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Research Article

On Subclasses of Analytic Functions with respect to Symmetrical Points

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In our present investigation, motivated from Noor work, we define the class $\mathcal{R}_k^s(b)$ of functions of bounded radius rotation of complex order b with respect to symmetrical points and learn some of its basic properties. We also apply this concept to define the class $\mathcal{A}_k^s(\alpha, b, \delta)$. We study some interesting results, including arc length, coefficient difference, and univalence sufficient condition for this class.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic function satisfying the condition $f(0) = 0$, $f'(0) - 1 = 0$ in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and in more simple form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{1.1}$$

By $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{S}^*$, we means the well-known subclasses of $\mathcal{A}$ which consists of univalent, convex, and starlike functions, respectively. In [1], Sakaguchi introduced the class $\mathcal{S}_s^*$ of starlike functions with respect to symmetrical points and is defined as follows: a function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_s^*$, if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{1.2}$$

Motivated from Sakaguchi work, Das and Singh [2] extend the concepts of $\mathcal{S}_s^*$ to other class in $\mathcal{U}$, namely, convex functions with respect to symmetrical points. Let $\mathcal{C}_s$ denote the class of convex functions with respect to symmetrical points and satisfying the following condition:

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)-f'(-z)}\right\}>0 \quad (z\in\mathcal{U}). \tag{1.3}$$

Let $\mathcal{P}_k(\delta)$, $0\le\delta<1$, be the class of functions $p(z)$ analytic in $\mathcal{U}$ with $p(0)=1$ and

$$\int_0^{2\pi}\left|\frac{\operatorname{Re}p(z)-\delta}{1-\delta}\right|d\theta\le k\pi, \quad z=re^{i\theta},\ k\ge 2. \tag{1.4}$$

This class was introduced in [3]. For $\delta=0$, we obtain the class $\mathcal{P}_k$ defined by Pinchuk [4], and for $k=2$, the class $\mathcal{P}_k$ reduces to the class $\mathcal{P}$ of functions with positive real part.

Now, with the help of the aforementioned concepts, we define the class $\mathcal{R}_k^s(b)$ of functions of bounded radius rotation of complex order b with respect to symmetrical points as follows.

Definition 1.1. Let $f(z)\in\mathcal{A}$ in $\mathcal{U}$. Then $f(z)\in\mathcal{R}_k^s(b)$, if and only if

$$1+\frac{1}{b}\left\{\frac{2zf'(z)}{f(z)-f(-z)}-1\right\}\in\mathcal{P}_k \quad (z\in\mathcal{U}), \tag{1.5}$$

where $k\ge 2$ and $b\in\mathbb{C}-\{0\}$.

Using the class $\mathcal{R}_k^s(b)$, we define the class $\mathcal{H}_k^s(\alpha,b,\delta)$ as follows.

Definition 1.2. Let $f(z)\in\mathcal{A}$ in $\mathcal{U}$. Then $f(z)\in\mathcal{H}_k^s(\alpha,b,\delta)$, if and only if there exists $g(z)\in\mathcal{R}_k^s(b)$ such that

$$\frac{zf'(z)}{f(z)}\left(\frac{2f(z)}{g(z)-g(-z)}\right)^{\alpha}\in\mathcal{P}(\delta), \tag{1.6}$$

where $\alpha>0$, $0\le\delta<1$, and $b\in\mathbb{C}-\{0\}$.

It is noticed that, by giving specific values to α, b, δ, and k in $\mathcal{R}_k^s(b)$ and $\mathcal{H}_k^s(\alpha,b,\delta)$, we obtain many well-known as well as new subclasses of analytic and univalent functions; for details see [5–11].

Throughout this paper, we will assume, unless otherwise stated, that $k\ge 2, \alpha>0, 0\le\delta<1$, and $b\in\mathbb{C}-\{0\}$.

Lemma 1.3. *Let $p(z)$ be analytic in $\mathcal{U}$ where $p(0)=1$ belongs to $P(\delta)$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2d\theta\le\frac{1+\left(4(1-\delta)^2-1\right)r^2}{1-r^2} \tag{1.7}$$

(see [8, 12]).

Lemma 1.4. *Let $s_1(z)$ be univalent function in $\mathcal{U}$. Then there exists ξ with $|\xi| = r$ such that for all z, $|z| = r$,*

$$|z-\xi||s_1(z)| \leq \frac{2r^2}{1-r^2} \tag{1.8}$$

(see [13]).

2. Some Properties of the Classes $\mathcal{R}_k^s(b)$ and $\mathcal{H}_k^s(\alpha, b, \delta)$

Theorem 2.1. *Let $f(z) \in \mathcal{R}_k^s(b)$. Then the odd function*

$$\phi(z) = \frac{1}{2}\left[f(z) - f(-z)\right] \tag{2.1}$$

belongs to $\mathcal{R}_k(b)$ in $\mathcal{U}$.

Proof. Let $f(z) \in \mathcal{R}_k^s(b)$ and consider

$$\phi(z) = \frac{1}{2}\left[f(z) - f(-z)\right]. \tag{2.2}$$

From logarithmic differentiation of the previous relation, we have

$$\frac{\phi'(z)}{\phi(z)} = \frac{f'(z) - f'(-z)}{f(z) - f(-z)}, \tag{2.3}$$

or, equivalently,

$$\frac{z\phi'(z)}{\phi(z)} = \frac{1}{2}\left[p_1(z) + p_2(z)\right] \tag{2.4}$$

with

$$p_1(z) = \frac{2zf'(z)}{f(z) - f(-z)}, \qquad p_2(z) = \frac{2(-z)f'(-z)}{f(-z) - f(z)} \tag{2.5}$$

belongs to $\mathcal{P}_k(b)$. Since $\mathcal{P}_k(b)$ is a convex set, we have

$$\frac{z\phi'(z)}{\phi(z)} \in \mathcal{P}_k(b) \quad (z \in \mathcal{U}), \tag{2.6}$$

and hence $\phi(z) \in \mathcal{R}_k(b)$. □

Theorem 2.2. *Let $f(z) \in \mathcal{R}_k^s(b)$. Then*

$$f'(z) = \frac{1}{2}\left[b\left(p(z)-1\right)+1\right] \exp\left\{\frac{b}{2}\int_0^z \frac{1}{\xi}\left(p(\xi)-p(-\xi)-\frac{2}{b}\right)d\xi\right\}. \tag{2.7}$$

Proof. Let $f(z) \in \mathcal{R}_k^s(b)$. Then by definition we have

$$1+\frac{1}{b}\left[\frac{2zf'(z)}{f(z)-f(-z)}-1\right] = p(z), \quad p(z) \in \mathcal{P}_k. \tag{2.8}$$

Simple computation yields us

$$\frac{f(z)-f(-z)}{z} = \exp\left\{\frac{b}{2}\int_0^z \frac{1}{\xi}\left[p(\xi)-p(-\xi)-\frac{2}{b}\right]d\xi\right\}. \tag{2.9}$$

Using (2.8) in (2.9), we can easily obtain (2.7).

If we put $b = 1$ and $k = 2$ in Theorem 2.1, we obtain the integral representation for $\mathcal{S}_s^*$ given by Stankiewiez in [14]. □

Theorem 2.3. *Let $f(z) \in \mathcal{R}_k^s(b)$. Then*

$$|a_2| \leq \frac{k|b|}{2}. \tag{2.10}$$

The function $f_0(z) \in \mathcal{R}_k^s(b)$ defined by

$$f_0'(z) = \frac{\left(1+z^2\right)^{(k-2)/4}}{\left(1-z^2\right)^{(k+2)/4}}\left[b\left\{\left(\frac{k+2}{4}\right)\left(\frac{1-z}{1+z}\right)-\left(\frac{k-2}{4}\right)\left(\frac{1+z}{1-z}\right)\right\}+(1-b)\right] \tag{2.11}$$

shows that this bound is sharp.

Proof . Since $f(z) \in \mathcal{R}_k^s(b)$, there exists an odd function $\phi(z) \in \mathcal{R}_k(b)$ with

$$\phi(z) = \frac{1}{2}\left[f(z)-f(-z)\right], \tag{2.12}$$

such that

$$zf'(z) = \phi(z)p(z), \tag{2.13}$$

with $p(z) \in \mathcal{P}_k(b)$. Let

$$\phi(z) = z+\sum_{n=2}^{\infty} b_{2n-1}\, z^{2n-1}, \qquad p(z) = 1+\sum_{n=1}^{\infty} c_n z^n. \tag{2.14}$$

Then (2.13) implies that

$$z+\sum_{n=2}^{\infty} n a_n z^n = \left[z+\sum_{n=2}^{\infty} b_{2n-1}\ z^{2n-1}\right]\left[1+\sum_{n=1}^{\infty} c_n z^n\right]. \tag{2.15}$$

Equating the coefficients of z^2, we have $2a_2 = c_1$, and so

$$|a_2| \le \frac{k|b|}{2}, \tag{2.16}$$

where we have used the coefficient bounds $|c_1| \le k|b|$ for the class $\mathcal{P}_k(b)$. □

Corollary 2.4. *The range of every univalent function $f(z) \in \mathcal{R}_k^s(b)$ contains the disc*

$$|w| < \frac{2}{4+k|b|}. \tag{2.17}$$

Proof. The Koebe one-quarter theorem states that each omitted value w of the univalent function $f(z)$ of the form (1.1) satisfies

$$|w| > \frac{1}{2+|a_2|}. \tag{2.18}$$

Using (2.18) and Theorem 2.3, we obtain the required result. □

By using the same method as in [1], we obtain the following result.

Theorem 2.5. *Let $f(z) \in \mathcal{R}_k^s(b)$. Then, for $z = re^{i\theta}$ and $0 \le \theta_1 < \theta_2 \le 2\pi$,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} d\theta > -(k-1)|b|\pi. \tag{2.19}$$

Theorem 2.6. *Let $f(z) \in \mathcal{M}_k^s(\alpha, b, 0)$. Then, for $z = re^{i\theta}$,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} J(\alpha, f(z)) d\theta > -(\alpha|b|(k-1)+1)\pi, \tag{2.20}$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$ and

$$J(\alpha, f(z)) = \left(1+\frac{zf''(z)}{f'(z)}\right) + (\alpha-1)\frac{zf'(z)}{f(z)}. \tag{2.21}$$

Proof. We can define, for $z = re^{i\theta}$, $r < 1$, θ real, the following:

$$S(r,\theta) = \arg\left[zf'(z)f^{\alpha-1}(z)\right], \tag{2.22}$$

$$V(r,\theta) = \arg\left[\frac{g(z)-g(-z)}{2}\right]^{\alpha}. \tag{2.23}$$

The functions $S(z)$ and $V(z)$ are periodic and continuous with period 2π. Since $f(z) \in \mathcal{A}_k^s(\alpha,b,0)$, therefore from (2.22), it follows that we can choose the branches of argument of $S(z)$ and $V(z)$ as

$$|S(r,\theta) - V(r,\theta)| \le \frac{\pi}{2}. \tag{2.24}$$

Now we have from (2.22)

$$V(r,\theta_2) - V(r,\theta_1) = \alpha\int_{\theta_1}^{\theta_2} \operatorname{Re}\frac{z\phi'(z)}{\phi(z)}d\theta, \tag{2.25}$$

where $\phi(z)$ is an odd function of the following form:

$$\phi(z) = \frac{1}{2}\left[g(z) - g(-z)\right]. \tag{2.26}$$

Since $g(z) \in \mathcal{R}_k^s(b)$, therefore by using Theorem 2.5, we have

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\frac{z\phi'(z)}{\phi(z)}d\theta > -(k-1)|b|\pi. \tag{2.27}$$

From (2.22), (2.23), (2.24), and (2.27), we have

$$\begin{aligned} |S(r,\theta_2) - S(r,\theta_1)| &= |S(r,\theta_2) - V(r,\theta_2) - (S(r,\theta_1) - V(r,\theta_1)) + (V(r,\theta_2) - V(r,\theta_1))| \\ &< \frac{\pi}{2} + \frac{\pi}{2} + \alpha(k-1)|b|\pi = (\alpha|b|(k-1)+1)\pi. \end{aligned} \tag{2.28}$$

Moreover, from (2.22)

$$\frac{d}{d\theta}S(r,\theta) = \operatorname{Re}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) + (\alpha-1)\frac{zf'(z)}{f(z)}\right]. \tag{2.29}$$

Therefore

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} J\big(\alpha, f(z)\big)d\theta > -(\alpha|b|(k-1)+1)\pi. \tag{2.30}$$

□

Theorem 2.7. *Let $f(z) \in \mathscr{H}_k^s(\alpha, b, \delta)$. Then for $\alpha(k/2+1)\operatorname{Re} b > 1$,*

$$L_r f(z) \leq \begin{cases} C(\alpha,b,\delta,k) M^{1-\alpha}(r) \left(\dfrac{1}{1-r}\right)^{\alpha(k/2+1)\operatorname{Re} b}, & 0 < \alpha \leq 1, \\ C(\alpha,b,\delta,k) m^{\alpha-1}(r) \left(\dfrac{1}{1-r}\right)^{\alpha(k/2+1)\operatorname{Re} b}, & \alpha > 1, \end{cases} \tag{2.31}$$

where $m(r) = \min_{|z|=r}|f(z)|$, $M(r) = \max_{|z|=r}|f(z)|$, and $C(\alpha,b,\delta,k)$ is a constant depending upon α, b, δ, and k only.

Proof. We know that

$$L_r f(z) = \int_0^{2\pi} \left|z f'(z)\right| d\theta, \quad z = re^{i\theta},\ 0 < r < 1. \tag{2.32}$$

Since $f(z) \in \mathscr{H}_k^s(\alpha, b, \delta)$, therefore

$$\frac{zf'(z)}{f(z)} \left(\frac{2f(z)}{g(z) - g(-z)}\right)^{\alpha} = p(z), \quad p(z) \in \mathcal{P}(\delta). \tag{2.33}$$

By Theorem 2.1, we have, for $g(z) \in \mathcal{R}_k^s(b)$, the odd function $\phi(z) = (1/2)[g(z) - g(-z)] \in \mathcal{R}_k(b)$. This implies that

$$zf'(z) = \left(f(z)\right)^{1-\alpha} \left(\phi(z)\right)^{\alpha} p(z). \tag{2.34}$$

Therefore, we have

$$\begin{aligned} L_r f(z) &\leq \int_0^{2\pi} |f(z)|^{1-\alpha} |\phi(z)|^{\alpha} |p(z)| d\theta, \\ &\leq M^{1-\alpha}(r) \int_0^{2\pi} |\phi(z)|^{\alpha} |p(z)| d\theta. \end{aligned} \tag{2.35}$$

Since $\phi(z) \in \mathcal{R}_k(b)$, therefore we have for odd functions $s_1(z)$, $s_1(z) \in \mathcal{S}^*$,

$$\begin{aligned} &\leq M^{1-\alpha}(r) \int_0^{2\pi} \left| \frac{(s_1(z))^{(k/4+1/2)b}}{(s_2(z))^{(k/4-1/2)b}} \right|^{\alpha} |p(z)| d\theta, \\ &\leq cM^{1-\alpha}(r) \int_0^{2\pi} \frac{|(s_1(z))|^{\alpha(k/4+1/2)\operatorname{Re} b}}{|(s_2(z))|^{\alpha(k/4-1/2)\operatorname{Re} b}} |p(z)| d\theta, \quad c = e^{(\pi/2)\operatorname{Im} b}. \\ &\leq cM^{1-\alpha}(r) 2^{\alpha(k/2-1)\operatorname{Re} b} r^{-\alpha(k/4-1/2)\operatorname{Re} b} \int_0^{2\pi} |(s_1(z))|^{\alpha(k/4+1/2)\operatorname{Re} b} |p(z)| d\theta. \end{aligned} \tag{2.36}$$

Now using Cauchy Schwarz inequality, we have

$$L_r f(z) \leq cM^{1-\alpha}(r) 2^{\alpha(k/2-1)\operatorname{Re} b} r^{-\alpha(k/4-1/2)\operatorname{Re} b} \left(\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \right)^{1/2} \times \left(\frac{1}{2\pi} \int_0^{2\pi} |(s_1(z))|^{2\alpha(k/4+1/2)\operatorname{Re} b} d\theta \right)^{1/2}. \tag{2.37}$$

By Lemma 1.3 and distortion results for the class $\mathcal{S}^*$ with a subordination result, we obtain

$$\begin{aligned} L_r f(z) &\leq cM^{1-\alpha}(r) 2^{\alpha(k/2-1)\operatorname{Re} b} r^{\alpha \operatorname{Re} b} \left(\frac{1}{(1-r)^{4\alpha(k/4+1/2)\operatorname{Re} b-1}} \right)^{1/2} \left(\frac{1 + \left(4(1-\delta)^2 - 1\right) r^2}{1-r^2} \right)^{1/2} \\ &= C(\alpha, b, \delta, k) M^{1-\alpha}(r) \left(\frac{1}{1-r} \right)^{\alpha(k/2+1)\operatorname{Re} b}. \end{aligned} \tag{2.38}$$

Similarly for $\alpha > 1$, we have

$$L_r f(z) \leq C(\alpha, b, \delta, k) m^{\alpha-1}(r) \left(\frac{1}{1-r} \right)^{\alpha(k/2+1)\operatorname{Re} b}. \tag{2.39}$$

□

Theorem 2.8. *Let $f(z) \in \mathcal{H}_k^s(\alpha, b, \delta)$. Then for $\alpha(k/2+1)\operatorname{Re} b > 1$*

$$|a_n| \leq \begin{cases} C_1(\alpha, b, \delta, k) M^{1-\alpha}(n)(n)^{\alpha(k/2+1)\operatorname{Re} b-1}, & 0 < \alpha \leq 1, \\ C_1(\alpha, b, \delta, k) m^{\alpha-1}(n)(n)^{\alpha(k/2+1)\operatorname{Re} b-1}, & \alpha > 1, \end{cases} \tag{2.40}$$

where m and M are the same as in Theorem 2.7 and $C_1(\alpha, b, \delta, k)$ is a constant depending upon α, b, δ, and k only.

Proof. Since, with $z = re^{i\theta}$ Cauchy theorem gives

$$na_n = \frac{1}{2\pi r^n} \int_0^{2\pi} zf'(z) e^{-in\theta} d\theta, \tag{2.41}$$

therefore

$$n|a_n| \leq \frac{1}{2\pi r^n} L_r f(z). \tag{2.42}$$

Now using Theorem 2.7 for $0 < \alpha \leq 1$, we have

$$n|a_n| \leq \frac{1}{2\pi r^n} C(\alpha, b, \delta, k) M^{1-\alpha}(r) \left(\frac{1}{1-r}\right)^{\alpha(k/2+1)\operatorname{Re} b}. \tag{2.43}$$

Putting $r = 1 - 1/n$, we have

$$|a_n| \leq C_1(\alpha, b, \delta, k) M^{1-\alpha}(r)(n)^{\alpha(k/2+1)\operatorname{Re} b - 1}. \tag{2.44}$$

Similarly we obtain the required result for $\alpha > 1$. □

Theorem 2.9. *Let $f(z) \in \mathscr{K}_k^s(\alpha, b, \delta)$. Then for $\alpha(k/2+1)\operatorname{Re} b > 3$,*

$$||a_{n+1}| - |a_n|| \leq \begin{cases} C_2(\alpha, b, \delta, k) M^{1-\alpha}(r)(n)^{\alpha(k/2+1)\operatorname{Re} b - 2}, & 0 < \alpha \leq 1, \\ C_2(\alpha, b, \delta, k) m^{\alpha-1}(r)(n)^{\alpha(k/2+1)\operatorname{Re} b - 2}, & \alpha > 1. \end{cases} \tag{2.45}$$

Proof. We know that for $\xi \in \mathcal{U}$ and $n \geq 1$,

$$|(n+1)\xi a_{n+1} - n a_n| \leq \int_0^{2\pi} |z - \xi| \left| z f'(z) \right| d\theta, \quad z = re^{i\theta},\ 0 < r < 1,\ 0 \leq \theta \leq 2\pi. \tag{2.46}$$

Since $f(z) \in \mathscr{K}_k^s(\alpha, b, \delta)$, therefore

$$\frac{z f'(z)}{f(z)} \left(\frac{2 f(z)}{g(z) - g(-z)}\right)^{\alpha} = p(z), \quad p(z) \in \mathcal{P}(\delta). \tag{2.47}$$

By Theorem 2.1, we have, for $g(z) \in \mathcal{R}_k^s(b)$, the odd function $\phi(z) = (1/2)[g(z) - g(-z)] \in \mathcal{R}_k(b)$. This implies that

$$z f'(z) = \left(f(z)\right)^{1-\alpha} \left(\phi(z)\right)^{\alpha} p(z). \tag{2.48}$$

Thus, for $\xi \in \mathcal{U}$ and $n \geq 1$, we have

$$|(n+1)\xi a_{n+1} - n a_n| \leq M^{1-\alpha}(r) \int_0^{2\pi} |z - \xi| \left|\phi(z)\right|^{\alpha} \left|p(z)\right| d\theta. \tag{2.49}$$

Since $\phi(z) \in \mathcal{R}_k(b)$, therefore we have for odd functions $s_1(z)$, $s_1(z) \in \mathcal{S}^*$,

$$|(n+1)\xi a_{n+1} - n a_n| \leq \frac{2^{\alpha(k/2-1)\operatorname{Re} b} e^{\operatorname{Im} b(\pi/2)} M^{1-\alpha}(r)}{2\pi r^{n+1-\alpha(k/2-1)\operatorname{Re} b}} \int_0^{2\pi} |z - \xi| |(s_1(z))|^{\alpha(k/4+1/2)\operatorname{Re} b} \left|p(z)\right| d\theta. \tag{2.50}$$

By using Lemma 1.4, we have

$$\leq \frac{2^{\alpha(k/2-1)\operatorname{Re} b} e^{\operatorname{Im} b(\pi/2)} M^{1-\alpha}(r)}{2\pi r^{n-1-\alpha(k/2-1)\operatorname{Re} b}(1-r)} \int_0^{2\pi} |(s_1(z))|^{\alpha(k/4+1/2)\operatorname{Re} b-1} |p(z)| d\theta. \tag{2.51}$$

Now using Cauchy Schwarz inequality, we have

$$\begin{aligned} |(n+1)\xi a_{n+1} - n a_n| \leq{}& \frac{2^{\alpha(k/2-1)\operatorname{Re} b} e^{\operatorname{Im} b(\pi/2)} M^{1-\alpha}(r)}{2\pi r^{n-1-\alpha(k/2-1)\operatorname{Re} b}(1-r)} \left(\frac{1}{2\pi} \int_0^{2\pi} |(s_1(z))|^{2\alpha(k/4+1/2)\operatorname{Re} b-2} d\theta \right)^{1/2} \\ & \times \left(\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \right)^{1/2}. \end{aligned} \tag{2.52}$$

By Lemma 1.3 and distortion result for the class S^* with a subordination result, we obtain

$$\begin{aligned} |(n+1)\xi a_{n+1} - n a_n| \leq{}& c M^{1-\alpha}(r) 2^{\alpha(k/4)-\operatorname{Re} b} r^{\alpha \operatorname{Re} b-n+1} \left(\frac{1}{(1-r)} \right)^{\alpha(k/2+1)\operatorname{Re} b-5/2+1} \\ & \times \left(\frac{1 + \left(4(1-\delta)^2 - 1\right) r^2}{1-r^2} \right)^{1/2}. \end{aligned} \tag{2.53}$$

Now putting $|\xi| = r = n/(n+1)$, we obtain

$$||a_{n+1}| - |a_n|| \leq C_2(\alpha, b, \delta, k) M^{1-\alpha}(r)(n)^{\alpha(k/2+1)\operatorname{Re} b-2}. \tag{2.54}$$

Similarly for $\alpha > 1$, we have

$$||a_{n+1}| - |a_n|| \leq C_2(\alpha, b, \delta, k) m^{\alpha-1}(r)(n)^{\alpha(k/2+1)\operatorname{Re} b-2}. \tag{2.55}$$

□

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