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Research Article

Abundant Explicit and Exact Solutions for the Variable Coefficient mKdV Equations

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This paper is concerned with the variable coefficients mKdV (VC-mKdV) equation. First, through some transformation we convert VC-mKdV equation into the constant coefficient mKdV equation. Then, using the first integral method we obtain the exact solutions of VC-mKdV equation, such as rational function solutions, periodic wave solutions of triangle function, bell-shape solitary wave solution, kink-shape solitary wave solution, Jacobi elliptic function solutions, and Weierstrass elliptic function solution. Furthermore, with the aid of Mathematica, the extended hyperbolic functions method is used to establish abundant exact explicit solution of VC-mKdV equation. By the results of the equation, the first integral method and the extended hyperbolic function method are extended from the constant coefficient nonlinear evolution equations to the variable coefficients nonlinear partial differential equation.

1. Introduction

It is well known that the KdV equation plays an important role in the soliton theory. Many properties of the KdV equation, such as symmetry, Bäcklund transformation, infinite conservation laws, Lax pairs, and Painleve analysis, have been studied. Miura transformation links the KdV equation with the mKdV equation. Therefore, as the KdV equation, mKdV equation is also important in mathematical physics field. In recent years, some authors considered the constant coefficients mKdV equation [1–4]. However, in practical applications, the coefficients of nonlinear evolution equations vary with time and space. Therefore, the exact solution of the variable coefficient nonlinear evolution equations has a greater application value.

This paper will discuss the variable coefficients mKdV equation (VC-mKdV):

$$u_t + a(t) u_x + b(t) u^2 u_x + r(t) u_{xxx} = 0, \quad (1)$$

where $a(t), b(t)$, and $r(t)$ are arbitrary function of variable t. More recently, some properties of the variable coefficients mKdV equation have been studied [5–18]. The aim of this paper is to apply the first integral method and the extended hyperbolic function method for constructing a series of explicit exact solutions to the VC-mKdV equation (1), such as rational function solutions, periodic wave solutions of triangle functions, bell-shape solitary wave solution, kink-shape solitary wave solution, Jacobi elliptic function solutions, Weierstrass elliptic function solution, and many exact explicit solutions in form of the rational function of hyperbolic function and the rational function of triangle function.

The rest of this paper is organized as follows. In Section 2, the outline of the first integral method will be given. In Section 3 we introduce a transformation to transform the VC-mKdV equation (1) into a constant coefficients mKdV equation. Section 4 is the main part of this paper; the methods are employed to seek the explicit and exact solutions of the VC-mKdV equation (1). In the last section, some conclusion is given.

2. The First Integral Method

The first-integral method, which is based on the ring theory of commutative algebra, was first proposed by Professor Feng Zhaosheng [19] in 2002. The method has been applied by Feng to solve Burgers-KdV equation, the compound Burgers-KdV equation, an approximate Sine-Gordon equation in $(n+1)$-dimensional space, and two-dimensional Burgers-KdV equation [20–24].

In the recent years, many authors employed this method to solve different types of nonlinear partial differential equations in physical mathematics. More information about these applications can be found in [25] and references therein. The most advantage is that the first integral method has not many sophisticated computation in solving nonlinear algebra equations compared to other direct algebra method. For completeness, we briefly outline the main steps of this method.

The main steps of this method are summarized as follows.

Given a system of nonlinear partial differential equations, for example, in two independent variables,

$$P\left(u_t, u_x, u_{xx}, u_{xt}, \ldots\right) = 0. \tag{2}$$

Using traveling wave transformation $u(x,t) = f(\xi)$, $\xi = kx + \omega t + \xi_0$ and some other mathematical operations, the systems (2) can be reduced to a second-order nonlinear ordinary differential equation:

$$D\left(f, f', f''\right) = 0. \tag{3}$$

By introducing new variables $X = f(\xi)$, $Y = f'(\xi)$ or making some other transformations, we reduce ordinary differential equation (3) to a system of the first order ordinary differential equation:

$$\begin{aligned} X' &= Y \\ Y' &= H(X,Y). \end{aligned} \tag{4}$$

Suppose that the first integral of (4) has a form as follows:

$$P(X,Y) = \sum_{i=0}^{m} a_i(X)\, Y^i = 0. \tag{5}$$

(In general $m = 1$ or $m = 2$), where $a_i(X)$ $(i = 0, 1, \ldots, m)$ are real polynomials of X.

According to the Division theorem from ring theory of commutative algebra, there exists polynomials $\alpha(X)$, $\beta(X)$ of variable X in $\mathfrak{R}[X]$ such that

$$\frac{dP}{d\xi} = \left[\alpha(X) + \beta(X) Y\right] P(X,Y). \tag{6}$$

We determine polynomials $\alpha(X)$, $\beta(X)$, $a_i(X)$ $(i = 0, 1, 2, \ldots)$ from (6), furthermore, obtain $P(X,Y)$.

Then substituting $X = f(\xi)$, $Y = f'(\xi)$ or other transformations into (5), exact solutions to (2) are established, through solving the resulting first-order integrable differential equation.

3. A Transformation to the VC-mKdV Equation

In order to transfer (1) into the form of (3), we firstly do some transformations for (1). Since (1) is a variable coefficients equation and we need to transform it to the constant coefficients mKdV equation, we introduce a transformation

$$u(x,t) = U(X,T)\, p(x,t), \tag{7}$$

where $X = X(x,t)$, $T = T(x,t)$. Through this transformation, we hope that (1) be changed into the form of constant coefficients mKdV equation:

$$U_T + 24U^2 U_X + U_{XXX} = 0. \tag{8}$$

In order to obtain the above transformation equation, substituting (7) into (1) and assuming $T_x = 0$, and simultaneously on both side of the formulas by dividing $r(t)X_x^3$, we have

$$\begin{aligned} &\frac{pbU^3 p_x}{rX_x^3} + \frac{p^2 bU^2 U_X}{rX_x^2} + \left(\frac{p_t}{rX_x^3 p} + \frac{p_{xxx}}{X_x^3 p} + \frac{a p_x}{rX_x^3 p}\right) U \\ &\quad + \left(\frac{a}{rX_x^2} + \frac{X_t}{rX_x^3} + \frac{X_{xxx}}{X_x^3} + \frac{2p_{xx}}{X_x^2 p} + \frac{3X_{xx} p_x}{X_x^3 p}\right) U_X \\ &\quad + \frac{T_t}{rX_x^3} U_T + \left(\frac{3X_{xx}}{X_x^2} + \frac{3p_x}{X_x p}\right) U_{XX} + U_{XXX} = 0. \end{aligned} \tag{9}$$

Comparing the coefficients of (9) with (8), such as $U^2 U_X$, U, U_x, and so on, we have

$$\frac{pbp_x}{rX_x^3} = 0, \tag{10}$$

$$\frac{p^2 b}{rX_x^2} = 24, \tag{11}$$

$$\frac{p_t}{rX_x^3 p} + \frac{p_{xxx}}{X_x^3 p} + \frac{a p_x}{rX_x^3 p} = 0, \tag{12}$$

$$\frac{a}{rX_x^2} + \frac{X_t}{rX_x^3} + \frac{X_{xxx}}{X_x^3} + \frac{2p_{xx}}{X_x^2 p} + \frac{3X_{xx} p_x}{X_x^3 p} = 0, \tag{13}$$

$$\frac{T_t}{rX_x^3} = 1, \tag{14}$$

$$\frac{3X_{xx}}{X_x^2} + \frac{3p_x}{X_x p} = 0. \tag{15}$$

From (11) and (14), we have $X_x \neq 0$ and

$$b = \frac{24 r X_x^2}{p^2}, \qquad r = \frac{T_t}{X_x^3}. \tag{16}$$

Substituting (16) into (10), (12), (13), and (15), we obtain

$$\frac{24p_x}{X_x p} = 0, \tag{17}$$

$$\frac{X_x^3 p_t + p_{xxx}T_t + X_x^3 a p_x}{T_t X_x^3 p} = 0, \tag{18}$$

$$\frac{aX_x^4 P + X_x^3 X_t p + X_{xxx}T_t p + 2p_{xx}T_t X_x + 3X_{xx}p_x T_t}{T_t X_x^3 p} = 0, \tag{19}$$

$$\frac{3\left(X_{xx}p + p_x X_x\right)}{X_x^2 p} = 0. \tag{20}$$

Form (18), we have $T_t \neq 0$. Also from (20), we have $X_{xx} = 0$ and $X_{xxx} = 0$. From (17) and (18), we have that p can only be a constant. For simplicity, we take $p = C$. Substituting this into (19), we obtain

$$X = F\left(x - \int a(t)\,dt\right). \tag{21}$$

For simplicity, we take

$$X = x - \int a(t)\,dt. \tag{22}$$

Substituting (23) into (16), we have

$$T = \int r(t)\,dt + C_1, \tag{23}$$

where C_1 is an arbitrary constant. Thus we get s transform between VC-mKdV equation (1) with constant coefficients mKdV equation (8).

4. Explicit and Exact Solutions of the VC-mKdV Equation

We firstly obtain explicit and exact solutions of the constant coefficients mKdV equation (8) and then obtain explicit and exact solutions of the constant coefficients mKdV equation (1). In the view of (8), we suppose $\xi = kX + \omega T + \xi_0$, and then we have

$$\omega U' + 24kU^2U' + k^3U''' = 0. \tag{24}$$

Integrating (24) once with respect to ξ, we obtain

$$U'' + mU^3 + lU = E, \tag{25}$$

where $l = \omega/k^3, m = 8/k^2$, and E is arbitrary integration constant. Let $R = U(\xi), S = R'$; (25) can be converted to a system of nonlinear ODEs as follows:

$$\begin{aligned} R' &= S, \\ S' &= -mR^3 - lR + E. \end{aligned} \tag{26}$$

Now we employ the Division theorem to seek the first integral to (26). Suppose that $R = R(\xi), S = S(\xi)$ are the nontrivial solution to the system (26), and its first integral is an irreducible polynomial in $\mathfrak{R}[R, S]$:

$$P(R(\xi), S(\xi)) = \sum_{i=0}^{2} a_i(R) S^i = 0, \tag{27}$$

where a_i, $i = 0, 1, 2$ are polynomial of R. According to the Division theorem, there exists polynomials $\alpha(R), \beta(R)$ of variable R in $\mathfrak{R}[R]$ such that

$$\begin{aligned} \frac{dP}{d\xi} &= a_2'(R) S^3 + 2a_2(R) SS' + a_1'(R) S^2 \\ &\quad + a_1(R) S' + a_0'(R) S \\ &= \left(\alpha(R) + \beta(R) S\right)\left(a_2(R) S^2 + a_1(R) S + a_0(R)\right). \end{aligned} \tag{28}$$

Collecting all the terms with the same power of S together and equating each coefficient to zero yield a set of nonlinear algebraic equations as follows:

$$a_2'(R) = \beta(R) a_2(R), \tag{29}$$

$$a_1'(R) = \beta(R) a_1(R) + \alpha(R) a_2(R), \tag{30}$$

$$\begin{aligned} a_0'(R) &= \beta(R) a_0(R) + \alpha(R) a_1(R) \\ &\quad + 2a_2(R)\left(mR^3 + lR - E\right), \end{aligned} \tag{31}$$

$$a_1(R)\left(-mR^3 - lR + E\right) = \alpha(R) a_0(R). \tag{32}$$

Because $a_i(R)$, $(i = 1, 2)$ are polynomials, from (29) we can deduce $\deg[a_2(R)] = 0$, $\beta(R) = 0$; that is $a_2(R)$ is a constant. For simplicity, we take $\beta(R) = 0$, $a_2(R) = 1$. Then we determine $a_0(R)$, $a_1(R)$, and $\alpha(R)$. From (30), we have $\deg[a_1(R)] - 1 = \deg[\alpha(R)]$ or $a_1(R) = \alpha(R) = 0$. In what follows we will discuss these two situations.

(1) In the case of $\alpha(R) = 0$, $a_1(R) = 0$.

In this case, (30) and (32) are satisfied. From (31), we can derive $a_0(R) = (m/2)R^4 + lR^2 - 2ER + d$, where d is an integral constant. Substituting $a_2(R), a_1(R), a_0(R)$ into (27), one obtains that

$$R' = \pm\sqrt{-\frac{m}{2}R^4 - lR^2 + 2ER - d}. \tag{33}$$

Based on the discussion for different parameters, we can obtain the solutions of the nonlinear ordinary differential equation (33).

(a) For $d = E = 0$, (33) admits the following three general solutions:

$$\begin{aligned} R_1 &= \pm\sqrt{\frac{2l}{m}}\operatorname{csch}\sqrt{-l}\xi, \quad lm > 0 \\ R_2 &= \pm\sqrt{-\frac{2l}{m}}\operatorname{sech}\sqrt{-l}\xi, \quad l < 0,\ m > 0 \\ R_3 &= \pm\frac{1}{\sqrt{-m/2}\xi}, \quad l = 0. \end{aligned} \tag{34}$$

Combining (7), (22), (23), (34), and $p(x,t) = C, R = U(\xi)$, one can get the following three sets of explicit exact solutions to (1):

$$u_1(x,t) = \pm iC\sqrt{\frac{\omega}{4k}}\operatorname{csch}\sqrt{\frac{\omega}{k^3}}\xi, \quad k\omega > 0,$$
$$u_2(x,t) = \pm C\sqrt{-\frac{\omega}{4k}}\operatorname{sech}\sqrt{-\frac{\omega}{k^3}}\xi, \quad k\omega < 0, \tag{35}$$
$$u_3(x,t) = \pm \frac{Ck}{2i\left[k\left(x - \int a(t)\,dt\right) + \xi_0\right]}, \quad \omega = 0,$$

where $\xi = k(x - \int a(t)dt) + \omega(\int r(t)dt + C_1) + \xi_0, C, C_1$ are arbitrary parameters, and ξ_0 is an arbitrary constant.

(b) For $E = 0$, $d = l^2/2m$, we can obtain the following four sets of explicit exact solutions to (33)

$$R_4 = \pm\sqrt{\frac{l}{m}}\tan\sqrt{-\frac{l}{2}}\xi, \quad lm > 0,$$
$$R_5 = \pm\sqrt{\frac{l}{m}}\cot\sqrt{-\frac{l}{2}}\xi, \quad lm > 0,$$
$$R_6 = \pm\sqrt{-\frac{l}{m}}\tanh\sqrt{\frac{l}{2}}\xi, \quad lm < 0, \tag{36}$$
$$R_7 = \pm\sqrt{-\frac{l}{m}}\coth\sqrt{\frac{l}{2}}\xi, \quad lm < 0.$$

Combining (7), (22), (23), (36), and $p(x,t) = C, R = U(\xi)$, we can get the following four explicit exact solutions of (1):

$$u_4(x,t) = \pm C\sqrt{\frac{\omega}{8k}}\tan\sqrt{-\frac{\omega}{2k^3}}\xi, \quad k\omega > 0,$$
$$u_5(x,t) = \pm C\sqrt{\frac{\omega}{8k}}\cot\sqrt{-\frac{\omega}{2k^3}}\xi, \quad k\omega > 0,$$
$$u_6(x,t) = \pm C\sqrt{-\frac{\omega}{8k}}\tanh\sqrt{\frac{\omega}{2k^3}}\xi, \quad k\omega < 0, \tag{37}$$
$$u_7(x,t) = \pm C\sqrt{-\frac{\omega}{8k}}\coth\sqrt{\frac{\omega}{2k^3}}\xi, \quad k\omega < 0,$$

where $\xi = k(x - \int a(t)dt) + \omega(\int r(t)dt + C_1) + \xi_0, C, C_1$ are arbitrary parameters, and ξ_0 is an arbitrary constant.

(c) For $E = l = 0$, from (33) we have

$$R' = \pm\sqrt{-\frac{m}{2}R^4 - d}. \tag{38}$$

Let $Z = R^2$; (38) becomes

$$Z' = \pm\sqrt{-\frac{m}{2}Z^3 - dZ}. \tag{39}$$

While $m < 0$, the above equation possesses a Weierstrass elliptic function doubly periodic wave type solution:

$$Z = \wp\left(\sqrt{-\frac{m}{8}}\xi, -\frac{8d}{m}, 0\right). \tag{40}$$

Combining (7), (22), (23), (40), $Z = R^2$, $p(x,t) = C$, and $R = U(\xi)$, we derive that (1) admits a Weierstrass elliptic function doubly periodic wave type solution:

$$u_8(x,t) = \pm C\sqrt{\wp\left(\pm\frac{1}{ki}\xi, -dk^2, 0\right)}, \tag{41}$$

where $\xi = k(x - \int a(t)dt), C, C_1$ are arbitrary parameters, and ξ_0 is an arbitrary constant.

(d) For $E = 0, d \neq 0$, we obtain elliptic function solutions for (33) as follows:

$$R_9 = \pm\sqrt{\frac{1-l}{m}}\operatorname{cn}\sqrt{\frac{1-l}{2}}\xi$$
$$d = \frac{l^2 - 1}{2m}$$
$$R_{10} = \pm\sqrt{\frac{2}{m}}\operatorname{dn}\sqrt{2+l}\xi$$
$$d = -\frac{2(l+1)}{m}$$
$$R_{11} = \pm\sqrt{-\frac{1+l}{m}}\operatorname{nc}\sqrt{\frac{l-l}{2}}\xi$$
$$d = \frac{l^2 - 1}{2m} \tag{42}$$
$$R_{12} = \pm\sqrt{-\frac{2(1+l)}{m}}\operatorname{nd}\sqrt{2+l}\xi$$
$$d = -\frac{2(l+1)}{m}$$
$$R_{13} = \pm\sqrt{\frac{1-l^2}{2m}}\operatorname{sd}\sqrt{\frac{1-l}{2}}\xi$$
$$d = \frac{l^2 - 1}{2m}$$
$$R_{14} = \pm\sqrt{\frac{2(1+l)}{m}}\operatorname{sc}\sqrt{l+2}\xi$$
$$d = -\frac{2(l+1)}{m}.$$

Combining (7), (22), (23), the above result (42), and $p(x,t) = C, R = U(\xi)$, we can get the following six Jacobi elliptic doubly periodic wave solutions of (1):

$$\begin{aligned}
u_9(x,t) &= \pm\sqrt{\frac{k^3-\omega}{8k}}\,\mathrm{cn}\sqrt{\frac{k^3-\omega}{2k^3}}\xi \\
d &= \frac{\omega^3-k^6}{16k^4} \\
u_{10}(x,t) &= \pm\frac{k}{2}\mathrm{dn}\sqrt{\frac{2k^3+\omega}{k^3}}\xi \\
d &= -\frac{\omega+k^3}{4k} \\
u_{11}(x,t) &= \pm\sqrt{-\frac{k^3+\omega}{8k}}\,\mathrm{nc}\sqrt{\frac{k^3-\omega}{2k^3}}\xi \\
d &= \frac{\omega^2-k^6}{16k^4} \\
u_{12}(x,t) &= \pm\sqrt{-\frac{k^3+\omega}{4k}}\,\mathrm{nd}\sqrt{\frac{2k^3+\omega}{k^3}}\xi \\
d &= -\frac{k^3+\omega}{4k} \\
u_{13}(x,t) &= \pm\sqrt{\frac{k^6-\omega^2}{16k^4}}\,\mathrm{sd}\sqrt{\frac{\omega-k^3}{2k^3}}\xi \\
d &= \frac{\omega^2-k^6}{16k^4} \\
u_{14}(x,t) &= \pm\sqrt{\frac{\omega+k^3}{4k^3}}\,\mathrm{sc}\sqrt{\frac{\omega+2k^3}{k^3}}\xi \\
d &= -\frac{\omega+k^3}{4k^3},
\end{aligned} \tag{43}$$

where $\xi = k(x - \int a(t)dt) + \omega(\int r(t)dt + C_1) + \xi_0$, C, C_1 are arbitrary parameters, and ξ_0 is an arbitrary constant.

(2) In the case of $\deg[a_1(R)] - 1 = \deg[\alpha(R)]$.

In this case, we assume that $\deg[\alpha(R)] = k_1$, $\deg[a_0(R)] = k_2$; then we have $\deg[a_1(R)] = k_1 + 1$. Now, by balancing the degrees of both sides of (32), we can deduce that $k_2 = 4$. By balancing the degrees of both sides of (31), we can also conclude that $k_1 = 1$ or $k_1 = 0$. If $k_1 = 0$, assuming that $\alpha(R) = A_0$, $a_1(R) = A_1R + A_2$, $a_0(R) = C_4R^4 + C_3R^3 + C_2R^2 + C_1R + C_0$ and substituting them into (30)–(32), by equating the coefficients of the different powers of X on both sides of (30) to (32), we can get that $\alpha(R) = a_1(R) = 0$. This is contradicting with our assumption. It indicates that $k_1 \neq 0$. While $k_1 = 1$, assuming that $a_0(R) = C_4R^4 + C_3R^3 + C_2R^2 + C_1R + C_0$, $a_1 = A_2R^2 + A_1R + A_0$, $\alpha(X) = B_1R + B_0$, then substituting these representations into (30)–(32), and by equating the coefficients of the different powers of R on both sides of (30) to (32), we can obtain an overdetermined system of nonlinear algebraic equations:

$$\begin{aligned}
2A_2 &= B_1 \\
A_1 &= B_0 \\
4C_4 - 2m &= A_2B_1 \\
3C_3 &= A_1B_1 + A_2B_0 \\
2C_2 - 2l &= A_1B_0 + A_0B_1 \\
C_1 &= A_2B_0 - 2E \\
-mA_2 &= B_1C_4 \\
-mA_1 &= B_1C_3 + C_4B_0 \\
-mA_0 - lA_2 &= B_1C_2 + B_0C_3 \\
EA_2 - lA_1 &= B_1C_1 + B_0C_2 \\
EA_1 - lA_2 &= B_1C_0 + B_0C_1 \\
EA_0 &= B_0C_0.
\end{aligned} \tag{44}$$

By analyzing all kinds of possibilities, we have the following.

(a) While $B_0 = C_0 = 0$, it leads to a contradiction.

(b) While $B_0 \neq 0, C_0 = 0$, it also leads to a contradiction.

(c) While $B_0 = 0, C_0 \neq 0$, we can derive that

$$E = 0, \qquad A_1 = C_1 = C_3 = 0, \qquad C_4 = -\frac{m}{2},$$
$$C_2 = -l, \qquad C_0 = -\frac{l^2}{2m}, \qquad A_2^2 = -2m, \qquad A_0 = \frac{l}{m}A_2. \tag{45}$$

Setting (45) in (27) yields

$$\frac{dR}{R^2 + (l/m)} = \pm\frac{\sqrt{-2m}}{2}d\xi. \tag{46}$$

Solving (46), we can obtain solutions R_4, R_5, R_6, and R_7 again. Consequently, we obtain explicit exact solutions u_4, u_5, u_6, and u_7 to (1). Here we will not list them one by one.

We obtain various of explicit and exact solutions of (1) by using the extended hyperbolic functions method presented in [26] by author. we can get the following explicit exact solutions to (1):

$$R_{15} = \pm\sqrt{-\frac{2}{m}\frac{a\sinh(\xi) + b\cosh(\xi)}{a\cosh(\xi) + b\sinh(\xi)}}, \tag{47}$$

when $l = 2$, where a, b, and r are arbitrary constants

$$R_{16} = \pm\sqrt{\frac{2\left(a^2 - b^2\right)}{m}}\frac{1}{a\cosh(\xi) + b\sinh(\xi)}, \tag{48}$$

when $l = -1$, where a, b, and r are arbitrary constants such that $(a^2 - b^2)\, m > 0$

$$R_{17} = \pm \sqrt{\frac{a^2 - b^2 - r^2}{2m}} \frac{1}{a \cosh(\xi) + b \sinh(\xi)} \pm \sqrt{-\frac{1}{2m}} \frac{a \sinh(\xi) + b \cosh(\xi)}{a \cosh(\xi) + b \sinh(\xi) + r}, \tag{49}$$

when $l = 1/2$, where a, b, and r are arbitrary constants

$$R_{18} = \pm \sqrt{\frac{a^2 - b^2 - r^2}{2m}} \frac{1}{a \cosh(\xi) + b \sinh(\xi)} \pm \sqrt{-\frac{1}{2m}} \frac{b \cosh(\xi) - a \sinh(\xi)}{a \cosh(\xi) + b \sinh(\xi) + r}, \tag{50}$$

when $l = -1/2$, where a, b, and r are arbitrary constants

$$R_{19} = \pm \sqrt{-\frac{2(a^2 + b^2)}{m}} \frac{1}{a \cosh(\xi) + b \sinh(\xi)}, \tag{51}$$

when $l = 1$, where a, b, and r are arbitrary constants

$$R_{20} = \pm \sqrt{-\frac{2}{m}} \frac{b \cosh(\xi) - a \sinh(\xi)}{a \cosh(\xi) + b \sinh(\xi)}, \tag{52}$$

when $l = -2$, where a, b, and r are arbitrary constants.

Combining (7), (22), (23), the above result (47)–(52), and $p(x,t) = C, R = U(\xi)$, the VC-mKdV equation (1) has explicit and exact solitary wave solutions:

$$u_{15}(x,t) = \pm \frac{ki(a \sinh(\xi) + b \cosh(\xi))}{2(a \cosh(\xi) + b \sinh(\xi))}, \tag{53}$$

where k, ω, a, and b are arbitrary constants such that $\omega = 2k^3$, $\xi = k(x - \int a(t)dt) + \omega(\int r(t)dt + C_1) + \xi_0$

$$u_{16}(x,t) = \pm \frac{k\sqrt{a^2 - b^2}}{2} \frac{1}{a \cosh(\xi) + b \sinh(\xi)}, \tag{54}$$

where k, ω, a, and b are arbitrary constants such that $\omega = -k^3$, $\xi = k(x - \int a(t)dt) + \omega(\int r(t)dt + C_1) + \xi_0$

$$u_{17}(x,t) = \pm \frac{k\sqrt{a^2 - b^2 - r^2}}{4} \frac{1}{a \cosh(\xi) + b \sinh(\xi)} \pm \sqrt{-\frac{1}{2m}} \frac{a \sinh(\xi) + b \cosh(\xi)}{a \cosh(\xi) + b \sinh(\xi) + r}, \tag{55}$$

where k, ω, a, and b are arbitrary constants such that $2\omega = k^3$, $\xi = k(x - \int a(t)dt) + \omega(\int r(t)dt + C_1) + \xi_0$

$$u_{18}(x,t) = \pm \frac{k\sqrt{a^2 - b^2 - r^2}}{2(a \cosh(\xi) + b \sinh(\xi))} \pm \sqrt{-\frac{1}{2m}} \frac{b \cosh(\xi) - a \sinh(\xi)}{a \cosh(\xi) + b \sinh(\xi) + r}, \tag{56}$$

where k, ω, a, and b are arbitrary constants such that $-2\omega = k^3$, $\xi = k(x - \int a(t)dt) + \omega(\int r(t)dt + C_1) + \xi_0$

$$u_{19}(x,t) = \pm \sqrt{-\frac{2(a^2 + b^2)}{m}} \frac{1}{a \cosh(\xi) + b \sinh(\xi)}, \tag{57}$$

where k, ω, a, and b are arbitrary constants such that $\omega = k^3$, $\xi = k(x - \int a(t)dt) + \omega(\int r(t)dt + C_1) + \xi_0$

$$u_{20}(x,t) = \pm \frac{ki(b \cosh(\xi) - a \sinh(\xi))}{2(a \cosh(\xi) + b \sinh(\xi))}, \tag{58}$$

where k, ω, a, and b are arbitrary constants such that $\omega = -2k^3$, $\xi = k(x - \int a(t)dt) + \omega(\int r(t)dt + C_1) + \xi_0$.

5. Summary and Conclusions

In summary, motivated by [27], we establish a transform from VC-mKdV equation to the constant coefficient mKdV equation firstly. Then we employ the first integral method and the extended hyperbolic function method to uniformly construct a series of explicit exact solutions for VC-mKdV equations. Abundant explicit exact solutions to VC-mKdV equations are obtained through an exhaustive analysis and discussion for different parameters. The exact solutions obtained in this paper include that of the solitary wave solutions of kink-type, singular traveling wave solutions, periodic wave solutions of triangle functions, Jacobi elliptic function doubly periodic solutions, Weierstrass elliptic function doubly periodic wave solutions, and so forth. In particular, the six explicit exact Jacobi elliptic function doubly periodic solutions R_9–R_{14} and Weierstrass elliptic function doubly periodic wave solution R_8 are uniformly obtained. Some known results of previous references are enriched greatly. The results indicate that the first integral method and extended hyperbolic function method are very effective methods to solve nonlinear differential equation. The methods also are readily applicable to a large variety of other nonlinear evolution equations in physical mathematics.

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