However, in our study, we prove that it is not possible to obtain $\lambda$-symmetries of the drag equation by selecting $\lambda$-function in a linear form. So we study another approach to obtain $\lambda$-symmetries based on using Lie point symmetries of the path equation. Thus, we have derived $\lambda$-symmetries, integrating factors, first integrals, and the reduced form of the original path equation. Based on using these new $\lambda$-symmetries, we present some new different invariant solutions by calculating new reduced forms and first integrals.

In our study, additionally, the Jacobi last multiplier concept is presented as a new and an alternative approach to construct $\lambda$-symmetries of the path equation algorithmically. In this method, first, $\lambda$-function is determined by taking divergence of the governing equation and then the infinitesimals functions $\xi$ and $\eta$ are determined from the determining equations, then we calculate new $\lambda$-symmetries. In this study we generate first-order equations by using these new symmetries, which provide invariant solutions of path equation. After all calculations we present that all methods discussed in this study have their own important properties to find first integrals and invariant solutions of ordinary differential equations, and the advantages of these approaches are given for specific cases. Furthermore, all symmetry classifications are presented in tables.

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