

Research Article

Existence Results for a Class of Fractional Differential Equations with Periodic Boundary Value Conditions and with Delay

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We discuss the existence and uniqueness of solution for two types of fractional order ordinary and delay differential equations. Fixed point theorems are the main tool used here to establish the existence and uniqueness results. First we use Banach contraction principle to prove the uniqueness of solution and then Krasnoselskii's fixed point theorem to show the existence of the solution under certain conditions in a Banach space.

1. Introduction

In mathematics delay differential equations are a type of differential equation in which the derivative of unknown function at a certain time is given in terms of the values of the function at previous times.

While physical events such as acceleration and deceleration take little time compared to the times needed to travel most distances, times involved in biological processes such as gestation and maturation can be substantial when compared to the data-collection times in most population studies. Therefore, it is often imperative to explicitly incorporate these process times into mathematical models of population dynamics. These process times are often called delay times, and the models that incorporate such delay times are referred as delay differential equation models [1, 2].

Recently theory of fractional differential equations attracted many scientists and mathematicians to work on them [3–12]. For the existence of solutions for fractional differential equations, one can see [13–30] and references therein. The results have been obtained by using fixed point theorems like Picard's, Schauder fixed-point theorem, and Banach contraction mapping principle. About the development of existence theorems for fractional functional differential equations, many publications exist [31–35]. Many

applications of fractional calculus amount to replacing the time derivative in a given evolution equation by a derivative of fractional order. The results of several studies clearly stated that the fractional derivatives seem to arise generally and universally from important mathematical reasons. Recently, interesting attempts have been made to give physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives which could be found in [8, 9, 36, 37].

Recently Benchohra et al. [28] studied existence of solutions for a class of fractional differential equations with infinite delay; namely,

$$\begin{aligned} D^\alpha u(t) &= u(t) + f(t, u_t), \quad \text{for each } t \in [0, b], \quad 0 < \alpha < 1, \\ u(t) &= \phi(t), \quad t \in [-\infty, 0], \end{aligned} \quad (1)$$

where D^α is the standard Riemann-Liouville fractional derivative and f satisfies some assumptions.

First, in this paper we consider nonlinear delayed fractional differential equations:

$$\begin{aligned} D^\alpha u(t) &= u(t) + f(t, u_t), \quad t \in (0, 1), \quad 0 < \alpha < 1, \\ u(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned} \quad (2)$$

associated with boundary conditions

$$u(0) = \lim_{t \rightarrow 0^+} t^{1-\alpha} u(t) = u(1), \quad (3)$$

where D^α is the standard Riemann-Liouville fractional derivative and f is a continuous function. Here $u_t(\cdot)$ represents the properitoneal state from time $-\tau$ up to time t which is defined by $u_t(\theta) = u(t + \theta)$, $-\tau \leq \theta \leq 0$. We proved the uniqueness of existence solutions for (2) with periodic boundary condition (3) under some further conditions.

For investigating to establish an existence theorem, we also consider a class of nonlinear delayed fractional differential equations of the form

$$\begin{aligned} D^\alpha u(t) &= u(t) + f(t, u(t), u(t - \tau)), \quad t \in (0, 1), \\ u(t) &= \phi(t), \quad t \in [-\tau, 0], \quad 0 < \alpha < 1, \end{aligned} \quad (4)$$

with periodic boundary condition

$$\begin{aligned} u(0) &= \lim_{t \rightarrow 0^+} t^{1-\alpha} u(t) = c = u(1), \\ D^{-(1-\alpha)} u(t) \Big|_{t=0} &= c \Gamma(\alpha). \end{aligned} \quad (5)$$

The paper has been organized as follows. In Section 2 we give basic definitions and preliminary. Unique solution of (2)-(3) under some conditions is proved in Section 3. The existence solution of (4)-(5) under some assumptions has been presented in Section 4.

2. Preliminaries

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory and functional analysis. These definitions and results can be found in the literature [3, 7, 38].

Let $C[0, 1]$ be the Banach space of all continuous real functions defined on $[0, 1]$ with the norm

$$\|u\| =: \max \{|u(t)| : t \in [0, 1]\}. \quad (6)$$

Let $C_r[0, 1]$, $r \geq 0$, be the space of all functions f such that $t^r u(t) \in C[0, 1]$ which is a Banach space when endowed with the norm

$$\|u\|_r =: \max \{t^r |u(t)| : t \in [0, 1]\}. \quad (7)$$

Definition 1. For a function u defined on an interval $[a, b]$, the Riemann-Liouville fractional integral of u of order $\alpha > 0$ is defined by

$$D_{a^+}^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad t > a, \quad (8)$$

and Riemann-Liouville fractional derivative of $u(t)$ of order $\alpha > 0$ defined by

$$D_{a^+}^\alpha u(t) = \frac{d^n}{dt^n} \{D_{a^+}^{-(\alpha-n)} u(t)\}, \quad n-1 < \alpha \leq n, \quad (9)$$

provided that the right-hand side of the pervious equation is pointwise defined on $(a, +\infty)$.

We denote $D_{a^+}^\alpha u(t)$ as $D_a^\alpha y(t)$ and $D_{a^+}^{-\alpha} u(t)$ as $D_a^{-\alpha} u(t)$. Further $D_{0^+}^\alpha u(t)$ and $D_{0^+}^{-\alpha} u(t)$ are referred as $D^\alpha u(t)$ and $D^{-\alpha} u(t)$, respectively.

Definition 2. A two-parameter function of the Mittag-Leffler type is defined by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta > 0). \quad (10)$$

Definition 3. The beta function is usually defined by

$$\begin{aligned} B(z, w) &= \int_0^1 \tau^{z-1} (1-\tau)^{w-1} d\tau, \\ (\operatorname{Re}(z) > 0, \operatorname{Re}(w) > 0). \end{aligned} \quad (11)$$

and we have also the following expression for the beta function:

$$B(z, w) = \frac{\Gamma(z) \Gamma(w)}{\Gamma(z+w)}. \quad (12)$$

Theorem 4 (Arzela-Ascoli's theorem). *A subset of $C[a, b]$ is compact if and only if it is closed, bounded, and equicontinuous.*

Theorem 5 (Banach's fixed point theorem). *Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is complete and $T : X \rightarrow X$ is a contraction on X . Then T has precisely one fixed point.*

Theorem 6 (Krasnoselskii's fixed point theorem). *Let B be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that T_1 , and T_2 map B into X such that*

- (1) *for any $x, y \in B$ we have $T_1 x + T_2 y \in B$,*
- (2) *T_1 is a contraction,*
- (3) *T_2 is continuous and $T_2(B)$ is contained in a compact set.*

Then there exists $z \in B$ such that $z = T_1 z + T_2 z$.

3. Uniqueness of Solution

In this section we prove (2) with boundary condition (3) and another condition on f has a unique solution. Before proving, we need to introduce some notations that will be provided in the following.

Let $\Omega = \{u : [-\tau, 1] \rightarrow \mathbb{R}, u \in C_{1-\alpha}[0, 1]\}$. Consider the operator $N : \Omega \rightarrow \Omega$ defined by

$$Nu(t) = \begin{cases} \phi(t), & t \in [-\tau, 0], \\ \int_0^1 G_{1,\alpha}(t, s) f(s, u_s) ds, & t \in (0, 1), \end{cases} \quad (13)$$

where

$$G_{1,\alpha}(t, s) = \begin{cases} \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t^\alpha) E_{\alpha,\alpha}((1-s)^\alpha) t^{\alpha-1} (1-s)^{\alpha-1}}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)} \\ + (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha), \\ 0 \leq s \leq t \leq 1, \\ \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t^\alpha) E_{\alpha,\alpha}((1-s)^\alpha) t^{\alpha-1} (1-s)^{\alpha-1}}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)}, \\ 0 \leq t \leq s \leq 1, \end{cases} \quad (14)$$

is given in Section 4. Let $x(\cdot) : [-\tau, 1] \rightarrow \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} 0, & t \in (0, 1), \\ \phi(t), & t \in [-\tau, 0]. \end{cases} \quad (15)$$

For each $z \in C_{1-\alpha}([0, 1], \mathbb{R})$ with $\lim_{t \rightarrow 0^+} t^{1-\alpha} z(t) = c$ we denote \bar{z} the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & t \in (0, 1), \\ 0, & t \in [-\tau, 0]. \end{cases} \quad (16)$$

If $u(\cdot)$ satisfies the integral equation,

$$u(t) = \int_0^1 G_{1,\alpha}(t, s) f(s, u_s) ds, \quad (17)$$

we can decompose $u(\cdot)$ as $u(t) = \bar{z}(t) + x(t)$, $0 < t < 1$, which implies $u_t = \bar{z}_t + x_t$ for every $0 < t < 1$, and function $z(\cdot)$ satisfies

$$z(t) = \int_0^1 G_{1,\alpha}(t, s) f(s, \bar{z}_s + x_s) ds. \quad (18)$$

Set $C_0 = \{z \in (C_{1-\alpha}[0, 1], \mathbb{R}) : \lim_{t \rightarrow 0^+} t^{1-\alpha} z(t) = c\}$. C_0 is Banach space with the norm $\|\cdot\|_r$. Let $T : C_0 \rightarrow C_0$ be defined by

$$Tz(t) = \int_0^1 G_{1,\alpha}(t, s) f(s, \bar{z}_s + x_s) ds, \quad 0 < t < 1. \quad (19)$$

Note that as operator N has a fixed point, equivalently T has a fixed point and so instead we try to prove that T has a fixed point.

Theorem 7. Assume that there exists a constant $K > 0$ such that $|f(t, u) - f(t, v)| \leq K|u - v|$ for each $t \in [0, 1]$ and all $u, v \in \mathbb{R}$. Then the problem (2)-(3) has a unique solution in $C_0 \subseteq C_{1-\alpha}[0, 1]$ provided that

$$K \left(\frac{\Gamma(\alpha) E_{\alpha,\alpha}(1)}{|1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)|} + 1 \right) E_{\alpha,\alpha}(1) \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)} < 1. \quad (20)$$

Proof. We prove that $T : C_0 \rightarrow C_0$ is a contraction map. For each $t \in [0, 1]$ and for $z_1, z_2 \in C_0$ we have

$$\begin{aligned} & t^{1-\alpha} |Tz_1(t) - Tz_2(t)| \\ & \leq t^{1-\alpha} \int_0^1 |G_{1,\alpha}(t, s)| \\ & \quad \times |f(s, \bar{z}_{1s} + x_s) - f(s, \bar{z}_{2s} + x_s)| ds, \\ & \leq K t^{1-\alpha} \int_0^1 |G_{1,\alpha}(t, s)| |\bar{z}_{1s} - \bar{z}_{2s}| ds; \end{aligned} \quad (21)$$

using the definition of $\|\cdot\|_{1-\alpha}$ we get

$$\begin{aligned} & \|Tz_1(t) - Tz_2(t)\|_{1-\alpha} \\ & \leq K \max_{t \in [0, 1]} \left\{ t^{1-\alpha} \int_0^1 |G_{1,\alpha}(t, s)| s^{\alpha-1} ds \right\} \\ & \quad \times \|z_1 - z_2\|_{1-\alpha}. \end{aligned} \quad (22)$$

Moreover,

$$\begin{aligned} & t^{1-\alpha} \int_0^1 |G_{1,\alpha}(t, s)| s^{\alpha-1} ds \\ & \leq \left(\frac{\Gamma(\alpha) E_{\alpha,\alpha}(1)}{|1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)|} + 1 \right) E_{\alpha,\alpha}(1) \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}. \end{aligned} \quad (23)$$

Indeed we have

$$\begin{aligned} & t^{1-\alpha} \int_0^1 |G_{1,\alpha}(t, s)| s^{\alpha-1} ds \\ & \leq \frac{\Gamma(\alpha) |E_{\alpha,\alpha}(t^\alpha)|}{|1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)|} \\ & \quad \times \sum_{i=0}^{\infty} \frac{1}{\Gamma(\alpha i + \alpha)} \int_0^1 (1-s)^{\alpha-1} s^{\alpha-1} ds \\ & \quad + t^{\alpha-1} \sum_{i=0}^{\infty} \frac{1}{\Gamma(\alpha i + \alpha)} \\ & \quad \times \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds \frac{\Gamma(\alpha) |E_{\alpha,\alpha}(t^\alpha)|}{|1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)|} \\ & \quad \times E_{\alpha,\alpha}(1) \int_0^1 (1-s)^{\alpha-1} s^{\alpha-1} ds \\ & \quad + t^{1-\alpha} E_{\alpha,\alpha}(1) \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds. \end{aligned} \quad (24)$$

Note that

$$\int_0^t s^{\alpha-1} (t-s)^{\alpha-1} ds = t^{2\alpha-1} \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}. \quad (25)$$

Hence we have

$$\begin{aligned}
 & t^{1-\alpha} \int_0^1 |G_{1,\alpha}(t, s)| s^{\alpha-1} ds \\
 & \leq \frac{\Gamma(\alpha) |E_{\alpha,\alpha}(t^\alpha)|}{|1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)|} E_{\alpha,\alpha}(1) \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)} \\
 & \quad + t^{1-\alpha} E_{\alpha,\alpha}(1) t^{2\alpha-1} \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)} \\
 & = \left(\frac{\Gamma(\alpha) |E_{\alpha,\alpha}(t^\alpha)|}{|1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)|} + t^\alpha \right) E_{\alpha,\alpha}(1) \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}.
 \end{aligned} \tag{26}$$

Using (22) and (23) we get

$$\begin{aligned}
 \|Tz_1 - Tz_2\|_{1-\alpha} & \leq K \left(\frac{\Gamma(\alpha) E_{\alpha,\alpha}(1)}{|1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)|} + 1 \right) \\
 & \quad \times E_{\alpha,\alpha}(1) \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)} \|z_1 - z_2\|_{1-\alpha}.
 \end{aligned} \tag{27}$$

This completes the proof. \square

4. Existence of Solution

In this section, by using Krasnoselskii's theorem, we discuss the existence solution of (4) under some assumptions on f and further conditions. Before proving this theorem, we prove the following lemma which will be used in the next theorem.

Lemma 8. Consider the following nonlinear fractional differential equation of the form

$$D^\alpha u(t) = u(t) + h(t), \quad 0 < t < 1, \quad 0 < \alpha < 1, \tag{28}$$

with periodic boundary conditions

$$\begin{aligned}
 u(0) &= \lim_{t \rightarrow 0^+} t^{1-\alpha} u(t) = c = u(1), \\
 D^{-(1-\alpha)} u(t) \Big|_{t=0} &= c \Gamma(\alpha),
 \end{aligned} \tag{29}$$

where h is a continuous function. Then the periodic boundary value problem (28)-(29) is equivalent to an integral equation given by $u(t) = \int_0^1 G_{1,\alpha}(t, s) h(s) ds \in C_{1-\alpha}[0, 1]$, where

$$G_{1,\alpha}(t, s) = \begin{cases} \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t^\alpha) E_{\alpha,\alpha}((1-s)^\alpha) t^{\alpha-1} (1-s)^{\alpha-1}}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)} \\ \quad + (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha), & 0 \leq s \leq t \leq 1, \\ \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t^\alpha) E_{\alpha,\alpha}((1-s)^\alpha) t^{\alpha-1} (1-s)^{\alpha-1}}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{30}$$

Proof. We consider the following fractional differential equation:

$$D^\alpha y(t) - y(t) = h(t), \quad (0 < t < 1, \quad 0 < \alpha < 1), \tag{31}$$

with

$$[D^{\alpha-1} y(t)]_{t=0} = b_1 = c \Gamma(\alpha), \tag{32}$$

where $\lim_{t \rightarrow 0^+} t^{1-\alpha} y(t) = c$. Laplace transform of (31) yields

$$s^\alpha Y(s) - Y(s) = H(s) + b_1 \tag{33}$$

from which

$$Y(s) = \frac{H(s)}{s^\alpha - 1} + \frac{b_1}{s^\alpha - 1} \tag{34}$$

and the inverse Laplace transform gives the solution

$$\begin{aligned}
 y(t) &= b_1 t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha) \\
 & \quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s)^\alpha h(s) ds.
 \end{aligned} \tag{35}$$

Hence we have

$$\begin{aligned}
 y(t) &= c \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha) \\
 & \quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s)^\alpha h(s) ds.
 \end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned}
 y(1) &= c \Gamma(\alpha) + E_{\alpha,\alpha}(1) \\
 & \quad + \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}((1-s)^\alpha) h(s) ds,
 \end{aligned} \tag{37}$$

which leads to

$$\begin{aligned}
 & c(1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)) \\
 & = \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}((1-s)^\alpha) h(s) ds;
 \end{aligned} \tag{38}$$

since $\Gamma(\alpha) E_{\alpha,\alpha}(1) \neq 1$ we have

$$c = \frac{1}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)} \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}((1-s)^\alpha) h(s) ds. \tag{39}$$

Then the solution of the problem (28)-(29) is given by

$$\begin{aligned}
 y(t) &= \frac{\Gamma(\alpha)}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)} t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha) \\
 & \quad \times \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}((1-s)^\alpha) h(s) ds \\
 & \quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha) h(s) ds.
 \end{aligned} \tag{40}$$

This completes the proof. \square

Now we prove our main result using Lemma 8 and two more assumptions which follow next.

(H1) We assume that $f(t, u, v)$ can be written as $f_1(t, u) + f_2(t, u, v)$, where f_1, f_2 are Lipschitz continuous. Moreover assume that the function f_1, f_2 satisfies the following relations:

$$\begin{aligned} |f_1(t, u(t))| &\leq M_1 \|u(t)\|_{1-\alpha}, \\ |f_2(t, u(t), v(t))| &\leq M_2 \|u(t)\|_{1-\alpha} \|v(t)\|_{1-\alpha}. \end{aligned} \quad (41)$$

(H2) Let $\Omega = C_{1-\alpha}([-\tau, 1], \mathbb{R})$ denote collection of the space of all function u such that $t^{1-\alpha}u(t) \in C[0, 1]$. Define the set $B = \{u \in \Omega : t^{1-\alpha}|u| \leq r\}$, where r satisfies

$$\begin{aligned} &(M_1 \|u\|_{1-\alpha} + M_2 \|u\|_{1-\alpha}^2) \\ &\times \left(\frac{|\Gamma(\alpha) E_{\alpha, \alpha}(t^\alpha)|}{|1 - \Gamma(\alpha) E_{\alpha, \alpha}(1)|} E_{\alpha, \alpha+1}(1) + t E_{\alpha, \alpha+1}(t^\alpha) \right) \leq r. \end{aligned} \quad (42)$$

Furthermore we assume that

$$L_{f_1} \left(\frac{\Gamma(\alpha) E_{\alpha, \alpha}(1)}{|1 - \Gamma(\alpha) E_{\alpha, \alpha}(1)|} + 1 \right) E_{\alpha, \alpha}(1) \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)} < 1, \quad (43)$$

where L_{f_1} is a Lipschitz constant of f .

Theorem 9. *If the assumptions (H1) and (H2) satisfied, then the problem (4) with periodic boundary value condition (5) has at least one solution.*

Proof. (i) Note that by Lemma 8, (4)-(5) is equivalent to integral equation (17). Define $T_1, T_2 : \Omega \rightarrow \Omega$ by

$$\begin{aligned} T_1 u(t) &= \int_0^1 G_{1, \alpha}(t, s) f_1(s, u(s)) ds, \\ T_2 u(t) &= \int_0^1 G_{1, \alpha}(t, s) f_2(s, u(s), u(s - \tau)) ds. \end{aligned} \quad (44)$$

For $u \in B$ we have,

$$\begin{aligned} &t^{1-\alpha} |T_1 u(t) + T_2 u(t)| \\ &\leq t^{1-\alpha} \int_0^1 |G_{1, \alpha}(t, s)| \\ &\quad \times |f_1(s, u(s)) \\ &\quad + f_2(s, u(s), u(s - \tau))| ds \\ &\leq (M_1 \|u\|_{1-\alpha} + M_2 \|u\|_{1-\alpha}^2) t^{1-\alpha} \\ &\quad \times \int_0^1 |G_{1, \alpha}(t, s)| ds \end{aligned}$$

$$\begin{aligned} &\leq (M_1 \|u\|_{1-\alpha} + M_2 \|u\|_{1-\alpha}^2) \\ &\quad \times \left(\frac{|\Gamma(\alpha) E_{\alpha, \alpha}(t^\alpha)|}{|1 - \Gamma(\alpha) E_{\alpha, \alpha}(1)|} E_{\alpha, \alpha+1}(1) \right. \\ &\quad \left. + t E_{\alpha, \alpha+1}(t^\alpha) \right) \leq r. \end{aligned} \quad (45)$$

(ii) We will prove that T_1 is a contraction:

$$\begin{aligned} &\|T_1 u - T_1 v\|_{1-\alpha} \\ &= t^{1-\alpha} |T_1 u(t) - T_1 v(t)|, \\ &\leq t^{1-\alpha} \int_0^1 |G_{1, \alpha}(t, s)| \\ &\quad \times |f_1(s, u(s)) - f_1(s, v(s))| ds, \\ &\leq L_{f_1} t^{1-\alpha} \int_0^1 |G_{1, \alpha}(t, s)| |u - v| ds, \\ &\leq L_{f_1} t^{1-\alpha} \int_0^1 |G_{1, \alpha}(t, s)| s^{\alpha-1} s^{1-\alpha} |u - v| ds \\ &\leq L_{f_1} \left(\frac{\Gamma(\alpha) E_{\alpha, \alpha}(1)}{|1 - \Gamma(\alpha) E_{\alpha, \alpha}(1)|} + 1 \right) \\ &\quad \times E_{\alpha, \alpha}(1) \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)} \|u - v\|_{1-\alpha}. \end{aligned} \quad (46)$$

Then T_1 is a contraction.

(iii) Finally we prove that T_2 is continuous and $T_2(B)$ is contained in a compact set. To prove the continuity of T_2 let us consider a sequence u_n converging to u . Taking the norm of $T_2 u_n(t) - T_2 u(t)$ we have

$$\begin{aligned} &\|T_2 u_n(t) - T_2 u(t)\|_{1-\alpha} \\ &= t^{1-\alpha} |T_2 u_n(t) - T_2 u(t)| \\ &\leq t^{1-\alpha} \int_0^1 |G_{1, \alpha}(t, s)| \\ &\quad \times |f_2(s, u_n(s), u_n(s - \tau)) \\ &\quad - f_2(s, u(s), u(s - \tau))| ds \\ &\leq t^{1-\alpha} L_{f_2} \int_0^1 |G_{1, \alpha}(t, s)| \\ &\quad \times (|u_n(s) - u(s)| \\ &\quad + |u_n(s - \tau) - u(s - \tau)|) ds \\ &\leq 2L_{f_2} t^{1-\alpha} \int_0^1 |G_{1, \alpha}(t, s)| s^{\alpha-1} \\ &\quad \times ds \|u_n - u\|_{1-\alpha}. \end{aligned} \quad (47)$$

Hence whenever $u_n \rightarrow u$ we have $T_2 u_n \rightarrow T_2 u$. This proves the continuity of T_2 .

On the other hand for $0 \leq t_1 < t_2 \leq 1$ and $u \in B$ we have

$$\begin{aligned}
& |t_1^{1-\alpha} T_2 u(t_1) - t_2^{1-\alpha} T_2 u(t_2)| \\
&= \left| t_1^{1-\alpha} \int_0^1 G_{1,\alpha}(t_1, s) f_2(s, u(s), u(s-\tau)) ds \right. \\
&\quad \left. - t_2^{1-\alpha} \int_0^1 G_{1,\alpha}(t_2, s) f_2(s, u(s), u(s-\tau)) ds \right| \\
&\leq \left| \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t_1^\alpha)}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)} \right. \\
&\quad \times \int_0^{t_1} E_{\alpha,\alpha}((1-s)^\alpha) (1-s)^{\alpha-1} \\
&\quad \times f_2(s, u(s), u(s-\tau)) ds \\
&\quad + t_1^{1-\alpha} \int_0^{t_1} E_{\alpha,\alpha}((t_1-s)^\alpha) (t_1-s)^{\alpha-1} \\
&\quad \times f_2(s, u(s), u(s-\tau)) ds \\
&\quad - \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t_2^\alpha)}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)} \\
&\quad \times \int_0^{t_1} E_{\alpha,\alpha}((1-s)^\alpha) (1-s)^{\alpha-1} \\
&\quad \times f_2(s, u(s), u(s-\tau)) \\
&\quad - t_2^{1-\alpha} \int_0^{t_1} E_{\alpha,\alpha}((t_2-s)^\alpha) (t_2-s)^{\alpha-1} \\
&\quad \times f_2(s, u(s), u(s-\tau)) \\
&\quad + \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t_1^\alpha)}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)} \\
&\quad \times \int_{t_1}^{t_2} E_{\alpha,\alpha}((1-s)^\alpha) (1-s)^{\alpha-1} \\
&\quad \times f(s, u(s), u(s-\tau)) ds \\
&\quad - \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t_2^\alpha)}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)} \\
&\quad \times \int_{t_1}^{t_2} E_{\alpha,\alpha}((1-s)^\alpha) (1-s)^{\alpha-1} \\
&\quad \times f_2(s, u(s), u(s-\tau)) ds \\
&\quad - t_2^{1-\alpha} \int_{t_1}^{t_2} E_{\alpha,\alpha}(t_2-s)^\alpha (t_2-s)^{\alpha-1} \\
&\quad \times f_2(s, u(s), u(s-\tau)) ds \\
&\quad + \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t_1^\alpha)}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)}
\end{aligned}$$

$$\begin{aligned}
& \times \int_{t_2}^1 E_{\alpha,\alpha}((1-s)^\alpha) (1-s)^{\alpha-1} \\
& \quad \times f_2(s, u(s), u(s-\tau)) ds \\
& \quad - \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t_2^\alpha)}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)} \\
& \quad \times \int_{t_2}^1 E_{\alpha,\alpha}((1-s)^{\alpha-1}) (1-s)^{\alpha-1} \\
& \quad \times f_2(s, u(s), u(s-\tau)) ds \Big|.
\end{aligned} \tag{48}$$

Then we have

$$\begin{aligned}
& |t_1^{1-\alpha} T_2 u(t_1) - t_2^{1-\alpha} T_2 u(t_2)| \\
&\leq M_2 \|u\|_{1-\alpha}^2 \\
&\quad \times \left(\frac{\Gamma(\alpha)}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)} |E_{\alpha,\alpha}(t_1^\alpha) - E_{\alpha,\alpha}(t_2^\alpha)| \right. \\
&\quad \times \int_0^{t_1} |E_{\alpha,\alpha}((1-s)^\alpha)| (1-s)^{\alpha-1} ds \\
&\quad + \int_0^{t_1} |t_1^{1-\alpha} E_{\alpha,\alpha}((t_1-s)^\alpha) (t_1-s)^{\alpha-1} \\
&\quad \quad - t_2^{1-\alpha} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}((t_2-s)^\alpha)| ds \\
&\quad + \frac{\Gamma(\alpha)}{|1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)|} |E_{\alpha,\alpha}(t_1^\alpha) - E_{\alpha,\alpha}(t_2^\alpha)| \\
&\quad \times \int_{t_1}^{t_2} |E_{\alpha,\alpha}((1-s)^\alpha)| (1-s)^{\alpha-1} ds \\
&\quad - t_2^{1-\alpha} \int_{t_1}^{t_2} |E_{\alpha,\alpha}((t_2-s)^\alpha)| (t_2-s)^{\alpha-1} ds \\
&\quad + \frac{\Gamma(\alpha)}{|1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)|} \\
&\quad \times |E_{\alpha,\alpha}(t_1^\alpha) - E_{\alpha,\alpha}(t_2^\alpha)| \\
&\quad \times \int_{t_2}^1 |E_{\alpha,\alpha}((1-s)^\alpha)| (1-s)^{\alpha-1} ds \Big).
\end{aligned} \tag{49}$$

Hence we deduce that if $|t_1 - t_2| \rightarrow 0$ then $|t_1^{1-\alpha} T_2(t_2) - t_2^{1-\alpha} T_2(t_2)| \rightarrow 0$.

Then $\{t^{1-\alpha} T_2 u : u \in B\}$ is equicontinuous. Moreover we show that $\{t^{1-\alpha} T_2 u : u \in B\}$ is a bounded set in $C[0, 1]$. Indeed we have

$$\begin{aligned}
& t^{1-\alpha} |T_2 u(t)| \\
&\leq M_2 \|u\|_{1-\alpha}^2 t^{1-\alpha} \int_0^1 |G_{1,\alpha}(t, s)| ds \\
&\leq M_2 \|u\|_{1-\alpha}^2 \\
&\quad \times \left(\frac{\Gamma(\alpha) |E_{\alpha,\alpha}(t^\alpha)|}{|1 - \Gamma(\alpha) E_{\alpha,\alpha}(1)|} E_{\alpha,\alpha+1}(1) + t E_{\alpha,\alpha+1}(t^\alpha) \right).
\end{aligned} \tag{50}$$

Then by Arzela-Ascoli's theorem we conclude that T_2 is compact. By using Krasnoselskii's theorem there exists $z \in B$ such that z is a fixed point of T . This completes the proof. \square

5. Conclusions

We considered two types of nonlinear delay fractional differential equations (FDE) with periodic boundary conditions involving Remann-Liouville fractional derivative possessing with a lower terminal at 0. In order to obtain the results in this paper, we have shown the existence and the uniqueness of solution for a class of nonlinear delayed FDE by Banach contraction principle. Then using Krasnoselskii's fixed point theorem we established an existence theorem for a different type of the equation that we have proven its uniqueness theorem.

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