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Research Article

A Note on Some Best Proximity Point Theorems Proved under *P*-Property

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We show that some recent results concerning the existence of best proximity points can be obtained from the same results in fixed point theory.

1. Introduction

Let A and B be two nonempty subsets of a metric space (X, d). In this paper, we adopt the following notations and definitions:

$$\begin{aligned} D(x, B) &:= \inf \{d(x, y) : y \in B\}, \quad \forall x \in X, \\ A_0 &:= \{x \in A : d(x, y) = \operatorname{dist}(A, B), \text{ for some } y \in B\}, \\ B_0 &:= \{y \in B : d(x, y) = \operatorname{dist}(A, B), \text{ for some } x \in A\}. \end{aligned} \tag{1}$$

The notion of *best proximity point* is defined as follows.

Definition 1. Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$ a non-self-mapping. A point $x^* \in A$ is called a best proximity point of T if $d(x^*, Tx^*) = \operatorname{dist}(A, B)$, where

$$\operatorname{dist}(A, B) := \inf \{d(x, y) : (x, y) \in A \times B\}. \tag{2}$$

Similarly, for a multivalued non-self-mapping $T : A \to 2^B$, where (A, B) is a nonempty pair of subsets of a metric space (X, d), a point $x^* \in A$ is a best proximity point of T provided that $D(x^*, Tx^*) = \operatorname{dist}(A, B)$.

Recently, the notion of P-property was introduced in [1] as follows.

Definition 2 (see [1]). Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have P-property if and only if

$$\begin{matrix} d(x_1, y_1) = \operatorname{dist}(A, B) \\ d(x_2, y_2) = \operatorname{dist}(A, B) \end{matrix} \Longrightarrow d(x_1, x_2) = d(y_1, y_2), \tag{3}$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

By using this notion, some best proximity point results were proved for various classes of non-self-mappings. Here, we state some of them.

Theorem 3 (see [1]). *Let (A, B) be a pair of nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a weakly contractive non-self-mapping; that is,*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \quad \forall x, y \in A, \tag{4}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ϕ is positive on $(0, \infty)$, $\phi(0) = 0$, and $\lim_{t\to\infty} \phi(t) = \infty$. Assume that the pair (A, B) has the P-property and $T(A_0) \subseteq B_0$. Then, T has a unique best proximity point.

Theorem 4 (see [2]). *Let (A, B) be a pair of nonempty closed subsets of a Banach space X such that A is compact and A_0 is nonempty. Let $T : A \to B$ be a nonexpansive mapping; that is,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in A. \tag{5}$$

Assume that the pair (A, B) has the P-property and $T(A_0) \subseteq B_0$. Then, T has a best proximity point.

Theorem 5 (see [3]). *Let (A, B) be a pair of nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a Meir-Keeler non-self-mapping; that is, for all $x, y \in A$ and for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies } d(Tx, Ty) \leq \varepsilon. \tag{6}$$

Assume that the pair (A, B) has the P-property and $T(A_0) \subseteq B_0$. Then, T has a unique best proximity point.

Theorem 6 (see [4]). *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) satisfies the P-property. Let $T : A \to 2^B$ be a multivalued contraction non-self-mapping; that is,*

$$H(Tx, Ty) \leq \alpha d(x, y), \tag{7}$$

for some $\alpha \in (0, 1)$ and for all $x, y \in A$. If Tx is bounded and closed in B for all $x \in A$ and Tx_0 is included in B_0 for each $x_0 \in A_0$, then T has a best proximity point in A.

Theorem 7 (see [5]). *Let (A, B) be a pair of nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a Geraghty-contraction non-self-mapping; that is,*

$$d(Tx, Ty) \leq \beta(d(x, y)), d(x, y), \quad \forall x, y \in A, \tag{8}$$

where $\beta : [0, \infty) \to [0, 1)$ is a function which satisfies the following condition:

$$\beta(t_n) \longrightarrow 1 \Longrightarrow t_n \longrightarrow 0. \tag{9}$$

Suppose that the pair (A, B) has the P-property and $T(A_0) \subseteq B_0$. Then, T has a unique best proximity point.

2. Main Result

In this section, we show that the existence of a best proximity point in the main theorems of [1–5] can be obtained from the existence of the fixed point for a self-map. We begin our argument with the following lemmas.

Lemma 8 (see [6]). *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and (A, B) has the P-property. Then, (A_0, B_0) is a closed pair of subsets of X.*

Lemma 9. *Let (A, B) be a pair of nonempty closed subsets of a metric space (X, d) such that A_0 is nonempty. Assume that the pair (A, B) has the P-property. Then there exists a bijective isometry $g : A_0 \to B_0$ such that $d(x, gx) = \operatorname{dist}(A, B)$.*

Proof. Let $x \in A_0$; then there exists an element $y \in B_0$ such that

$$d(x, y) = \operatorname{dist}(A, B). \tag{10}$$

Assume that there exists another point $\acute{y} \in B_0$ such that

$$d(x, \acute{y}) = \operatorname{dist}(A, B). \tag{11}$$

By the fact that (A, B) has the P-property, we conclude that $y = \acute{y}$. Consider the non-self-mapping $g : A_0 \to B_0$ such that $d(x, gx) = \operatorname{dist}(A, B)$. Clearly, g is well defined. Moreover, g is an isometry. Indeed, if $x_1, x_2 \in A_0$, then

$$d(x_1, gx_1) = \operatorname{dist}(A, B), \qquad d(x_2, gx_2) = \operatorname{dist}(A, B). \tag{12}$$

Again, since (A, B) has the P-property,

$$d(x_1, x_2) = d(gx_1, gx_2); \tag{13}$$

that is, g is an isometry. □

Here, we prove that the existence and uniqueness of the best proximity point in Theorem 3 are a sample result of the existence of fixed point for a weakly contractive self-mapping.

Theorem 10. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a weakly contractive mapping. Assume that the pair (A, B) has the P-property and $T(A_0) \subseteq B_0$. Then, T has a unique best proximity point.*

Proof. Consider the bijective isometry $g : A_0 \to B_0$ as in Lemma 9. Since $T(A_0) \subseteq B_0$, for the self-mapping $g^{-1}T : A_0 \to A_0$, we have

$$d\left(g^{-1}(Tx), g^{-1}(Ty)\right) = d(Tx, Ty) \leq \varphi(d(x, y)), \tag{14}$$

for all $x, y \in A_0$ which implies that the self-mapping $g^{-1}T$ is weakly contractive. Note that A_0 is closed by Lemma 8. Thus, $g^{-1}T$ has a unique fixed point [7]. Suppose that $x^* \in A_0$ is a unique fixed point of the self-mapping $g^{-1}T$; that is, $g^{-1}T(x^*) = x^*$. So, $Tx^* = gx^*$, and then

$$d(x^*, Tx^*) = d(x^*, gx^*) = \operatorname{dist}(A, B), \tag{15}$$

from which it follows that $x^* \in A_0$ is a unique best proximity point of the non-self weakly contractive mapping T. □

Remark 11. By a similar argument, using the fact that every nonexpansive self-mapping defined on a nonempty compact and convex subset of a Banach space has a fixed point, we conclude Theorem 4. Also, the existence and uniqueness of best proximity point for Meir-Keeler non-self-mapping T (Theorem 5) follow from the Meir-Keeler's fixed point theorem ([8]). Moreover, in Theorem 6, Nadler's fixed point theorem ([9]) ensures the existence of a best proximity point for multivalued non-self mapping T. Finally, Theorem 7 due to Caballero et al., is obtained from Geraghty's fixed point theorem ([10]).

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