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Research Article

Spectral Properties of Non-Self-Adjoint Perturbations for a Spectral Problem with Involution

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Full description of Riesz basis property for eigenfunctions of boundary value problems for first order differential equations with involutions is given.

1. Introduction

Differential equations with involutions were considered firstly in [1]. They are a particular case of functional differential equations that appear in several applications (see, for instance, monographs [2, 3] and papers [4–6]). Different spectral problems for equations of this form were considered in [7, 8].

In particular, main questions about the following spectral problem:

$$u'(x) = \lambda u(-x), \quad -1 < x < 1, \tag{1.1}$$

$$u(-1) = \gamma u(1), \tag{1.2}$$

were solved in [7]. Namely, (1.1)-(1.2) is a Volterra operator if and only if $\gamma^2 = -1$; furthermore, (1.1)-(1.2) is self-adjoint if and only if γ is a real number. For $\gamma^2 \neq -1$, the system of eigenfunctions for (1.1)-(1.2) is a Riesz basis in $L_2(-1,1)$. Observe also that for $\gamma^2 \neq -1$, (1.1)-(1.2) has no associated functions, that is, all eigenvalues are simple. Note that problem (1.1)-(1.2) is an example of a generalized spectral problem of the form $Au = \lambda Su$, with $A = d/dx$ and $(Sf)(x) = f(-x)$. In general, they were considered in [9] when A and S are operators in a Banach space. Equiconvergence questions for two different perturbations of (1.1)-(1.2) were deeply studied in [10]. The main goal of the paper is to study questions about Riesz basis property of eigenfunctions for the following non-self-adjoint spectral problem:

$$u'(-x) + \alpha u'(x) = \lambda u(x), \quad -1 < x < 1, \tag{1.3}$$

$$u(-1) = \gamma u(1). \tag{1.4}$$

Also note that problems similar to (1.3)-(1.4) appear when the Fourier method is applied for solving boundary value problems for partial differential equations with involution (see, for example, [11] and the bibliography therein).

2. Results

Theorem 2.1. *If* $\alpha^2 \neq 1$, $\gamma \neq \alpha \pm \sqrt{\alpha^2 - 1}$, *then the eigenfunctions system for* (1.3)-(1.4) *is a Riesz basis in* $L_2(-1, 1)$.

Proof. Before the proof, we need several facts about (1.3)-(1.4). First of all, it is easy to see that the general solution of (1.3)-(1.4) with $\alpha^2 \neq 1$ is given by the following formula:

$$u(x) = C\left(\sqrt{1+\alpha}\cos\frac{\lambda x}{\sqrt{1-\alpha^2}} + \sqrt{1-\alpha}\sin\frac{\lambda x}{\sqrt{1-\alpha^2}}\right). \tag{2.1}$$

Next, we observe that for $\alpha^2 \neq 1$, $\gamma^2 \neq 1$ eigenvalues are equal to

$$\lambda_k = \sqrt{1-\alpha^2}\left[k\pi + \arctan\frac{1-\gamma}{1+\gamma}\frac{\sqrt{1+\alpha}}{\sqrt{1-\alpha}}\right], \quad k = 0, \pm 1, \ldots \tag{2.2}$$

The related eigenfunctions are given by the following formula:

$$\begin{aligned} u_k(x) = {} & \sqrt{1+\alpha}\cos\left[k\pi + \arctan\frac{1-\gamma}{1+\gamma}\frac{\sqrt{1+\alpha}}{\sqrt{1-\alpha}}\right]x \\ & + \sqrt{1-\alpha}\sin\left[k\pi + \arctan\frac{1-\gamma}{1+\gamma}\frac{\sqrt{1+\alpha}}{\sqrt{1-\alpha}}\right]x, \quad k = 0, \pm 1, \ldots \end{aligned} \tag{2.3}$$

Observe also that for $\gamma = 1$, the eigenvalues are

$$\lambda_k = \sqrt{1-\alpha^2}\, k\pi, \quad k = 0, \pm 1, \pm 2, \ldots \tag{2.4}$$

The corresponding eigenfunctions are

$$u_k(x) = \sqrt{1+\alpha}\cos(k\pi x) + \sqrt{1-\alpha}\sin(k\pi x), \quad k = 0, \pm 1, \pm 2, \ldots; \tag{2.5}$$

and for $\gamma = -1$, the eigenvalues are

$$\lambda_k = \sqrt{1-\alpha^2}\left(k\pi + \frac{\pi}{2}\right), \quad k = 0, \pm 1, \ldots, \tag{2.6}$$

and the corresponding eigenfunctions are

$$u_k(x) = \sqrt{1+\alpha}\, \cos\left(k\pi + \frac{\pi}{2}\right)x + \sqrt{1-\alpha}\, \sin\left(k\pi + \frac{\pi}{2}\right)x, \quad k = 0, \pm 1, \ldots \tag{2.7}$$

We introduce the differential operator L by

$$Lu = u'(-x) + \alpha u'(x), \quad -1 < x < 1, \tag{2.8}$$

and by the boundary condition (1.4). Suppose that Lu belongs to the domain of L, $Lu \in D(L)$. Then we consider

$$L^2 u = L\left(u'(-x) + \alpha u'(x)\right) = -\left(1-\alpha^2\right)u''(x). \tag{2.9}$$

From the boundary condition (1.4), for Lu we deduce that L^2 is the second order differential operator generated by the following relations:

$$L^2 u = -\left(1-\alpha^2\right)u''(x), \quad -1 < x < 1, \tag{2.10}$$

$$\begin{gathered} (\alpha - \gamma)u'(-1) + (1-\alpha\gamma)u'(1) = 0, \\ u(-1) - \gamma u(1) = 0. \end{gathered} \tag{2.11}$$

Spectral problem (2.10)-(2.11) is a typical spectral problem for an ordinary second order differential operator. These problems are studied very well and they have numerous applications (see, for example, [12–14].) Recall [12, Chapter 2] that the following boundary conditions:

$$\begin{gathered} a_1 u'(-1) + b_1 u'(1) + c_1 u(-1) + d_1 u(1) = 0, \\ c_0 u(-1) + d_0 u(1) = 0, \end{gathered} \tag{2.12}$$

for an ordinary second order differential operator are regular if $b_1 c_0 + a_1 d_0 \neq 0$; and they are strongly regular if additionally $\theta_0^2 \neq 4\theta_{-1}\theta_1$, where

$$\begin{gathered} \theta_{-1} = \theta_1 = b_1 c_0 + a_1 d_0, \\ \theta_0 = 2(a_1 c_0 + b_1 d_0). \end{gathered} \tag{2.13}$$

Since $a_1 = (\alpha - \gamma)$, $b_1 = (1 - \alpha\gamma)$, $c_0 = 1$, $d_0 = -\gamma$, we obtain $\theta_{-1} = \theta_1 = \gamma^2 - 2\alpha\gamma + 1$, $\theta_0 = 2(\alpha - 2\gamma + \alpha\gamma^2)$. It follows from $\alpha^2 \neq 1$, $\gamma^2 \neq 1$ and $\gamma \neq \alpha \pm \sqrt{\alpha^2 - 1}$ that the boundary conditions

(2.11) are strongly regular. It is known [12, 13] that the eigenfunctions of an operator with strongly regular boundary conditions constitute a Riesz basis in $L_2(-1,1)$. By (2.2) numbers $-\lambda_k$ cannot be eigenvalues of L, hence any eigenfunction of L^2 which corresponds to λ_k^2 will be an eigenfunction L which corresponds to λ_k as

$$\left(L^2 - \lambda_k^2 E\right)u_k = (L + \lambda_k E)(L - \lambda_k E)u_k = 0. \tag{2.14}$$

Finally, we deduce the assertion of Theorem 2.1 in the case $\gamma^2 \neq 1$.

For the case $\gamma^2 = 1$ the explicit representations of eigenfunctions (2.5) and (2.7) give the Riesz basis property for these systems directly. □

Remark 2.2. If $\alpha = 0$, then (1.3)-(1.4) coincide with the unperturbed problem (1.1)-(1.2) which is a Volterra operator for $\gamma^2 = -1$, that is, $\gamma = \alpha \pm \sqrt{\alpha^2 - 1}$. If $\alpha \neq 0$ and $\gamma = \alpha \pm \sqrt{\alpha^2 - 1}$, then the boundary conditions (2.7)–(2.10) are nonregular and hence the system of eigenfunctions is incomplete [12, 13]. Finally, for $\alpha^2 = 1$, (1.3) has only trivial solution.

Now, we consider other types of non-selfadjoint perturbations of (1.1)-(1.2).

Theorem 2.3. *If $\gamma^2 \neq \pm 1$, then the eigenfunctions of the following spectral problem:*

$$\begin{gathered} u'(-x) + \alpha u(-x) = \lambda u(x), \quad -1 < x < 1, \\ u(-1) = \gamma u(1), \end{gathered} \tag{2.15}$$

constitute a Riesz basis of $L_2(-1,1)$.

Proof. The proof is analogous to the proof of Theorem 2.1. It uses the following spectral problem:

$$-u''(x) + \alpha^2 u(x) = \lambda u(x), \quad -1 < x < 1, \tag{2.16}$$

$$\begin{gathered} \gamma u'(-1) - u'(1) + \alpha\left(\gamma^2 - 1\right)u(1) = 0, \\ u(-1) - \gamma u(1) = 0. \end{gathered} \tag{2.17}$$

Boundary conditions (2.17) are regular for $\gamma^2 = -1$, and nonregular for $\gamma^2 = 1$. Then, basis property for eigenfunctions of an ordinary differential second order operator with constant coefficients gives the result for $\gamma^2 = 1$.

For $\gamma^2 \neq \pm 1$, boundary conditions (2.17) are strongly regular and the proof terminates analogously to the proof of Theorem 2.1. □

Remark 2.4. The perturbation $u'(-x) + \alpha u(x) = \lambda u(x)$ of (1.1)-(1.2) has the same form after the substitution $\lambda - \alpha = \mu$. Hence, the result of [7] gives full description of basis properties for the following spectral problem:

$$\begin{gathered} u'(-x) + \alpha u(x) = \lambda u(x), \quad -1 < x < 1, \\ u(-1) = \lambda u(1). \end{gathered} \tag{2.18}$$

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