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Research Article

On the Distribution of Zeros and Poles of Rational Approximants on Intervals

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The distribution of zeros and poles of best rational approximants is well understood for the functions $f(x) = |x|^{\alpha}$, $\alpha > 0$. If $f \in C[-1,1]$ is not holomorphic on $[-1,1]$, the distribution of the zeros of best rational approximants is governed by the equilibrium measure of $[-1,1]$ under the additional assumption that the rational approximants are restricted to a bounded degree of the denominator. This phenomenon was discovered first for polynomial approximation. In this paper, we investigate the asymptotic distribution of zeros, respectively, a-values, and poles of best real rational approximants of degree at most n to a function $f \in C[-1,1]$ that is real-valued, but not holomorphic on $[-1,1]$. Generalizations to the lower half of the Walsh table are indicated.

1. Introduction

Let B be a subset of $\mathbb{C}$; we denote by

$$m_1(B) := \inf \sum_{\nu} |U_\nu| \tag{1.1}$$

the m_1-*measure of* B, where the infimum is taken over all coverings $\{U_\nu\}$ of B by disks U_ν and $|U_\nu|$ is the radius of the disk U_ν.

Let D be a region in $\mathbb{C}$ and φ a function defined in D with values in $\overline{\mathbb{C}}$. A sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of meromorphic functions in D is said to *converge to a function* φ *with*

respect to the m_1-measure inside D if for every $\varepsilon > 0$ and any compact set $K \subset D$ we have

$$m_1\left(\left\{z \in K : \left|(\varphi - \varphi_n)(z)\right| \geq \varepsilon\right\}\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{1.2}$$

(cf. Gončar [1]).

The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is said to converge to φ *m_1-almost geometrically inside D* if for any $\varepsilon > 0$ there exists a set $\Omega(\varepsilon)$ in $\mathbb{C}$ with $m_1(\Omega(\varepsilon)) < \varepsilon$ such that

$$\limsup_{n\to\infty} \left\|\varphi - \varphi_n\right\|^{1/n}_{K\setminus\Omega(\varepsilon)} < 1 \tag{1.3}$$

for any compact set $K \subset D$. We note that $\|\cdot\|_B$ is the supremum norm on a subset B of $\mathbb{C}$.

For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we denote by $\mathcal{P}_n$ the collection of all polynomials of degree at most n, and let

$$\mathcal{R}_{n,m} := \left\{ r = \frac{p}{q} : p \in \mathcal{P}_n, q \in \mathcal{P}_m, q \not\equiv 0 \right\}. \tag{1.4}$$

In [2], sequences $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, on a region D were investigated if the number of poles of r_n in D is bounded. It turns out that the geometric convergence of $\{r_n\}_{n\in\mathbb{N}}$ on a continuum $S \subset D$ implies that the sequence converges m_1-almost geometrically inside D to a meromorphic function f in D with at most a finite number of poles in D.

To be precise, let $B \subset \mathbb{C}$ and let $\mathcal{M}_m(B)$ denote the subset of meromorphic functions in B with at most m poles in B, each pole counted with its multiplicity. The main result of [2] can be stated as follows.

Theorem A. *Let S be a continuum in $\mathbb{C}$ and D a region with $S \subset D$. Let $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, be a sequence of rational functions converging geometrically to a function f on S, that is,*

$$\limsup_{n\to\infty} \left\|f - r_n\right\|^{1/n}_S < 1, \tag{1.5}$$

and assume that $f \not\equiv 0$ on S. If there exists a fixed integer $m \in \mathbb{N}$ such that $r_n \in \mathcal{M}_m(D)$ for all n and

$$N_0(r_n, K) = o(n) \quad \text{as } n \longrightarrow \infty \tag{1.6}$$

for each compact set $K \subset D$, then the sequence $\{r_n\}_{n\in\mathbb{N}}$ converges m_1-almost geometrically inside D to a meromorphic function $f \in \mathcal{M}_m(D)$.

Here, the number $N_0(r_n, K)$ denotes the number of zeros of r_n in K, each zero counted with its multiplicity.

The above result was applied in [2] to Chebyshev approximation on $[-1,1]$. Let $G(z,\infty)$ be the Green function of $\Omega = \overline{\mathbb{C}} \setminus [-1,1]$ with pole at ∞, and let

$$\mathcal{E}_\rho := \{z \in \mathbb{C} : G(z,\infty) < \log \rho\}, \quad \rho > 1, \tag{1.7}$$

be the Green domain to the parameter ρ, that is, $\mathcal{E}_\rho$ is the open Joukowski-ellipse with foci at $+1$ and -1 and major axis $\rho + 1/\rho$.

Let $f \in C[-1,1]$ be real-valued on $[-1,1]$. For abbreviation, we will write $\|\cdot\|$ for $\|\cdot\|_{[-1,1]}$. Given $n, m \in \mathbb{N}_0$, let $r^*_{n,m} = r^*_{n,m}(f) \in \mathcal{R}_{n,m}$ denote the real rational function of best uniform approximation to $f \in C[-1,1]$ with respect to $\mathcal{R}_{n,m}$, that is,

$$E_{n,m}(f) := \|f - r^*_{n,m}\| = \inf\{\|f - r\| : r \in \mathcal{R}_{n,m},\ r \text{ real-valued on } \mathbb{R}\}. \tag{1.8}$$

Moreover, let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ with

$$\lim_{n\to\infty} m_n = \infty, \quad m_n = o\left(\frac{n}{\log n}\right) \quad \text{as } n \longrightarrow \infty, \tag{1.9}$$

and let us consider a function $f \in C[-1,1]$ that can be continued meromorphically into $\mathcal{E}_\rho$ for some $\rho > 1$. Then the sequence $\{r^*_{n,m_n}\}_{n\in\mathbb{N}}$ converges m_1-almost geometrically inside $\mathcal{E}_\rho$ to f [3]. Using Theorem A, we obtain results about the distribution of the a-values in the neighborhood of a point $z_0 \in \partial E_\rho$. For $a \in \overline{\mathbb{C}}$ and $B \subset \mathbb{C}$, we denote by

$$N_a(r,B) := \#\{z \in B : r(z) = a\} \tag{1.10}$$

the number of a-values of the rational function r in B and each a-value is counted with its multiplicity. If f cannot be continued meromorphically to z_0, then for any neighborhood U of z_0 and any $a \in \overline{\mathbb{C}}$, with at most one exception,

$$\limsup_{n\to\infty} N_a(r^*_{n,m_n}, U) = \infty. \tag{1.11}$$

Particulary, such a point z_0 is either an accumulation point of zeros or of poles of r^*_{n,m_n}.

On the other hand, if f is not holomorphic on $[-1,1]$, so far results about the distribution of the zeros of $r^*_{n,m_n}(f)$ are only known in the case that $m_n = 0$ for all $n \in \mathbb{N}$ (polynomial approximation) or in the case that $m_n = m \in \mathbb{N}$ is fixed (rational approximation with a bounded number of free poles). In the polynomial case, the normalized zero counting measures of $r^*_{n,0}(f)$ converge in the weak*-sense to the equilibrium measure of $[-1,1]$, at least for a subsequence $n \in \Lambda \subset \mathbb{N}$ [4]. This result was generalized to rational approximation with a bounded number of poles (cf. [5, Theorem 4.1]). Moreover, Stahl [6] and Saff and Stahl [7] have investigated for the function $f(x) = |x|^\alpha$, $\alpha > 0$, the distribution of zeros and poles of rational approximants, as well as the alternation points of the optimal error function.

In contrast to the distribution of zeros of r^*_{n,m_n}, the behavior of the alternation points of $f - r^*_{n,m_n}$ for $f \in C[-1,1]$ is well understood, not only in the polynomial case (cf. [8, 9]), but also for rational approximations (cf. [10–14]). The aim of the present paper is to investigate the distribution of the zeros of the rational approximants via the distribution of the alternation points.

2. Main Results

Let f be continuous on $[-1,1]$, possibly complex-valued. It is well known that the rate of approximation by rational functions does not guarantee the holomorphy of the function f. Gončar ([15], p. 101) pointed out the example

$$f(z) = \sum_{n=1}^{\infty} \frac{A_n}{z - \alpha_n}, \tag{2.1}$$

where the points α_n are situated in $\mathbb{C} \setminus [-1,1]$ such that any point of $[-1,1]$ is a limit point of the sequence $\{\alpha_n\}$ and the coefficients A_n converge to zero sufficiently fast. Hence, it is possible that there exists a sequence $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, such that

$$\limsup_{n\to\infty} \|f - r_n\|^{1/n} < 1, \tag{2.2}$$

and f is continuous on $[-1,1]$, but nowhere holomorphic on $[-1,1]$.

But it turns out that in this case Theorem A immediately yields the following.

Theorem 2.1. *Let $f \in C[-1,1]$ be not holomorphic on $[-1,1]$, and let $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, be a sequence such that*

$$\limsup_{n\to\infty} \|f - r_n\|^{1/n} < 1. \tag{2.3}$$

Then for any non holomorphic point $z_0 \in [-1,1]$ of f any neighborhood U of z_0 either

$$\limsup_{n\to\infty} N_\infty(r_n, U) = \infty \tag{2.4}$$

or

$$\limsup_{n\to\infty} \frac{N_a(r_n, U)}{n} > 0 \tag{2.5}$$

for all $a \in \mathbb{C}$.

In the following we consider functions $f \in C[-1,1]$ that are always real-valued on $[-1,1]$. Then the case that

$$\limsup_{n\to\infty} E_{n,n}^{1/n}(f) = 1 \tag{2.6}$$

is not covered by Theorem 2.1. By Bernstein's theorem, condition (2.6) implies that $f \in C[-1,1]$ is not holomorphic on $[-1,1]$. Examples for (2.6) are functions which are piecewise analytic on $[-1,1]$ (Newman [16], Gončar [15]).

In the following, we assume that $\{m_n\}_{n\in\mathbb{N}}$ is a sequence with

$$m_n \le n, \qquad m_n \le m_{n+1} \le m_n + 1. \tag{2.7}$$

For abbreviation, let

$$E_n := E_{n,m_n}(f), \qquad r_n^* := r_{n,m_n}^*(f) = \frac{p_n^*}{q_n^*}, \tag{2.8}$$

where $p_n^* \in \mathcal{P}_n$ and $q_n^* \in \mathcal{P}_{m_n}$ have no common factor. We define

$$\delta_n := \min\left(n - \deg p_n^*, m_n - \deg q_n^*\right) \tag{2.9}$$

as the defect of r_n^* and $d_n := n+m_n+1-\delta_n$. According to the alternation theorem of Chebyshev (cf. Meinardus [17], Theorem 98) there exist $d_n + 1$ points $x_k^{(n)}$,

$$-1 \le x_0^{(n)} < x_1^{(n)} < \cdots < x_{d_n}^{(n)} \le 1, \tag{2.10}$$

which satisfy

$$\lambda_n(-1)^k\left(f - r_n^*\right)\left(x_k^{(n)}\right) = \left\|f - r_n^*\right\|_{[-1,1]}, \quad 0 \le k \le d_n, \tag{2.11}$$

where $\lambda_n = +1$ or $\lambda_n = -1$ is fixed. For each pair (n, m_n) let

$$A_n = A_n(f) := \left\{x_k^{(n)}\right\}_{k=0}^{d_n} \tag{2.12}$$

denote an arbitrary, but fixed alternation set for the best approximation $r_n^* \in \mathcal{R}_{n,m_n}$, and let ν_n denote the normalized counting measure of A_n, that is,

$$\nu_n\left([\alpha,\beta]\right) := \frac{\#\left\{x_k^{(n)} : \alpha \le x_k^{(n)} \le \beta\right\}}{d_n + 1} \tag{2.13}$$

for any interval $[\alpha,\beta] \subset [-1,1]$. Since ν_n is a probability measure on $[-1,1]$, there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\nu_n \xrightarrow{*} \nu \quad \text{as } n \longrightarrow \infty, n \in \Lambda, \tag{2.14}$$

in the weak*-topology and ν is again a probability measure on $[-1,1]$.

Theorem 2.2. *Let $f \in C[-1,1]$ be real-valued, and let (2.6) hold. Moreover, let f be approximated with respect to $\mathcal{R}_{n,m_n}$, where the sequence $\{m_n\}_{n\in\mathbb{N}}$ satisfies (2.7). Then there exists a subsequence $\Lambda \subset \mathbb{N}$ with the following properties:*

(i)

$$\nu_n \xrightarrow{*} \nu \quad as \quad n \longrightarrow \infty, n \in \Lambda, \tag{2.15}$$

(ii) *let $z_0 \in \operatorname{supp}(\nu)$, $a \in \mathbb{C}$, and let U be a neighborhood of z_0 with $f(z) \not\equiv a$ on $U \cap [-1,1]$; then*

$$\begin{aligned} &\text{either} \quad \limsup_{n\in\Lambda, n\to\infty} N_\infty(r_n^*, U) = \infty \\ &\text{or} \quad \limsup_{n\in\Lambda, n\to\infty} \frac{N_a(r_n^*, U)}{n} > 0. \end{aligned} \tag{2.16}$$

Applying to the approximation in the upper half of the Walsh table, we obtain the following.

Corollary 2.3. *Let $f \in C[-1,1]$ with (2.6) and let the subsequence $\{m_n\}_{n\in\mathbb{N}}$ satisfy*

$$m_n \le cn \quad with\ 0 \le c < 1, \qquad m_n \le m_{n+1} \le m_n + 1. \tag{2.17}$$

Then there exists a subsequence $\Lambda \subset \mathbb{N}$ with the following property: Let $a \in \mathbb{C}$, $z_0 \in [-1,1]$, and let U be a neighborhood of z_0 with $f(z) \not\equiv a$ on $U \cap [-1,1]$; then either (i) or (ii) holds.

3. Auxiliary Tools

One of the essential tools for proving Theorem 2.2 is the interaction between alternation points and poles of best rational approximants.

Let τ_n denote the normalized counting measure of the poles of r_n^*, counted with their multiplicities, and let us denote by $\widehat{\tau}_n$ the balayage measure of τ_n onto $[-1,1]$. Then the following distribution results hold for the interaction between the alternation points of A_n and the poles of r_n^* and r_{n+1}^*.

Theorem B (See [11]). *Let f be not a rational function, and let $\{m_n\}_{n\in\mathbb{N}}$ satisfy (2.7). Then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that*

$$\nu_n - \alpha_n(\widehat{\tau}_n + \widehat{\tau}_{n+1}) - (1-\alpha_n)\mu \xrightarrow{*} 0 \quad as\ n \longrightarrow \infty, n \in \Lambda, \tag{3.1}$$

where

$$\alpha_n = \frac{\deg\, q_n^* + \deg\, q_{n+1}^*}{d_n + 1} \tag{3.2}$$

and μ is the equilibrium distribution of $[-1,1]$.

We remark that in the proof of Theorem B in [11], the subsequence $\Lambda \subset \mathbb{N}$ is defined by

$$\Lambda := \left\{ n \in \mathbb{N} : \frac{E_n + E_{n+1}}{E_n - E_{n+1}} \leq n^2 \right\}. \tag{3.3}$$

Inspecting the proof of (3.1) in [11], it turns out that we can modify the definition of Λ by

$$\Lambda := \left\{ n \in \mathbb{N} : E_{n+1} \leq \left(1 - \frac{1}{n^2}\right) E_n \right\}. \tag{3.4}$$

The existence of such sequences Λ is based on the divergence of the infinite product

$$\prod_{n=0}^{\infty} \frac{E_{n+1}}{E_n} = \prod_{n=0}^{\infty} \left(1 - \frac{E_n - E_{n+1}}{E_n}\right) \tag{3.5}$$

to 0 if f is not a rational function. This argument has already been used by Kadec [9] in his proof for the distribution of the alternation points in polynomial approximation.

Concerning the distribution of the zeros of best polynomial approximations p_n^* to f,

$$p_n^*(z) = a_n z^n + \cdots, \tag{3.6}$$

the asymptotic behavior of the highest coefficient a_n plays an essential role, namely,

$$\limsup_{n \to \infty} |a_n|^{1/n} = \frac{1}{\operatorname{cap}([-1,1]) \limsup_{n \to \infty} e_n^{1/n}}, \tag{3.7}$$

where

$$e_n = \left\| f - p_n^* \right\| = \inf_{p_n \in \mathcal{P}_n} \left\| f - p_n \right\| \tag{3.8}$$

and $\operatorname{cap}([-1,1]) = 1/2$ is the logarithmic capacity of $[-1,1]$.

If $f \in C[-1,1]$ is not holomorphic on $[-1,1]$, then $\limsup_{n \to \infty} e_n^{1/n} = 1$ and we can choose a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\lim_{n \in \Lambda, n \to \infty} e_n^{1/n} = 1 \tag{3.9}$$

and moreover,

$$\lim_{n \to \Lambda, n \to \infty} |a_n|^{1/n} = 2. \tag{3.10}$$

If $e_n \neq e_{n+1}$, then the polynomial

$$p_n(z) := \frac{p_n^*(z)}{a_n} \tag{3.11}$$

is monic and satisfies

$$\|p_n\| \leq \left(\frac{1}{2-\varepsilon}\right)^n \tag{3.12}$$

for all $n \in \Lambda$ which are sufficiently large, where $\varepsilon > 0$ can be chosen arbitrarily. Then the Erdős-Turán Theorem [18] (cf. [19]) implies a weak*-version of Kadec's result, namely, the weak*-convergence of the normalized counting measures of alternation sets of $f - p_n^*$ to the equilibrium measure μ of $[-1,1]$, at least for a subsequence Λ, $n \in \Lambda$.

The objective of this section is to show that there exists a subsequence $\Lambda \subset \mathbb{N}$ such that (3.4) and the analogue of (3.9) for rational approximation hold simultaneously with consequences for the behavior of the difference of two consecutive best approximants.

Lemma 3.1. *Let $f \in C[-1,1]$ with (2.6). Then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that*

$$\begin{gathered} E_{n+1} \leq \left(1 - \frac{1}{n^2}\right) E_n \quad \text{for } n \in \Lambda, \\ \lim_{n\in\Lambda, n\to\infty} E_n^{1/n} = 1. \end{gathered} \tag{3.13}$$

Moreover, let $\{\xi_n\}_{n\in\Lambda}$ be a sequence in $[-1,1]$ with $|(f - r_n^*)(\xi_n)| = \|f - r_n^*\|$; then

$$\lim_{n\in\Lambda, n\to\infty} \left|\left(r_n^* - r_{n+1}^*\right)(\xi_n)\right|^{1/n} = 1. \tag{3.14}$$

Proof . Using the above arguments of the beginning of this section, there exists a subsequence $\Lambda_1 \subset \mathbb{N}$ such that

$$E_{n+1} \leq \left(1 - \frac{1}{n^2}\right) E_n \quad \text{for } n \in \Lambda_1. \tag{3.15}$$

First, we show that there exists $\Lambda \subset \mathbb{N}$ such that (3.13) holds.

For proving this, we define

$$\widetilde{\Lambda} := \left\{ n \in \mathbb{N} : E_{n+1} \leq \left(1 - \frac{1}{n^2}\right) E_n \right\}. \tag{3.16}$$

Since $\Lambda_1 \subset \widetilde{\Lambda}$, $\widetilde{\Lambda} \neq \emptyset$, and $\widetilde{\Lambda}$ is not finite, hence the complement

$$\widetilde{\Lambda}^c := \mathbb{N} \setminus \widetilde{\Lambda} \tag{3.17}$$

of $\widetilde{\Lambda}$ in $\mathbb{N}$ has the property that

$$E_{n+1} > \left(1 - \frac{1}{n^2}\right) E_n \quad \text{for } n \in \widetilde{\Lambda}^c. \tag{3.18}$$

If $\widetilde{\Lambda}^c$ is a finite set, then there exists $m \in \mathbb{N}$ such that

$$\Lambda := \{n \in \mathbb{N} : n \geq m\} \tag{3.19}$$

satisfies property (3.13).

If $\widetilde{\Lambda}^c$ is an infinite set, then observing that $\widetilde{\Lambda}$ is not a finite set, we can define subsequences $\{m_j\}_{j\in\mathbb{N}}$ and $\{n_j\}_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\begin{aligned} n_{j-1} < m_j \leq n_j < m_{j+1}, \\ \widetilde{\Lambda} = \{n \in \mathbb{N} : m_j \leq n \leq n_j, j \geq 1\}. \end{aligned} \tag{3.20}$$

Next, we consider a fixed integer $m \geq m_1$. If

$$n_{j-1} < m < m_j, \quad j \geq 2, \tag{3.21}$$

then $m \notin \widetilde{\Lambda}$ and we deduce

$$\begin{aligned} E_{m_j} &> \left(1 - \frac{1}{(m_j - 1)^2}\right) E_{m_j - 1} > \left(1 - \frac{1}{(m_j - 1)^2}\right)\left(1 - \frac{1}{(m_j - 2)^2}\right) E_{m_j - 2} \\ &> \cdots > \prod_{k=0}^{m_j - m - 1} \left(1 - \frac{1}{(m + k)^2}\right) E_m. \end{aligned} \tag{3.22}$$

Since the infinite product

$$S = \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) \tag{3.23}$$

converges, there exists a constant β, $0 < \beta < 1$, such that all partial products

$$S_{\nu,\mu} := \prod_{n=\nu}^{\mu} \left(1 - \frac{1}{n^2}\right), \quad 2 \leq \nu < \mu, \tag{3.24}$$

of S are bounded by β from below, that is, $S_{\nu,\mu} \geq \beta$.

By (3.22), $E_{m_j} > \beta E_m$ and

$$E_{m_j}^{1/m_j} \geq E_{m_j}^{1/m} > \beta^{1/m} E_m^{1/m} \text{ for } E_{m_j} \leq 1. \tag{3.25}$$

Let us define for $m \geq m_1$

$$\nu(m) := \begin{cases} m, & \text{if } m \in \widetilde{\Lambda} \\ m_j, & \text{if } n_{j-1} < m < m_j. \end{cases} \tag{3.26}$$

Next, we choose a subsequence $\Lambda_2 = \{k_j\}_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that $k_1 \geq m_1$ and

$$\lim_{j\to\infty} E_{k_j}^{1/k_j} = 1. \tag{3.27}$$

If $\Lambda_2 \subset \widetilde{\Lambda}$, then we are done. As for the general case, let us define

$$\Lambda := \bigcup_{j=1}^{\infty} \{\nu(k_j)\}; \tag{3.28}$$

then $\Lambda \subset \widetilde{\Lambda}$ and (3.25)–(3.27) imply

$$\lim_{j\to\infty} E_{\nu(k_j)}^{1/\nu(k_j)} = 1. \tag{3.29}$$

Hence, (3.13) is proved.

Moreover, for $n \in \Lambda$,

$$\begin{aligned}
\left|\left(r_n^* - r_{n+1}^*\right)(\xi_n)\right| &\geq \left|\left(f - r_n^*\right)(\xi_n)\right| - \left|\left(f - r_{n+1}^*\right)(\xi_n)\right| \\
&\geq E_n - E_{n+1} \geq E_n - \left(1 - \frac{1}{n^2}\right)E_n = \frac{1}{n^2}E_n, \\
1 \geq \limsup_{n\to\infty} \left|\left(r_n^* - r_{n+1}^*\right)(\xi_n)\right|^{1/n} &\geq \limsup_{n\in\Lambda, n\to\infty} \left|\left(r_n^* - r_{n+1}^*\right)(\xi_n)\right|^{1/n} \\
&\geq \limsup_{n\in\Lambda, n\to\infty} \left(\left(\frac{1}{n^2}\right)^{1/n} E_n^{1/n}\right) = \lim_{n\in\Lambda, n\to\infty} E_n^{1/n} = 1.
\end{aligned} \tag{3.30}$$

Hence,

$$\lim_{n\in\Lambda, n\to\infty} \left|\left(r_n^* - r_{n+1}^*\right)(\xi_n)\right|^{1/n} = 1, \tag{3.31}$$

and (3.14) is proved. □

4. Proofs

Proof of Theorem 2.2. First we will prove the theorem for $a = 0$.

According to the lemma in Section 3, there exists a subsequence $\Lambda \subset \mathbb{N}$ such that (3.13)–(3.14) hold. Then Theorem B applies and (3.1) holds for $n \in \Lambda$. Because ν_n are probability measures on $[-1, 1]$, we may assume that

$$\nu_n \xrightarrow{*} \nu \quad \text{as } n \longrightarrow \infty, n \in \Lambda. \tag{4.1}$$

Let $z_0 \in \operatorname{supp}(\nu)$ and U a neighborhood of z_0 such that $f(z) \not\equiv 0$ on $U \cap [-1, 1]$.

Let us assume that (ii) of Theorem 2.2 does not hold. Hence, there exists $m \in \mathbb{N}$

$$N_\infty(r_n^*, U) \leq m \quad \forall n \in \mathbb{N}, \tag{4.2}$$

$$N_0(r_n^*, U) = o(n) \quad \text{as } n \longrightarrow \infty. \tag{4.3}$$

Of course, we may assume that U is a bounded symmetric region with respect to the real axis. Let l_n be the number of poles $\xi_{n,i}$ of r_n^* in U counted with their multiplicities. Then we define

$$q_n(z) := \begin{cases} \prod_{i=1}^{l_n}(z - \xi_{n,i}), & l_n \geq 1, \\ 1, & l_n = 0. \end{cases} \tag{4.4}$$

Because $q_n, q_{n+1} \in \mathcal{P}_m$, there exists a subsequence $\Lambda_1 \subset \Lambda$ and $\tilde{q}_0, \tilde{q}_1 \in \mathcal{P}_m$ such that

$$\lim_{n \in \Lambda_1, n \to \infty} q_{n+i} = \tilde{q}_i \quad \text{for } i = 0, 1. \tag{4.5}$$

Together with $f(z) \not\equiv 0$ for $z \in U \cap [-1, 1]$, this implies that there exists an interval $[\alpha, \beta] \subset U \cap [-1, 1]$, $\alpha \neq \beta$, and a constant $\kappa > 0$ such that

$$\left|\tilde{q}_i(x)\right| \geq \kappa \quad \text{for } x \in [\alpha, \beta], i = 0, 1, \tag{4.6}$$

$$\left|f(x)\right| \geq \kappa \quad \text{for } x \in [\alpha, \beta]. \tag{4.7}$$

Let k_n be the number of zeros (with multiplicities) of r_n^* in U. If $k_n \geq 1$, let $\eta_{n,i}$, $1 \leq i \leq k_n$, be the zeros of r_n^* in U and let

$$\pi_n(z) := \begin{cases} \prod_{i=1}^{k_n}(z - \eta_{n,i}), & k_n > 0 \\ 1, & k_n = 0. \end{cases} \tag{4.8}$$

Because of (4.3), $k_n = o(n)$ as $n \to \infty$ and we obtain

$$\limsup_{n \to \infty} \|\pi_n\|_K^{1/n} \leq 1 \tag{4.9}$$

for any compact set K in $\mathbb{C}$. Now, let us define

$$h_n(z) := \frac{1}{n} \log|\Phi_n(z)| \tag{4.10}$$

with

$$\Phi_n(z) := \frac{\pi_n(z)}{r_n^*(z)q_n(z)}. \tag{4.11}$$

Then Φ_n is holomorphic in U and h_n harmonic in U.

Consider $z \in [\alpha, \beta]$ and Λ_1 as before. Then by (4.5)–(4.7) there exists $\tilde{n} \in \mathbb{N}$ such that

$$\left|r_{n+i}^*(z)\right| \geq \frac{\kappa}{2}, \qquad \left|q_{n+i}(z)\right| \geq \frac{\kappa}{2} \tag{4.12}$$

for $z \in [\alpha, \beta]$, $i = 0, 1$, and $n \in \Lambda_1$, $n \geq \tilde{n}$. Then for $i = 0, 1$

$$\left\| \frac{\pi_{n+i}}{r_{n+i}^* q_{n+i}} \right\|_{[\alpha,\beta]} \leq \frac{4(d+1)^{k_{n+i}}}{\kappa^2}, \tag{4.13}$$

where

$$d = \sup_{z \in U} |z|. \tag{4.14}$$

According to a Lemma of Gončar [20, Lemma 1, page 153], for any compact set $K \subset U$ there exists a constant $\lambda = \lambda([\alpha, \beta], U, K) > 1$ such that

$$\left\| \frac{\pi_{n+i}}{r_{n+i}^* q_{n+i}} \right\|_K \leq \lambda^{n+i} \left\| \frac{\pi_{n+i}}{r_{n+i}^* q_{n+i}} \right\|_{[\alpha,\beta]} \tag{4.15}$$

for $i = 0, 1$. For example, $\lambda([\alpha, \beta], U, K)$ can be chosen as

$$\lambda\left([\alpha, \beta], U, K\right) := \max_{z \in K} \sup_{t \in \overline{\mathbb{C}} \setminus U} \exp\left(G_{[\alpha,\beta]}(z, t)\right), \tag{4.16}$$

where $G_{[\alpha,\beta]}(z, t)$ is the Green function of $\overline{\mathbb{C}} \setminus [\alpha, \beta]$ with pole at t.

Next, we choose a region $W \subset U$, W symmetric to the real axis, with $z_0 \in W$, $\overline{W} \subset U$ and $[\alpha, \beta] \subset W$, then

$$h_{n+i}(z) \leq \lambda\left([\alpha, \beta], U, \overline{W}\right) + \frac{1}{n+i} \log \frac{4}{\kappa^2} + \frac{k_n + i}{n+i} \log(1+d) \tag{4.17}$$

for $i = 0, 1$. Hence for $i = 0, 1$, the sequences $\{h_{n+i}\}_{n \in \Lambda_1}$ are uniformly bounded in W from above as $n \to \infty$, $n \in \Lambda_1$, $i = 0, 1$. By Harnack's theorem, either

$$h_n(z) \longrightarrow -\infty \quad \text{locally uniformly in } W \text{ as } n \longrightarrow \infty, n \in \Lambda_1, \tag{4.18}$$

or there exists a subsequence $\Lambda_2 \subset \Lambda_1$ such that $\{h_n\}_{n \in \Lambda_2}$ converges locally uniformly to h_0 as $n \to \infty$, $n \in \Lambda_2$, in the region W and the function h_0 is harmonic in W.

Next, let us show that the first situation cannot occur: if $C > 0$ is such that

$$\max_{z\in[\alpha,\beta]} h_n(z) \le -C \tag{4.19}$$

for $n \in \Lambda_1$ and n sufficiently large, then

$$n \cdot \max_{z\in[\alpha,\beta]} |h_n(z)| \le \max_{z\in[\alpha,\beta]} \log\left|\frac{\pi_n(z)}{r_n^*(z)q_n(z)}\right| \le -nC. \tag{4.20}$$

Hence, by (4.5)–(4.7) there exists a constant $c_1 > 0$ such that

$$\max_{z\in[\alpha,\beta]} |\pi_n(z)| \le c_1 e^{-nC}. \tag{4.21}$$

Since $\pi_n \in \mathcal{P}_{k_n}$ is a monic polynomial and $k_n = o(n)$ as $n \to \infty$, this is a contradiction to

$$\|\pi_n\|_{[\alpha,\beta]} \ge 2\left(\frac{(\beta-\alpha)}{4}\right)^{k_n}. \tag{4.22}$$

Next, we consider (4.17) for $i = 1$. Again by Harnack's theorem, either

$$h_{n+1}(z) \longrightarrow -\infty \quad \text{locally uniformly in } W \text{ as } n \longrightarrow \infty, n \in \Lambda_2, \tag{4.23}$$

or there exists a subsequence $\Lambda_3 \subset \Lambda_2$ such that $\{h_{n+1}\}_{n\in\Lambda_3}$ converges locally uniformly to a function h_1 in W and h_1 is harmonic in W.

As above for $\{h_n\}_{n\in\Lambda_1}$, the first situation cannot occur. Consequently,

$$\max_{z\in[\alpha,\beta]} h_i(z) \ge 0 \quad \text{for } i = 0, 1. \tag{4.24}$$

On the other hand, using (4.13) we deduce for $i = 0, 1$ that

$$\limsup_{n\in\Lambda_1, n\to\infty} \max_{z\in[\alpha,\beta]} h_{n+i}(z) \le 0. \tag{4.25}$$

Summarized, we have for $i = 0, 1$ that

$$h_i(z) \equiv 0 \quad \text{for } z \in [\alpha,\beta]. \tag{4.26}$$

By definition, the regions U, W are symmetric to $\mathbb{R}$ as well as the functions

$$\left|r_{n+i}^*(z)\right|, \qquad |\pi_{n+i}(z)|, \qquad \left|q_{n+i}(z)\right| \tag{4.27}$$

for $i = 0, 1$. This symmetry, together with (4.26), implies that

$$h_i(z) \equiv 0 \quad \forall z \in W \tag{4.28}$$

for $i = 0, 1$. Hence,

$$\lim_{n\in\Lambda_3, n\to\infty} \left\| r^*_{n+i} q_{n+i} \right\|_K^{1/n} \leq 1 \tag{4.29}$$

for all compact sets K in W, $i = 0, 1$.

Combining (4.29) for $i = 0, 1$, we obtain

$$\lim_{n\in\Lambda_3, n\to\infty} \left\| \left(r^*_n - r^*_{n+1}\right) q_n q_{n+1} \right\|_K^{1/n} \leq 1 \tag{4.30}$$

for all compact sets $K \subset W$. Hence, the function $V(z) \equiv 0$ is a harmonic majorant for the sequence $\{F_n\}_{n\in\Lambda_3}$ of subharmonic functions in W, where

$$F_n(z) := \frac{1}{n} \log\left| \left(r^*_n - r^*_{n+1}\right)(z) q_n(z) q_{n+1}(z) \right|, \quad n \in \mathbb{N}. \tag{4.31}$$

Next, we want to show that $V(z) \equiv 0$ is an *exact* harmonic majorant for $\{F_n\}_{n\in\Lambda_3}$ and also for any $\{F_n\}_{n\in\Lambda_4}$ for any subsequence $\Lambda_4 \subset \Lambda_3$.

Let us assume that this assertion would be false: then there exists a subsequence $\Lambda_4 \subset \Lambda_3 \subset \Lambda$ (Λ as in the Corollary of Section 3) and a continuum $K \subset W$ such that

$$\limsup_{n\in\Lambda_4, n\to\infty} \max_{z\in K} F_n(z) < 0. \tag{4.32}$$

Since $V(z) \equiv 0$ is a harmonic majorant for $\{F_n\}_{n\in\Lambda_4}$ in W, then (4.32) implies that the inequality (4.32) holds for any continuum $K \subset W$.

First, let us note that under the condition (4.2) a point $\xi \in U \cap [-1, 1]$ cannot be an isolated point of $\operatorname{supp}(v)$.

To prove this, let us denote by δ_z the Dirac measure of the point $z \in \overline{\mathbb{C}}$, and let $\widehat{\delta}_z$ be the associated balayage measure of δ_z to the interval $[-1, 1]$. For $z \notin [-1, 1]$ the density of the balayage measure $\widehat{\delta}_z$ at the point $x \in (-1, 1)$ is given by

$$\frac{d}{dx}\widehat{\delta}_z(x) = \frac{\partial}{\partial_{n_+}} G(x, z) + \frac{\partial}{\partial_{n_-}} G(x, z), \tag{4.33}$$

where n_+ (resp., n_-) denotes the normal at the point x to the upper half (resp., lower half) plane and $G(\xi, z)$ is the Green function for $\xi \in \overline{\mathbb{C}} \setminus [-1, 1]$ with pole at z, continuously extended by $G(x, z) = 0$ to $\xi = x \in [-1, 1]$.

Then for any interval $[\alpha,\beta] \subset [-1,1]$

$$0 \le \widehat{\delta}_z\left([\alpha,\beta]\right) \le 1,$$
$$\lim_{\substack{z\to\eta \\ \eta\in[-1,1]\setminus[\alpha,\beta]}} \widehat{\delta}_z\left([\alpha,\beta]\right) = 0. \tag{4.34}$$

Let $z \in \overline{\mathbb{C}} \setminus [-1,1]$, $\xi \in U \cap [-1,1]$, and $\varepsilon > 0$; then

$$\lim_{\varepsilon\to 0} \widehat{\delta}_z([\xi-\varepsilon,\xi+\varepsilon]) = 0. \tag{4.35}$$

Consider the exterior of the ε-neighborhood of $[-1,1]$; that is, let

$$W_\varepsilon := \left\{ z \in \overline{\mathbb{C}} : \operatorname{dist}(z,[-1,1]) \ge \varepsilon \right\}. \tag{4.36}$$

Then we can obtain a sharpening of (4.35), namely,

$$\lim_{\varepsilon\to 0} \max_{z\in W_\varepsilon\setminus U} \widehat{\delta}_z([\xi-\varepsilon,\xi+\varepsilon]) = 0. \tag{4.37}$$

Since $\xi \in U \cap [-1,1]$ and (4.2) holds, (4.34)–(4.37) imply

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \widetilde{\tau}_n([\xi-\varepsilon,\xi+\varepsilon]) = 0. \tag{4.38}$$

Because (3.1) and (4.1) hold for $n \in \Lambda$, ξ cannot be an isolated point of $\operatorname{supp}(\nu)$.

Consequently, since $z_0 \in \operatorname{supp}(\nu)$ there exists a sequence $\{\xi_k\}_{k\in\mathbb{N}}$ in U, $\xi_k \in \operatorname{supp}(\nu)$, such that

$$z_0 = \lim_{k\to\infty} \xi_k \tag{4.39}$$

and each ξ_k is not an isolated point of $\operatorname{supp}(\nu)$. Hence, for any $k \in \mathbb{N}$ and any open interval (α,β) with $\xi_k \in (\alpha,\beta)$ we have $\nu((\alpha,\beta)) > 0$. Taking into account (4.39) and the fact that the zero set

$$Z := \left\{ z \in \mathbb{C} : \widetilde{p}_0(z) = 0 \text{ or } \widetilde{p}_i(z) = 0 \right\} \tag{4.40}$$

of the polynomials $\widetilde{p}_0$, $\widetilde{p}_1$ in (4.5) is finite, there exists an interval $[\widetilde{\alpha},\widetilde{\beta}] \subset U \cap [-1,1]$, $\widetilde{\alpha} < \widetilde{\beta}$, with

$$\nu\left(\left[\widetilde{\alpha},\widetilde{\beta}\right]\right) > 0, \qquad \left[\widetilde{\alpha},\widetilde{\beta}\right] \cap Z = \emptyset. \tag{4.41}$$

Using (4.5) we conclude that there exists $n_1 \in \mathbb{N}$ and a constant $\widetilde{\kappa} > 0$ such that

$$\left|q_{n+i}(z)\right| \geq \widetilde{\kappa} \quad \text{for } z \in \left[\widetilde{\alpha}, \widetilde{\beta}\right], \tag{4.42}$$

where $n \in \Lambda_1$, $n \geq n_1$, and $i = 0, 1$.

Let us choose for K in (4.32) the interval $[\widetilde{\alpha}, \widetilde{\beta}]$. Then there exists, by definition of $F_n(z)$ in (4.31), a constant δ, $0 < \delta < 1$, and $n_2 \in \mathbb{N}$, $n_2 \geq n_1$, such that

$$\max_{z \in [\widetilde{\alpha}, \widetilde{\beta}]} \left|\left(r_n^* - r_{n+1}^*\right)(z) q_n(z) q_{n+1}(z)\right| \leq \delta^n. \tag{4.43}$$

for all $n \in \Lambda_4$, $n \geq n_2$. By (4.42) we obtain

$$\max_{z \in [\widetilde{\alpha}, \widetilde{\beta}]} \left|\left(r_n^* - r_{n+1}^*\right)(z)\right| \leq \frac{\delta^n}{\widetilde{\kappa}^2}, \tag{4.44}$$

$$\limsup_{n \in \Lambda_4, n \to \infty} \left\|r_n^* - r_{n+1}^*\right\|_{[\widetilde{\alpha}, \widetilde{\beta}]}^{1/n} \leq \delta < 1 \tag{4.45}$$

contradicting the property (3.14) and $\nu([\widetilde{\alpha}, \widetilde{\beta}]) > 0$.

Hence, $V(z) \equiv 0$ is an exact harmonic majorant for $\{F_n\}_{n \in \Lambda_3}$ and for any subsequence $\{F_n\}_{n \in \Lambda_4}$, $\Lambda_4 \subset \Lambda_3$, in the region W.

This is now the situation that a distribution result of Walsh about the zeros of the sequence

$$\left\{\left(r_n^* - r_{n+1}^*\right) q_n q_{n+1}\right\}_{n \in \Lambda_3} \tag{4.46}$$

of holomorphic functions in W can be applied (Walsh [21], Theorem 16, page 221): for every compact set K in W we have

$$N_0\left(\left(r_n^* - r_{n+1}^*\right) q_n q_{n+1}, K\right) = o(n) \quad \text{as } n \in \Lambda_3, n \longrightarrow \infty. \tag{4.47}$$

Choosing for K the interval $[\widetilde{\alpha}, \widetilde{\beta}]$, then the number of alternations of $f - r_n^*$ in $[\widetilde{\alpha}, \widetilde{\beta}]$ is a lower bound for the number

$$N_0\left(\left(r_n^* - r_{n+1}^*\right) q_n q_{n+1}, \left[\widetilde{\alpha}, \widetilde{\beta}\right]\right) \tag{4.48}$$

of zeros of $(r_n^* - r_{n+1}^*) q_n q_{n+1}$ in $[\widetilde{\alpha}, \widetilde{\beta}]$. Because of (4.1) and $\nu([\widetilde{\alpha}, \widetilde{\beta}]) > 0$,

$$\lim_{n \in \Lambda_3, n \to \infty} \nu_n\left(\left[\widetilde{\alpha}, \widetilde{\beta}\right]\right) = \nu\left(\left[\widetilde{\alpha}, \widetilde{\beta}\right]\right) > 0 \tag{4.49}$$

which contradicts (4.47).

Hence, the theorem is proved for $a = 0$. The case $a \neq 0$ can be reduced to $a = 0$ by defining

$$\widetilde{f}(z) := f(z) + a, \quad z \in [-1,1], \tag{4.50}$$

$$\widetilde{r}(z) := r(z) + a \quad \text{for } r \in \mathcal{R}_{n,n}, z \in \mathbb{C}. \tag{4.51}$$

If $a \in \mathbb{C}$, we note that the inequality (4.30) is equivalent to

$$\lim_{n\in\Lambda_3, n\to\infty} \left\| (\widetilde{r}_n^* - \widetilde{r}_{n+1}^*) q_n q_{n+1} \right\|_K^{1/n} \leq 1 \tag{4.52}$$

and (3.14) is equivalent to

$$\lim_{n\in\Lambda, n\to\infty} \left| (\widetilde{r}_n^* - \widetilde{r}_{n+1}^*)(\xi_n) \right|^{1/n} = 1, \tag{4.53}$$

where $\{\xi_n\}_{n\in\Lambda}$, $\xi_n \in [-1,1]$, and $|(\widetilde{f} - \widetilde{r}_n^*)(\xi_n)| = \|\widetilde{f} - \widetilde{r}_n^*\|$. Therefore, all arguments for the sequence $\{F_n\}$ are invariant by replacing in definition (4.10) the functions r_n^*, r_{n+1}^* by $\widetilde{r}_n^*, \widetilde{r}_{n+1}^*$. Hence, Theorem 2.2 is true for all $a \in \mathbb{C}$. □

Proof of the Corollary. In the proof of Theorem 2.2, the subsequence Λ was chosen such that

$$\nu_n - \alpha_n(\widehat{\tau}_n + \widehat{\tau}_{n+1}) - (1-\alpha_n)\mu \xrightarrow{*} 0 \quad \text{as } n \longrightarrow \infty, n \in \Lambda, \tag{4.54}$$

where

$$\alpha_n = \frac{\deg q_n^* + \deg q_{n+1}^*}{d_n + 1}. \tag{4.55}$$

Since $\{m_n\}$ fulfills (2.17), we obtain

$$\begin{aligned} \alpha_n &= \frac{\deg q_n^* + \deg q_{n+1}^*}{n + m_n + 1 - \delta_n} \\ &\leq \frac{\deg q_n^* + \deg q_{n+1}^*}{n + m_n + 1 - (m_n - \deg q*_n)} \\ &= 1 - \frac{n + 1 - \deg q_{n+1}^*}{n + 1 + \deg q_n^*} \\ &< 1 - \frac{n + 1 - c(n+1)}{n + 1 + c(n+1)} = 1 - \frac{1-c}{1+c}. \end{aligned} \tag{4.56}$$

Hence, by (3.1)

$$\nu_n([\alpha,\beta]) \geq \frac{1-c}{1+c}\mu([\alpha,\beta]) \tag{4.57}$$

for any interval $[\alpha, \beta] \subset C[-1,1]$. Therefore, property (i) of Theorem 2.2 implies that

$$\nu_n \xrightarrow{*} \nu, \quad \operatorname{supp}(\nu) = [-1,1] \tag{4.58}$$

and Theorem 2.2 holds for all $z_0 \in [-1,1]$. □

5. Generalization to the Lower-Half of the Walsh Table

Theorem 2.2 restricts the approximation to the upper half of the Walsh table. In the following, we also want to allow approximations in the lower half of the Walsh table. We assume that the pairs

$$(n(s), m(s)) \in \mathbb{N}_0 \times \mathbb{N}_0 \tag{5.1}$$

depend on parameters $s \in \mathbb{N}$. For abbreviation, let

$$E_s := E_{n(s),m(s)}(f), \qquad r_s^* = r_{n(s),m(s)}^*(f) = \frac{p_s^*}{q_s^*}, \tag{5.2}$$

where p_s^* and q_s^* have no common factor. As above, let

$$\delta_s := \min\bigl(n(s) - \deg p_s^*, m(s) - \deg p_s^*\bigr) \tag{5.3}$$

be the defect of r_s^*, and let $A_s = A_s(f) = \{x_k^{(s)}\}_{k=0}^{d(s)}$ be an alternation point set to $f - r_s^*$, where

$$d_s = n(s) + m(s) + 1 - \delta_s. \tag{5.4}$$

We denote by ν_s the normalized counting measure of A_s. Then Theorem 2.2 can be generalized in the following way.

Theorem 5.1. *Let $(n(s), m(s))$, $s \in \mathbb{N}$, be a strictly increasing subsequence of $\mathbb{N}_0 \times \mathbb{N}_0$ with*

$$n(s) \le n(s+1) \le n(s) + 1, \qquad m(s) \le m(s+1) \le m(s) + 1, \tag{5.5}$$

and let us approximate $f \in C[-1,1]$, with respect to $\mathcal{R}_{n(s),m(s)}$, where

$$\begin{aligned} m(s) &\le n(s) + \kappa(s), \quad s \in \mathbb{N}, \\ \kappa(s) &= o\left(\frac{s}{\log s}\right) \quad \text{as } s \longrightarrow \infty. \end{aligned} \tag{5.6}$$

If $f \in C[-1,1]$ satisfies (2.6), then there exists a subset $\Lambda \subset \mathbb{N}$ with the following properties:

(i) $\nu_s \xrightarrow{*} \nu$ *as* $s \to \infty$, $s \in \Lambda$.

(ii) *let* $a \in \mathbb{C}$*; then for any* $z_0 \in \operatorname{supp}(\nu)$ *and any neighborhood* U *of* z_0 *with* $f(z) \not\equiv a$ *on* $U \cap [-1,1]$ *either*

$$\limsup_{s\to\infty} N_\infty(r_s^*, U) = \infty \tag{5.7}$$

or

$$\limsup_{s\to\infty} \frac{N_a(r_s^*, U)}{s} > 0. \tag{5.8}$$

For the proof, we use a generalization of Theorem B to the previous situation (see [10]): if (5.5) and (5.6) hold, then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\nu_s - \alpha_s(\widehat{\tau}_s + \widehat{\tau}_{s+1}) - (1-\alpha_s)\mu \overset{*}{\to} 0 \quad \text{as } s \longrightarrow \infty, s \in \Lambda. \tag{5.9}$$

Again, we use in (5.9) the balayage measures of the normalized pole counting measures τ_s and τ_{s+1} of r_s^*, respectively, r_{s+1}^*, onto $[-1,1]$ and

$$\alpha_s := \frac{\deg q_s^* + \deg q_{s+1}^*}{d_s + 1}. \tag{5.10}$$

Then the proof of (5.7) and (5.8) follows the same lines as the proof of Theorem 2.2 if

$$\limsup_{s\to\infty} E_s^{1/s} = 1. \tag{5.11}$$

Because of (5.5), the index $n(s)$ runs from $n(1)$ to ∞. Moreover, let $M(s) := \max(n(s), m(s))$, $s \in \mathbb{N}$; then $M(s)$ runs from $M(1)$ to ∞ and

$$\begin{aligned} \limsup_{s\to\infty} E_s^{1/s} &= \limsup_{s\to\infty} E_{n(s),m(s)}^{1/s} \\ &\geq \limsup_{s\to\infty} \left(E_{M(s),M(s)}^{1/M(s)}\right)^{M(s)/s} = 1, \end{aligned} \tag{5.12}$$

since

$$s \geq M(s) - M(1). \tag{5.13}$$

6. Remarks

For the function $f(x) = |x|^\alpha$, $\alpha > 0$, the distribution of alternation points of the optimal error curves, as well as the zeros and poles of $r^*_{n,m}$ is very well investigated [7].

Let $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$, and let $(n, m_n) \in \mathbb{N} \times \mathbb{N}$ with

$$\lim_{n\to\infty} \frac{m_n}{n} = c \leq 1, \quad n \geq m_n + 2\left[\frac{\alpha}{2}\right]. \tag{6.1}$$

Since all best approximants of $f(x) = |x|^\alpha$ are even functions, we can assume that $n, m_n \in \mathbb{N}$ are even. Moreover, the error function $f - r^*_{n,m_n}$ has always exactly $n + m_n + 3$ points [7]. By $\nu_{A_n} = \nu_n$ we denote the normalized alternation counting measure and ν_{P_n} denotes the normalized pole counting measure of r^*_{n,m_n} and ν_{Z_n} the normalized zero counting measure of r^*_{n,m_n}. Then

$$\nu_{A_n} \overset{*}{\underset{n\to\infty}{\to}} \frac{2c}{1+c}\delta_0 + \frac{1-c}{1+c}\mu, \tag{6.2}$$

$$\nu_{P_n} \overset{*}{\underset{n\to\infty}{\to}} \delta_0, \tag{6.3}$$

$$\nu_{Z_n} \overset{*}{\underset{n\to\infty}{\to}} c\delta_0 + (1-c)\mu \tag{6.4}$$

(cf. Theorems 1.6 and 1.7 in [7]).

For $c < 1$, we would obtain by (3.1) and by the corollary of Theorem 2.2 that any point of $[-1, 1]$ is either a limit point of poles or of a-values of r^*_{n,m_n}, $a \in \mathbb{C}$, as $n \to \infty$. Since by (6.3) the normalized pole counting measures converge to the Dirac measure at 0, any point of $[-1, 1]$, with 0 as only possible exception, is a limit point of a-values.

For $c = 1$, $\nu_{A_n} \overset{*}{\to} \delta_0$. Hence Theorem 2.2 can only tell us that the point 0 is either a limit point of poles or of a-values, $a \in \mathbb{C}$. But (6.3) and (6.4) show that 0 is as well a limit point of zeros as of poles of r^*_{n,m_n}. Hence, the investigations in [7] for the special functions $f(x) = |x|^\alpha$ lead to deeper results for the zeros and poles of the best approximants.

But the example of $f(x) = |x|^\alpha$ shows an interesting area for further investigations, namely, a weak*-type analogue of relation (3.1) for the distribution of zeros, respectively, a-values, and poles of rational approximation would be desirable. Moreover, the approximation problem should be moved from the interval $[-1, 1]$ to more general compact sets E in $\mathbb{C}$.

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