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Research Article

On Generalized Localization of Fourier Inversion Associated with an Elliptic Operator for Distributions

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We study the behavior of Fourier integrals summed by the symbols of elliptic operators and pointwise convergence of Fourier inversion. We consider generalized localization principle which in classical L_p spaces was investigated by Sjölin (1983), Carbery and Soria (1988, 1997) and Alimov (1993). Proceeding these studies, in this paper, we establish sharp conditions for generalized localization in the class of finitely supported distributions.

1. Introduction

In this paper, we study the behavior of spherical Fourier integrals and pointwise convergence and summability of Fourier inversion.

Let

$$A(D) = \sum_{|\alpha|=m} c_\alpha D^\alpha \tag{1.1}$$

be a homogeneous elliptic differential operator of order m. Let us consider its symbol defined as polynomial:

$$A(x) = \sum_{|\alpha|=m} c_\alpha x^\alpha, \tag{1.2}$$

and assume that the Gaussian curvature of surface $S = \{x \in R^n : A(x) = 1\}$ is always strictly positive.

We recall that for $f \in L_2(R^n)$ its Fourier transform is defined as

$$\hat{f}(\xi) = \int f(y) e^{-iy\xi} dy \tag{1.3}$$

and partial Fourier integral associated with elliptic operator (1.1):

$$E_\lambda f(x) = (2\pi)^{-n} \int_{A(\xi)<\lambda} \hat{f}(\xi) d\xi \tag{1.4}$$

(note that throughout the paper we consider only Lebesque measure on R^n and $\int = \int_{R^n}$). For some functions, Fourier integrals do not converge pointwisely and various summation techniques are applied to recover convergence property. In this paper, we consider the method of the Riesz means. The Riesz means of order s are defined as

$$E_\lambda^s f(x) = (2\pi)^{-n} \int_{A(\xi)\le\lambda} \left(1 - \frac{|\xi|^2}{\lambda^2}\right)^s \hat{f}(\xi) e^{i\xi x} d\xi. \tag{1.5}$$

As an example, one can consider Laplacian $A(D) = \sum_{i=1}^n (\partial^2/\partial x_i^2)$, and note that the level surfaces of its symbol are Euclidean spheres. Thus, Fourier inversion associated with Laplace operator has the form:

$$E_\lambda f(x) = (2\pi)^{-n} \int_{|\xi|^2<\lambda} \hat{f}(\xi) e^{i\xi x} d\xi \tag{1.6}$$

and known as spherical partial Fourier integrals. The question of $E_\lambda f(x)$ convergence to $f(x)$ almost everywhere is not solved in R^n, $n \ge 2$ even for classical L_2 functions and presents one of the most challenging open problems of classical harmonic analysis, and even special cases of this problem are of particular interest. One of such special cases is the problem of generalized localization, which for the first time was formulated by V. Ii'in in [1]. For convenience, we give its definition for the Riesz means E_λ^s.

Definition 1.1. We say that, for the Riesz means of order s, the generalized localization principle in function class $\mathfrak{F}$ is satisfied, if for any function $f \in \mathfrak{F}$, the equality

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) = 0 \tag{1.7}$$

is true for a.e. $x \in R^n \setminus \operatorname{supp} f$.

This localization principle generalizes the classical Riemann localization principle and for L_p functions was intensively investigated by Sjölin [2], Carbery and Soria [3, 4], Bastis [5–7], and Ashurov et al. [8]. It was established that R^n localization holds true in L_p, where $p \in [2, 2n/(n-1)]$ and fails otherwise.

Over the last several years, a number of Fourier inversion studies considered distributions and investigated the behavior of their Fourier integrals (see, e.g., [9–12]). In particular, Alimov in [13] considered the classical Riemann localization principle for compactly supported distributions and established criteria for its validity (see also [14, 15]).

In this paper, we study generalized localization principle for compactly supported distributions and present conditions for its fulfillment.

2. Notation and Definitions

We define Schwartz space $S(R^n)$ as the function class of all infinitely differentiable functions that are rapidly decreasing at infinity along with all partial derivatives. It is well known that $S(R^n)$, being equipped with a family of seminorms

$$d_{\alpha,\beta}(\phi) = \sup_{x\in R^n}\left|x^\alpha D^\beta \phi(x)\right|, \tag{2.1}$$

is a Frechet space (here α, β are multi-indices and D is a partial derivative). As usual, we also consider class of tempered distributions S' defined as dual to S.

Let $\mathcal{E}$ be the space of infinitely differentiable functions with topology τ_E such that $\phi_n \to 0$ in τ_E if and only if for each multiindex α and compact K

$$\sup_{x\in K} D^\alpha \phi_n(x) \longrightarrow 0. \tag{2.2}$$

As usual we denote its conjugate space by $\mathcal{E}'$.

It is known (see, e.g., [16]) that each $f \in \mathcal{E}'$ has finite support and equivalent to the class of finitely supported tempered distributions. Thus, it follows from the Paley-Wienner theorem that, for each $f \in \mathcal{E}'$, its Fourier transform $\widehat{f} \in C^\infty$. Since $\widehat{f}$ is locally integrable, it is natural to define Fourier integral of $f \in \mathcal{E}'$ and its Riesz means by (1.4) and (1.5), respectively.

We also note that for $f \in L_2$ the Riesz mean $E_\lambda^s f$ can be considered as an integral operator:

$$E_\lambda^s f(x) = (2\pi)^{-n} \int f(y)\theta_\lambda^s(x-y)\,dy, \tag{2.3}$$

with kernel $\theta_\lambda^s(y) = \widehat{m}_\lambda^s(y)$ where

$$m_\lambda^s(y) = \left(1 - \frac{A(y)}{\lambda}\right)_+^s, \tag{2.4}$$

where $(1 - A(y)/\lambda)_+^s = (1 - A(y)/\lambda)^s \cdot \chi_{A(y)<\lambda}(y)$.

Representation (2.3) has its natural analogue for $f \in \mathcal{E}'$. Let ψ_n be a sequence of Schwartz functions such that $\psi_n(y) = 0$ as $|y| > \lambda$ and $\psi_n(y) \to m_\lambda^s(y)$ in L_1 norm. Then:

$$\begin{aligned} E_\lambda^s f(x) &= \lim_{n\to\infty} (2\pi)^{-n} \int \widehat{f}(\xi)\psi_n(\xi)e^{ix\xi}d\xi \\ &= (2\pi)^{-n} \lim_{n\to\infty} \left\langle \widehat{f}(\xi), \psi_n(\xi)e^{ix\xi} \right\rangle \\ &= (2\pi)^{-n} \lim_{n\to\infty} \langle f(y), \widehat{\psi}_n(x-y)\rangle. \end{aligned} \tag{2.5}$$

Note that inequality $\|\widehat{g}\|_\infty \le \|g\|_1$ implies that $\widehat{\psi}_n \to \widehat{m}_\lambda^s$ in $\mathcal{E}$ and since f is continuous on $\mathcal{E}$

$$E_\lambda^s f(x) = (2\pi)^{-n}\langle f(\cdot), \theta_\lambda^s(x-\cdot)\rangle. \tag{2.6}$$

We will need Sobolev's classes which can be defined for $l \in \mathbb{R}$ in the following way.

Definition 2.1. We say that tempered distribution f belongs to Sobolev class H^l if $\widehat{f}$ is a regular distribution such that

$$\|f\|_{H^l}^2 = \int \left|\widehat{f}(\xi)\right|^2 \left(1+|\xi|^2\right)^l d\xi < \infty. \tag{2.7}$$

One can see that, in particular, $H^0 = L_2$. We also remark that for every $f \in \mathcal{E}'$ there is $l \in \mathbb{R}$ such that $f \in H^l$ (for proof see, e.g., [16]).

In other respects, we make the following conventions:

(i) symbol J_ν is used to denote Bessel function of the first kind and order $\nu \ge 0$,

(ii) χ_E is preserved for an indicator function of $E \subset R^n$,

(iii) unless otherwise indicated, all functions are assumed to be defined on R^n and by definition $L_p(\Omega) \equiv \{f \in L_p(R^n) : \operatorname{supp} f \subset \Omega \subset R^n\}$.

3. Main Result

As has been mentioned above, every $f \in \mathcal{E}'$ belongs to some Sobolev classes H^l, in this paper, we use this fact to establish criterion of generalized localization for finitely supported distributions. The following theorems present major results of current study.

Theorem 3.1. *Let $f \in \mathcal{E}' \cap H^{-l}$, $l \ge 0$. Then, for integer $s \ge l$, equality*

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) = 0 \tag{3.1}$$

holds true a.e. on $R^n \setminus \operatorname{supp} f$.

Our approach is based on the methods by Carbery and Soria [3] and in order to prove Theorem 3.1, we will follow his idea first proving some auxiliary facts in the following section.

4. Dual Sets

Let $a(x) = [A(x)]^{1/m}$ and $K = \{x \in R^n : a(x) \leq 1\}$. Then, K is a symmetric body that is convex compact symmetric set. We recall that set $K^* = \{y : |x \cdot y| \leq 1, \forall x \in K\}$ is called polar set with respect to K.

As it is done in [17], we will introduce the norm $\|\cdot\|_a$ generated by $a(x)$ as

$$\|x\|_a = a(x) \tag{4.1}$$

and dual norm $\|\cdot\|_a^*$ as

$$\|y\|_a^* = \sup_{\|x\|_a \leq 1} |x \cdot y| = \sup_{\|x\|_a = 1} |x \cdot y|. \tag{4.2}$$

Next, let S and S^* be the boundaries of K and K^*, respectively.

It is not difficult to show that $S^* = \{\nabla a(x), x \in S\}$. Indeed on the one hand $a(\lambda x) = \lambda a(x)$ and, therefore, for $x \in S$

$$\begin{aligned} x \cdot \nabla a(x) &= \frac{da}{dr}(x) = a(x) = 1, \\ (-x) \cdot \nabla a(x) &= -\frac{da}{dr}(x) = a(x) = -1, \end{aligned} \tag{4.3}$$

which means that $\|\nabla a(x)\|_a^* \geq 1$. On the other hand, for any $y \in S$, one can consider $F(y) = y \cdot \nabla a(x)$ and examine its local extremums on the surface S. Since S is compact, $F(y)$ reaches its extremum values and it is known that, at extremum points, $\nabla F(y)$ must be parallel to the normal to S at point y, which is parallel to $\nabla a(y)$. Since $\nabla F(y) = \nabla a(x)$, we can conclude that $\nabla a(x) \| \nabla a(y)$ at the extremum points. Since S is strictly convex, it is possible only for $y = \pm x$, that implies $\|\nabla a(x)\|_a^* \leq 1$.

It is convenient for given $x \in R^n$ to use the notation $\theta(x)$ to denote the point on S such that the outer normal to S at $\theta(x)$ is parallel to x. Similarly, we denote $\eta(x)$ the point on S^* such that the outer normal to S^* at $\eta(x)$ is parallel to x. One can remark that we have just seen that for $y \in S^*$

$$y \cdot \theta(y) = 1. \tag{4.4}$$

5. Technical Lemmas for Theorem 3.1

We will need the asymptotic representation of $\theta_\lambda^s(y)$, which can be derived by stationary-phase method (see, e.g., [18]):

$$\theta_\lambda^s(y) = \lambda^{(n-1)/2} |y|^{-(n+1)/2} \cdot \left[R_+^s(y, \lambda) e^{i\lambda y \cdot \theta(y)} + R_-^s(y, \lambda) e^{-i\lambda y \cdot \theta(y)} \right], \tag{5.1}$$

where functions $R^s_\pm(y,\lambda) \in C^\infty(\{y : |y| > \epsilon\} \times [1,\infty))$ and

$$D^\alpha_y D^\beta_\lambda R^s_\pm(y,\lambda) = O\left(\lambda^{-s-\beta}\right), \tag{5.2}$$

uniformly on $|y| > \delta$ and $\lambda > \delta$.

Now, let us consider positive numbers ε and R, $\varepsilon < R$ and function $\phi(x) = \phi(\|x\|^*_a) \in C^\infty_0$ vanishing on $\{x : (\|x\|^*_a < \varepsilon) \vee (\|x\|^*_a > R)\}$. Then, for $s \geq 0$, we set by definition

$$\Theta^s_\lambda(x) = \phi(x)\theta^s_\lambda(x), \tag{5.3}$$

where θ^s_λ as in (2.3).

We will need some estimates for the Fourier transform of Θ^s_λ. With this aim, we will need the following lemmas.

Lemma 5.1. *Let $t \geq \delta > 0$ and $|\xi| < 1$. Then, for any $\alpha > 0$*

$$\left|\widehat{\Theta}^s_t(\xi)\right| \leq O(t^{-\alpha}). \tag{5.4}$$

Proof. This estimate easily follows from the definition of Θ^s_t. Indeed,

$$\begin{aligned}
\widehat{\Theta}^s_t(\xi) &= \int \theta^s_t(x)\phi(x)e^{-i\xi x}dx = \int \left(1 - \frac{\widehat{A(y)}}{t}\right)^s_+ (x)\phi(x)e^{-i\xi x}dx \\
&= \int \left(1 - \frac{A(x)}{t}\right)^s_+ \widehat{\phi}(x+\xi)dx = \int_{A(x)<t} \widehat{\phi}(x+\xi)dx \\
&\quad + \sum_{k=1}^s \frac{C_k}{t^k} \int_{A(x)<t} \widehat{\phi_k(\cdot,\xi)}(x)dx,
\end{aligned} \tag{5.5}$$

where $\phi_k(y,\xi) = B^k(D_y)[\phi(y)e^{-iy\xi}]$ and $B(D)$ is formally conjugate to operator $A(D)$. Since $\phi(0) = \phi_k(0,\xi) = 0$,

$$\widehat{\Theta}^s_t(\xi) = \int_{A(x)>t} \widehat{\phi}(x+\xi)dx + \sum_{k=1}^s \frac{C_k}{t^k} \int_{A(x)>t} \widehat{\phi_k(\cdot,\xi)}(x)dx. \tag{5.6}$$

Further, we notice that since $\phi_k(y,\xi) \in C^\infty_0$ then for any $\alpha > 0$ there is C_α such that functions $\widehat{\phi}_k(x,\xi) = C_\alpha/(1 + A(x))^\alpha$, uniformly for $k = 1,\ldots,s$ and $|\xi| < 1$. For the same reason, for any $\alpha > 0$, one has $\widehat{\phi}(x) \leq O((1+x)^{-\alpha})$. Now substituting these estimates into (5.6), we complete the proof. □

Lemma 5.2. *Let $t \geq \delta > 0$ and $|\xi| \geq 1$. Then, for any $\alpha > 0$,*

$$\left|\widehat{\Theta}^s_t(\xi)\right| = \frac{O(1)t^{-s}}{(1 + |\|\xi\|_a - t|)^\alpha}. \tag{5.7}$$

Proof. By definition,

$$\widehat{\Theta}_t^s(\xi) = \int_{\epsilon<\|y\|_a^*<R} \phi(y)\theta_t^s(y)e^{-i\xi\cdot y}dy. \tag{5.8}$$

Let us pass to a new coordinate system $y \to (r = \|y\|_a^*, \eta = \eta(y))$. Then,

$$\widehat{\Theta}_t^s(\xi) = \int_\epsilon^R \phi(r)r^{n-1}\int_{\eta\in S^*} \theta_t^s(r\eta)e^{-ir\xi\cdot\eta}d\sigma(\eta)dr, \tag{5.9}$$

where $d\sigma(\eta)$ is a Lebesgue surface measure of S^*.

Using (5.1), we have

$$\begin{aligned}\widehat{\Theta}_t^s(\xi) &= t^{(n-1)/2}\int_\epsilon^R \phi(r)r^{(n-3)/2}\int_{\eta\in S^*} |\eta|^{-(n+1)/2}e^{itr[\eta\cdot\theta(\eta)]}\widetilde{R}_+^s(r\eta,t)e^{-ir\xi\cdot\eta}d\sigma(\eta)dr \\ &\quad + t^{(n-1)/2}\int_\epsilon^R \phi(r)r^{(n-3)/2}\int_{\eta\in S^*} |\eta|^{-(n+1)/2}e^{-itr[\eta\cdot\theta(\eta)]}\widetilde{R}_-^s(r\eta,t)e^{-ir\xi\cdot\eta}d\sigma(\eta)dr.\end{aligned} \tag{5.10}$$

We will focus on the first term since the second one can be handled alike

$$I_t^s(\xi) = t^{(n-1)/2}\int_\epsilon^R \phi(r)r^{(n-3)/2}\int_{\eta\in S^*} |\eta|^{-(n+1)/2}e^{itr[\eta\cdot\theta(\eta)]}\widetilde{R}_+^s(r\eta,t)e^{-ir\xi\cdot\eta}d\sigma(\eta)dr \tag{5.11}$$

and note that due to (4.4) $\eta\cdot\theta(\eta) = 1$, and thus

$$I_t^s(\xi) = t^{(n-1)/2}\int_\epsilon^R \phi(r)r^{(n-3)/2}e^{itr}\int_{\eta\in S^*} |\eta|^{-(n+1)/2}\widetilde{R}_+^s(r\eta,t)e^{-ir\xi\cdot\eta}d\sigma(\eta)dr. \tag{5.12}$$

One can use the expression $\xi = \|\xi\|_a\theta(\xi)$ and employ stationary phase method to obtain

$$\begin{aligned}\int_{\eta\in S^*} |\eta|^{-(n+1)/2}\widetilde{R}_+^s(r\eta,t)e^{-ir\xi\cdot\eta}d\sigma(\eta) &= \|\xi\|_a^{-(n-1)/2}e^{ir\|\xi\|_a[\theta(\xi)\cdot\eta(\theta(\xi))]}P_+^s(r\xi,t) \\ &\quad + \|\xi\|_a^{-(n-1)/2}e^{-ir\|\xi\|_a[\theta(\xi)\cdot\eta(\theta(\xi))]}P_-^s(r\xi,t),\end{aligned} \tag{5.13}$$

where $P_\pm^s$ are smooth functions such that $D_t^\alpha D_z^\beta P_\pm^s(z,t) = O(t^{-s-\alpha})$. Using this expression, we have

$$\begin{aligned}I_t^s(\xi) &= \left(\frac{t}{\|\xi\|_a}\right)^{(n-1)/2}\int_\epsilon^R \phi(r)r^{(n-3)/2}e^{ir(t-\|\xi\|_a)}P_+^s(r\xi,t)dr \\ &\quad + \left(\frac{t}{\|\xi\|_a}\right)^{(n-1)/2}\int_\epsilon^R \phi(r)r^{(n-3)/2}e^{-ir(t-\|\xi\|_a)}P_-^s(r\xi,t)dr.\end{aligned} \tag{5.14}$$

Further integrating by parts the integrals, one can see that for any $N > 0$ both integrals are controlled by $(C_N t^{-s})/(1 + |t - \|\xi\|_a|)^N$. As a result, we have

$$\left|I_t^s(\xi)\right| \leq \left(\frac{t}{\|\xi\|_a}\right)^{(n-1)/2} \frac{C_N t^{-s}}{(1 + |t - \|\xi\|_a|)^N} \leq \frac{D_N t^{-s}}{(1 + |t - \|\xi\|_a|)^{N-((n-1)/2)}}, \tag{5.15}$$

uniformly for $|\xi| > 1$ and $t > \delta$. Finally, substituting into (5.10), we obtain (5.7). □

Now, combining Lemmas 5.1 and 5.2, we can claim that, in fact, for $t > \delta$ and any $\xi \in R^n$,

$$\left|\widehat{\Theta}_t^s(\xi)\right| \leq \frac{O(1) t^{-s}}{(1 + |\|\xi\|_a - t|)^{\alpha}}. \tag{5.16}$$

Lemma 5.3. *Let $\Theta_\lambda^s(x)$ be defined by* (5.3). *Then, for any $\delta > 0$ there is $C_\delta > 0$ such that*

$$\int_\delta^\infty \left|\widehat{\Theta}_t^s(\xi)\right|^2 dt \leq \frac{C_\delta}{(1 + |\xi|)^{2s}}. \tag{5.17}$$

Proof. As it follows from (5.16),

$$\int_\delta^\infty \left|\widehat{\Theta}_t^s(\xi)\right|^2 dt \leq O(1) \int_\delta^\infty \frac{t^{-2s} dt}{(1 + |\|\xi\|_a - t|)^{2\alpha}}, \tag{5.18}$$

where $\alpha > 0$ can be chosen arbitrary large. Changing the variables $u = \xi - t$, one has

$$\begin{aligned} \int_\delta^\infty \frac{|t|^{-2s} dt}{(1 + |\|\xi\|_a - t|)^{2\alpha}} &= \int_{\delta < |\|\xi\|_a - u| < \|\xi\|_a/2} \frac{|\|\xi\|_a - u|^{-2s} dt}{(1 + |u|)^{2\alpha}} \\ &\quad + \int_{\max(\delta, \|\xi\|_a/2) < |\|\xi\|_a - u|} \frac{|\|\xi\|_a - u|^{-2s} dt}{(1 + |u|)^{2\alpha}}. \end{aligned} \tag{5.19}$$

It is not difficult to see that for the values u in the first integral $|u| > |\|\xi\|_a - | u - \|\xi\|_a| > \|\xi\|_a/2$, and thus choosing $\alpha > \max(2s, 1)$

$$\int_{\delta < |\|\xi\|_a - u| < \|\xi\|_a/2} \frac{|\|\xi\|_a - u|^{-2s} dt}{(1 + |u|)^{2\alpha}} \leq \frac{O(1)}{(1 + \|\xi\|_a)^{\alpha}} \leq \frac{O(1)}{(1 + \|\xi\|_a)^{2s}}. \tag{5.20}$$

Moreover, it is clear that for such α

$$\int_{\delta < |\|\xi\|_a - u| < \|\xi\|_a/2} \frac{|\|\xi\|_a - u|^{-2s} dt}{(1 + |u|)^{2\alpha}} \leq \frac{O(1)}{(1 + \|\xi\|_a)^{2s}}. \tag{5.21}$$

Therefore,

$$\int_{\delta}^{\infty}\left|\widehat{\Theta}_t^s(\xi)\right|^2 dt \le \frac{O(1)}{\left(1+\|\xi\|_a\right)^{2s}}. \tag{5.22}$$

Since all norms in R^n are equivalent, the lemma is proved. □

Lemma 5.4. *Let* Θ_λ^s *be defined by* (5.3). *Then, for any* $\delta > 0$, *there is* C_δ *such that*

$$\int_{\delta}^{\infty}\left|\frac{d}{dt}\widehat{\Theta}_t^s(\xi)\right|^2 dt \le \frac{C_\delta}{\left(1+|\xi|\right)^{2s}}. \tag{5.23}$$

Proof. For any $t > 1$, using the Fubini theorem, one has

$$\int_1^t\int \frac{d}{du}\Theta_u^s(y)e^{-i\xi y}\,dy\,du = \int e^{-i\xi y}\int_1^t \frac{d}{du}\Theta_u^s(y)\,du\,dy = \widehat{\Theta}_t^s(\xi) - \widehat{\Theta}_1^s(\xi), \tag{5.24}$$

which implies $(d/dt)\widehat{\Theta}_t^s(\xi) = \widehat{(d/dt)\Theta_t^s}(\xi)$.

If $s > 0$

$$\begin{aligned}
\frac{d}{dt}\Theta_t^s(x) &= \phi(x)\frac{d}{dt}\theta_t^s(x)\\
&= \phi(x)\frac{d}{dt}\int_0^t\left(1-\frac{u^2}{t^2}\right)^s d\theta_u(x)\\
&= \frac{2s\phi(x)}{t}\int_0^t\left(1-\frac{u^2}{t^2}\right)^{s-1}\frac{u^2}{t^2}\,d\theta_u(x)\\
&= \frac{2s}{t}\left(\Theta_t^{s-1}(x) - \Theta_t^s(x)\right).
\end{aligned} \tag{5.25}$$

Thus, using inequality $(a+b)^2 \le 2a^2 + 2b^2$, one has

$$\int_{\delta}^{\infty}\left|\frac{d}{dt}\widehat{\Theta}_t^s(\xi)\right|^2 dt \le C\int_{\delta}^{\infty} t^{-2}\left|\widehat{\Theta}_t^{s-1}(\xi)\right|^2 dt + C\int_{\delta}^{\infty} t^{-2}\left|\widehat{\Theta}_t^s(\xi)\right|^2 dt. \tag{5.26}$$

Now, one can use estimate (5.16) to each integral on the right side and complete the proof.

If $s = 0$, then for any $\xi \in R^n$,

$$\begin{aligned}\left|\frac{d}{dt}\widehat{\Theta}_t(\xi)\right| &= \left|\frac{d}{dt}\int_{A(y)\le t}\widehat{\phi}(\xi+y)dy\right| \\ &= \left|\int_{A(y)=t}\widehat{\phi}(\xi+y)n(y)d\sigma(y)\right| \\ &\le \frac{O(1)}{(1+|\|\xi\|_a - t|)^{\alpha}}, \quad \forall\alpha > 0.\end{aligned} \tag{5.27}$$

Using this estimate and the reasoning presented in the previous lemma, we obtain the required estimate. □

6. Proof of Theorem 3.1

Let $f \in H^{-l} \cap \mathcal{E}'$ be such that supp $f \subset \Omega$. For $0 < \epsilon < 1/2$, we set

$$E_\epsilon = \left\{x : 2\epsilon < \operatorname{dist}(x,\Omega) < (2\epsilon)^{-1}\right\} \tag{6.1}$$

and consider an arbitrary radial function $\phi_\epsilon \in C_0^\infty$ such that

$$\phi_\epsilon(x) = \begin{cases} 1, & \dfrac{3\epsilon}{2} \le |x| \le \dfrac{1}{\epsilon} + \operatorname{diam}\ \Omega; \\ 0, & |x| \le \epsilon. \end{cases} \tag{6.2}$$

It is clear that to prove the theorem it is sufficient to show that for any $\epsilon > 0$, $\lim_{\lambda\to\infty} E_\lambda^s f(x) = 0$, a.e. $x \in E_\epsilon$

In this case, as $x \in E_\epsilon$ due to (2.6)

$$\begin{aligned} E_\lambda^s f(x) &= \int \widehat{f}(\xi)\left[\chi_\Omega(\cdot)\theta_\lambda^s(x-\cdot)\right]\hat{}(-\xi)d\xi \\ &= \int \widehat{f}(\xi)\left[\phi_\epsilon(x-\cdot)\theta_\lambda^s(x-\cdot)\right]\hat{}(-\xi)d\xi \\ &= \int \widehat{f}(\xi)\left[\phi_\epsilon\theta_\lambda^s\right]\hat{}(\xi)e^{i\xi x}d\xi, \end{aligned} \tag{6.3}$$

or using notation (5.3)

$$E_\lambda^s f(x) = \int \widehat{f}(\xi)\widehat{\Theta}_\lambda^s(\xi)e^{i\xi x}d\xi = \left(\widehat{\widehat{f}(\xi)\widehat{\Theta}_\lambda^s(\xi)}\right)(-x). \tag{6.4}$$

Further, we consider maximal operator:

$$E_*^s f(x) = \sup_{\lambda>1} \left| E_\lambda^s f(x) \right|. \tag{6.5}$$

We recall that to prove a.e. convergence on E_ϵ one can use the standard technique of Banach principle (see, e.g., [19]) according to which it is sufficient to estimate maximal operator on $E_\epsilon \subset R^n \setminus \operatorname{supp} f$ as

$$\left\| E_*^s f(x) \right\|_{L_2(E_\epsilon)} \leq C \| f \|_{H^{-l}}. \tag{6.6}$$

Let $\gamma(t) : \mathbb{R} \to \mathbb{R}_+$ be a C^∞ function such that

$$\gamma(t) = \begin{cases} 0, & t \leq \dfrac{1}{3}; \\ \\ 1, & t \geq \dfrac{2}{3}. \end{cases} \tag{6.7}$$

If we set $\widetilde{E}_\lambda^s f(x) = \gamma(\lambda) E_\lambda^s f(x)$, then by (6.4),

$$\widetilde{E}_\lambda^s f(x) = \gamma(\lambda) < f(\cdot), \qquad \Theta_\lambda^s(x - \cdot) >= \gamma(\lambda) \Big(\widehat{\widehat{f}(\xi) \widehat{\Theta}_\lambda^s(\xi)} \Big)(-x). \tag{6.8}$$

According to Sobolev's embedding theorem (see, e.g., [20]) for any $f \in H^1(R^1)$,

$$\| f \|_{L_\infty} \leq C \| f \|_{H^1}. \tag{6.9}$$

Using this fact, we have

$$E_*^s f(x) \leq \left\| \widetilde{E}_\lambda^s f(x) \right\|_{L^\infty(R)} \leq \left\| \widetilde{E}_\lambda^s f(x) \right\|_{H^1(R)}. \tag{6.10}$$

And, therefore, in order to obtain (6.6), it is sufficient to show that there are constants C_1, C_2 such that the following estimates are true:

$$\begin{gathered} \int \left\| \widetilde{E}_\lambda^s f(x) \right\|_{L_2(R)}^2 dx \leq C_1 \| f \|_{H^{-l}}, \\ \int \left\| \frac{d}{d\lambda} \widetilde{E}_\lambda^s f(x) \right\|_{L_2(R)}^2 dx \leq C_2 \| f \|_{H^{-l}}. \end{gathered} \tag{6.11}$$

First, we note that estimate (5.16) and $f \in H^{-l}$ imply that $\widehat{f} \widehat{\Theta}_\lambda^s \in L_2$ which in turn with (6.8) implies the fact $\widetilde{E}_\lambda^s f \in L_2$.

Further, using the Plancherel theorem, we have

$$\begin{aligned}\int\int\left|\widetilde{E}_\lambda^s f(x)\right|^2 d\lambda\, dx &\le \int \gamma^2(\lambda)\int\left|\widehat{f}(\xi)\right|^2\left|\widehat{\Theta}_\lambda^s(\xi)\right|^2 d\xi\, d\lambda\\ &\le \int_{1/3}^{\infty}\gamma^2(t)\int\left(1+|\xi|^2\right)^l\left|\widehat{\Theta}_t^s(\xi)\right|^2\left(1+|\xi|^2\right)^{-l}\left|\widehat{f}(\xi)\right|^2 d\xi\, dt\\ &\le \sup_{\xi\in R^n}\left(1+|\xi|^2\right)^l\int_{1/3}^{\infty}\left|\widehat{\Theta}_t^s(\xi)\right|^2 dt\times\|f\|_{H^{-l}}^2\le C\|f\|_{H^{-l}}^2\end{aligned} \tag{6.12}$$

(the last inequality follows from Lemma 5.3).

For the same reason, (6.11) can be proved using Lemmas 5.3 and 5.4:

$$\begin{aligned}\int\int_{1/3}^{\infty}\left|\frac{d}{dt}\left[\gamma(t)\Theta_t^s\right]*f(x)\right|^2 dt\, dx &\le \left\|\gamma'^2(\lambda)\right\|_\infty\int\int_{1/3}^{\infty}\left|\Theta_t^s*f(x)\right|^2 dt\, dx\\ &\quad+\int\int_{1/3}^{\infty}\left|\frac{d}{d\lambda}\Theta_\lambda^s*f(x)\right|^2 d\lambda\, dx\le C\|f\|_{H^{-l}}^2.\end{aligned} \tag{6.13}$$

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