Research Article

Duality for Multitime Multiobjective Ratio Variational Problems on First Order Jet Bundle

Mihai Postolache

Department of Mathematics and Informatics, Faculty of Applied Sciences,
University Politehnica of Bucharest, Splaiul Independenţei, No. 313, 060042 Bucharest, Romania

Correspondence should be addressed to Mihai Postolache, emscolar@yahoo.com

We consider a new class of multitime multiobjective variational problems of minimizing a vector of quotients of functionals of curvilinear integral type. Based on the efficiency conditions for multitime multiobjective ratio variational problems, we introduce a ratio dual of generalized Mond-Weir-Zalmai type, and under some assumptions of generalized convexity, duality theorems are stated. We prove our weak duality theorem for efficient solutions, showing that the value of the objective function of the primal cannot exceed the value of the dual. Direct and converse duality theorems are stated, underlying the connections between the values of the objective functions of the primal and dual programs. As special cases, duality results of Mond-Weir-Zalmai type for a multitime multiobjective variational problem are obtained. This work further develops our studies in (Pitea and Postolache (2011)).

1. Introduction

The duality for scalar variational problems involving convex functions has been formulated by Mond and Hanson in [1]. This study was further developed to various classes of convex and generalized convex functions. In [2], Hanson extended the duality results (in the sense of Wolfe) to a class of functions subsequently called invex. In order to weaken the convexity conditions, Bector et al. [3], introduced a dual to a variational problem (in the sense of Mond and Weir), different from that formulated by Mond and Hanson in [1]. Mond et al. [4], extended the concept of invexity to continuous functions and used it to generalize earlier Wolfe duality results for a class of variational problems. Duality theory for scalar optimization can be found in many other works and see Mond and Husain [5] and Preda [6].

In [7], using a vector-valued Lagrangian function, Tanino and Sawaragi introduced a duality theory for multiobjective optimization. This idea is employed by Bitran [8],

which associated a matrix of dual variables with the constraints in the primal problem. In [9], Mond and Weir considered a pair of symmetric dual nonlinear programs and developed a duality theory under assumptions of pseudoconvexity. Under various types of generalized convex functions, Mukherjee and Purnachandra [10], Preda [11] and Zalmai [12] established several weak efficiency conditions and developed different types of dualities for multiobjective variational problems. D. S. Kim and A. L. Kim [13] used the efficiency property of nondifferentiable multiobjective variational problems in duality theory. In a recent study [14], Pitea and Postolache considered a new class of multitime multiobjective variational problems of minimizing a vector of functionals of curvilinear integral type. Based on appropriate normal efficiency conditions, they studied duals of Mond-Weir type, generalized Mond-Weir-Zalmai type, and under appropriate assumptions of generalized convexity, stated duality theorems.

In time, several authors have been interested in the study of (vector) ratio programs in connection with generalized convexity. This study is motivated by many practical optimization problems whose objective functions are quotients of two functions. In [15], Jagannathan introduced a duality study using some results connecting solutions of a nonlinear ratio program with those of suitably defined parametric convex program. Concerning advances on single-objective ratio programs, see [16] by Khan and Hanson and [17] by Reddy and Mukherjee, which utilized invexity assumptions in the sense of Hanson [18] to obtain optimality conditions and duality results. As concerns vector ratio problems, Singh and Hanson [19] applied invex functions to derive duality results, while Jeyakumar and Mond [20] generalized these results to the class of V-invex functions. Later, Liang et al. [21] introduced a unified formulation of the generalized convexity to derive optimality conditions and duality results for vector ratio problems.

In this paper, we are motivated by previous published research articles (please refer to [22] by Antczak, [23] by Nahak and Mohapatra, and [14] by Pitea and Postolache) to consider a new class of multitime multiobjective variational problems of minimizing a vector of quotients of functionals of curvilinear integral type and, as a particular case, a vector of functionals of curvilinear integral type. We state and prove new duality results, of Mond-Weir-Zalmai type, for feasible solutions of our multitime multiobjective ratio variational problems, under assumptions of (ρ, b)-quasi-invexity. This study is encouraged by its possible application in mechanical engineering, where curvilinear integral objectives are extensively used due to their physical meaning as mechanical work. The objective vector function is of curvilinear integral type, the integrand depending on velocities; that is why we consider it adequate to introduce our results within the framework offered by the first order jet bundle, [24–26] by Pitea et al., [27] by Sounders, and [28] by Udriste et al.

2. Preliminaries

Before presenting our results, we need the following background, which is necessary for the completeness of the exposition. For more details, we address the reader to [27].

2.1. The First Order Jet Bundle

To make our presentation self-contained and reader-friendly, we recall the notion of jet bundle of the first order, $J^1(T, M)$.

Let be given the smooth manifolds (M, g) and (T, h), of dimensions n, and p, respectively. The corresponding local coordinates are $x = (x^i)$, $i = \overline{1, n}$, and $t = (t^\alpha)$, $\alpha = \overline{1, p}$, respectively.

In the following, the set $\{1, 2, \ldots, n\}$ will be indexed by Latin characters, while the set $\{1, 2, \ldots, p\}$ will be indexed by Greek characters.

Denote by $C^\infty(T, M)$ the set of mappings of class C^∞, from T to M.

Consider (t_0, x_0) an arbitrary point of the product manifold $T \times M$. On the set $C^\infty(T, M)$, define the equivalence relation

$$f \sim_{(t_0, x_0)} g \iff \begin{cases} f(t_0) = g(t_0) = x_0 \\ df_{t_0} = dg_{t_0}. \end{cases} \tag{2.1}$$

If f and g are arbitrary mappings from $C^\infty(T, M)$, we denote

$$t^\beta(t_0) = t_0^\beta, \quad \beta = \overline{1, p},$$
$$x^i = x^i \circ f,$$
$$y^i = x^i \circ g,$$
$$x^i(x_0) = x_0^i, \quad i = \overline{1, n}. \tag{2.2}$$

Thus, the equivalence relation $f \sim_{(t_0, x_0)} g$ has the local expression

$$x^i\left(t_0^\beta\right) = y^i\left(t_0^\beta\right) = x_0^i,$$
$$\frac{\partial x^i}{\partial t^\alpha}\left(t_0^\beta\right) = \frac{\partial y^i}{\partial t^\alpha}\left(t_0^\beta\right), \quad i = \overline{1, n}, \ \alpha = \overline{1, p}. \tag{2.3}$$

The equivalence class of a mapping $f \in C^\infty(T, M)$ will be denoted by

$$[f]_{(t_0, x_0)} = \left\{ g \in C^\infty(T, M) \mid g \sim_{(t_0, x_0)} f \right\}. \tag{2.4}$$

The quotient space obtained by the factorization of the space $C^\infty(T, M)$ by the equivalence relation "$\sim_{(t_0, x_0)}$",

$$J^1_{(t_0, x_0)}(T, M) = \frac{C^\infty(T, M)}{\sim_{(t_0, x_0)}}, \tag{2.5}$$

is called 1-jet at the point (t_0, x_0).

The total space of the set of 1-jets,

$$J^1(T, M) = \bigcup_{(t_0, x_0) \in T \times M} J^1_{(t_0, x_0)}(T, M), \tag{2.6}$$

can be organized as a vector bundle over the base space $T \times M$, endowed with the differentiable structure of the product space.

Let ς be the canonical projection, defined as

$$\varsigma : J^1(T, M) \longrightarrow T \times M, \qquad \varsigma\left([f]_{(t_0, x_0)}\right) = (t_0, f(t_0)). \tag{2.7}$$

The mapping ς is well defined, having the property to be onto.

For every local chart $U \times V$ of the product manifold $T \times M$, we define the bijection

$$\phi_{U \times V} : \varsigma^{-1}(U \times V) \longrightarrow U \times V \times \mathbb{R}^{np},$$

$$\phi_{U \times V}\left([f]_{(t_0, x_0)}\right) = \left(t_0, x_0, \frac{\partial x^i}{\partial t^\alpha}\left(t_0^\beta\right)\right), \qquad x_0 = f(t_0). \tag{2.8}$$

Therefore, the 1-jet space is endowed with a differentiable structure of dimension $n + p + np$, such that the mappings $\phi_{U \times V}$ are diffeomorphisms.

The local coordinates on the space $J^1(T, M)$ are $(t^\alpha, x^i, x^i_\alpha)$, where

$$t^\alpha\left([f]_{(t_0, x_0)}\right) = t^\alpha(t_0),$$

$$x^i\left([f]_{(t_0, x_0)}\right) = x^i(x_0), \tag{2.9}$$

$$x^i_\alpha\left([f]_{(t_0, x_0)}\right) = \frac{\partial x^i}{\partial t^\alpha}\left(t_0^\beta\right), \quad \alpha = \overline{1, p}, \ i = \overline{1, n}.$$

Remark 2.1. From physical viewpoint, the differentiable manifold T should be thought as a "temporal manifold" or a "multitime", while M is a "space manifold."

Remark 2.2. From geometrical viewpoint, an element $[f]$ of a fiber 1-jets $J^1(T, M)$ should be thought as a class of parametrized sheet. The sections of fiber 1-jets are "physical fields."

To simplify the notations, denote by $x_\gamma(t) = (\partial x / \partial t^\gamma)(t)$, $\gamma = \overline{1, p}$, the partial velocities. Also, in our subsequent theory, we will set $\pi_x(t) = (t, x(t), x_\gamma(t))$.

2.2. On Lagrange 1-Forms

A Lagrange 1-form of the first order on the jets space $J^1(T, M)$ has the form

$$\omega = L_\alpha(\pi_x(t))dt^\alpha + M_i(\pi_x(t))dx^i + N_i^\beta(\pi_x(t))dx^i_\beta, \tag{2.10}$$

where L_α, M_i and N_i^β are Lagrangians of the first order. The pullback

$$x^*\omega = \left(L_\alpha + M_i x^i_\alpha + N_i^\beta x^i_{\beta\alpha}\right)dt^\alpha \tag{2.11}$$

is a Lagrange 1-form of the second order on M. The coefficients $L_\alpha + M_i x_\alpha^i + N_i^\beta x_{\beta\alpha}^i$ are second order Lagrangians, which are linear in the partial accelerations.

A smooth Lagrangian $L(\pi_x(t))$, $t \in \mathbb{R}_+^m$ produces two smooth closed (completely integrable) 1-forms:

(i) the differential

$$dL = \frac{\partial L}{\partial x^i} dx^i + \frac{\partial L}{\partial x_\gamma^i} dx_\gamma^i + \frac{\partial L}{\partial t^\gamma} dt^\gamma \tag{2.12}$$

of components $(\partial L/\partial t^\gamma, \partial L/\partial x^i, \partial L/\partial x_\gamma^i)$, with respect to the basis $(dt^\gamma, dx^i, dx_\gamma^i)$;

(ii) the restriction of dL to $\pi_x(t)$, that is, the pullback

$$dL|_{\pi_x(t)} = \left(\frac{\partial L}{\partial x^i} \frac{\partial x^i}{\partial t^\beta} + \frac{\partial L}{\partial x_\gamma^i} \frac{\partial x_\gamma^i}{\partial t^\beta} + \frac{\partial L}{\partial t^\beta} \right) dt^\beta \tag{2.13}$$

of components containing partial accelerations

$$D_\beta L = \frac{\partial L}{\partial x^i}(\pi_x(t)) \frac{\partial x^i}{\partial t^\beta}(t) + \frac{\partial L}{\partial x_\gamma^i}(\pi_x(t)) \frac{\partial x_\gamma^i}{\partial t^\beta}(t) + \frac{\partial L}{\partial t^\beta}(\pi_x(t)), \tag{2.14}$$

with respect to the basis dt^β (for other ideas, see [28]).

3. Problem Description

Let (T, h) and (M, g) be Riemannian manifolds of dimensions p and n, respectively. Denote by $t = (t^\alpha)$, $\alpha = \overline{1, p}$, and $x = (x^i)$, $i = \overline{1, n}$, the local coordinates on T and M, respectively. Consider $J^1(T, M)$ the first order jet bundle associated to T and M.

To develop our theory, we recall the following relations between two vectors $v = (v^j)$ and $w = (w^j)$, $j = \overline{1, \delta}$:

$$v = w \Longleftrightarrow v^j = w^j, \quad j = \overline{1, \delta},$$

$$v < w \Longleftrightarrow v^j < w^j, \quad j = \overline{1, \delta},$$

$$v \leqq w \Longleftrightarrow v^j \leq w^j, \quad j = \overline{1, \delta} \quad \text{(product order relation)}, \tag{3.1}$$

$$v \leq w \Longleftrightarrow v \leqq w, \quad v \neq w.$$

Using the product order relation on $\mathbb{R}^p$, the hyperparallelepiped Ω_{t_0, t_1}, in $\mathbb{R}^p$, with diagonal opposite points $t_0 = (t_0^1, \ldots, t_0^p)$ and $t_1 = (t_1^1, \ldots, t_1^p)$, can be written as being the interval $[t_0, t_1]$. Suppose γ_{t_0, t_1} is a piecewise C^1-class curve joining the points t_0 and t_1.

The closed Lagrange 1-forms densities of C^∞-class

$$f_\alpha = \left(f_\alpha^\ell \right) : J^1(T, M) \longrightarrow \mathbb{R}^r, \quad k_\alpha = \left(k_\alpha^\ell \right) : J^1(T, M) \longrightarrow \mathbb{R}^r, \quad \ell = \overline{1, r}, \; \alpha = \overline{1, p}, \tag{3.2}$$

determine the following path independent curvilinear functionals (actions, mechanical work):

$$F^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} f_\alpha^\ell(\pi_x(t)) \, dt^\alpha, \qquad K^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} k_\alpha^\ell(\pi_x(t)) \, dt^\alpha. \tag{3.3}$$

The closeness conditions (complete integrability conditions) are

$$D_\beta f_\alpha^\ell = D_\alpha f_\beta^\ell, \qquad D_\beta k_\alpha^\ell = D_\alpha k_\beta^\ell, \quad \alpha, \beta = \overline{1, p}, \alpha \neq \beta, \ \ell = \overline{1, r}, \tag{3.4}$$

where D_β is the total derivative.

Suppose $K^\ell(x(\cdot)) > 0$, for all $\ell = \overline{1, r}$, and accept that the Lagrange matrix densities

$$g = \left(g_a^b \right) : J^1(T, M) \longrightarrow \mathbb{R}^{\bar{q}s}, \quad a = \overline{1, s}, \ \ b = \overline{1, \bar{q}}, \ \ \bar{q} < n,$$

$$h = \left(h_a^b \right) : J^1(T, M) \longrightarrow \mathbb{R}^{\tilde{q}s}, \quad a = \overline{1, s}, \ \ b = \overline{1, \tilde{q}}, \ \ \tilde{q} < n, \tag{3.5}$$

of C^∞-class define the partial differential inequations (PDIs) (of evolution)

$$g(\pi_x(t)) \leqq 0, \quad t \in \Omega_{t_0,t_1}, \tag{3.6}$$

and the partial differential equations (PDE) (of evolution)

$$h(\pi_x(t)) = 0, \quad t \in \Omega_{t_0,t_1}. \tag{3.7}$$

On the set $C^\infty(\Omega_{t_0,t_1}, M)$ of all functions $x : \Omega_{t_0,t_1} \to M$ of C^∞-class, we set the norm $\|x\| = \|x\|_\infty + \sum_{\alpha=1}^p \|x_\alpha\|_\infty$.

For each $\ell = \overline{1, r}$, suppose $K^\ell(x(\cdot)) > 0$, and consider

$$\frac{F(x(\cdot))}{K(x(\cdot))} = \left(\frac{F^1(x(\cdot))}{K^1(x(\cdot))}, \dots, \frac{F^r(x(\cdot))}{K^r(x(\cdot))} \right). \tag{3.8}$$

The aim of this work is to introduce and study the variational problem of minimizing a vector of quotients of functionals of curvilinear integral type:

$$\min_{x(\cdot)} \ \left(\frac{F(x(\cdot))}{K(x(\cdot))} \right)$$

$$\text{subject to} \quad x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1}), \tag{MFP}$$

where $\mathcal{F}(\Omega_{t_0,t_1})$ denotes the *set of all feasible solutions* of problem (MFP), defined as

$$\mathcal{F}(\Omega_{t_0,t_1}) = \left\{ x \in C^\infty(\Omega_{t_0,t_1}, M) \mid x(t_0) = x_0, x(t_1) = x_1, \text{or } x(t)|_{\partial\Omega_{t_0,t_1}} \right.$$

$$= \chi = \text{given, } g(\pi_x(t)) \leqq 0, h(\pi_x(t)) = 0, t \in \Omega_{t_0,t_1} \Big\}. \tag{3.9}$$

This kind of problems, of considerable interest, arises in various branches of mathematical, engineering, and economical sciences. We especially have in mind mechanical engineering, where curvilinear integral objectives are extensively used due to their physical meaning as mechanical work. These objectives play an essential role in mathematical modeling of certain processes in relation with robotics, tribology, engines, and much more.

4. Main Results

The following two definitions are crucial in developing our results. For more details, see [26] by Pitea et al., and [14], by Pitea and Postolache.

Definition 4.1. A feasible solution $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is called an efficient solution of (MFP) if there is no $x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$, $x(\cdot) \neq x^\circ(\cdot)$, such that $F(x(\cdot))/K(x(\cdot)) \leq F(x^\circ(\cdot))/K(x^\circ(\cdot))$.

Definition 4.2. Let ρ be a real number and let $b : C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M) \to [0, \infty)$ be a functional. To any closed 1-form $a = (a_\alpha)$ we associate the path independent curvilinear functional

$$A(x(\cdot)) = \int_{\gamma_{t_0,t_1}} a_\alpha(\pi_x(t)) dt^\alpha. \tag{4.1}$$

The functional A is called [strictly] (ρ, b)-quasi-invex at the point $x^\circ(\cdot)$ if there is a vector function $\eta : J^1(\Omega_{t_0,t_1}, M) \times J^1(\Omega_{t_0,t_1}, M) \to \mathbb{R}^n$, vanishing at the point $(\pi_{x^\circ}(t), \pi_{x^\circ}(t))$, and $\theta : C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M) \to \mathbb{R}^n$, such that for any $x(\cdot)$ in $C^\infty(\Omega_{t_0,t_1}, M)$, $[x(\cdot) \neq x^\circ(\cdot)]$, the following implication holds:

$$(A(x(\cdot)) \leq A(x^\circ(\cdot)))$$

$$\implies \left(b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \left[\left\langle \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial a_\alpha}{\partial x}(\pi_{x^\circ}(t)) \right\rangle \right. \right.$$

$$\left. + \left\langle D_\gamma \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial a_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) \right\rangle \right] dt^\alpha \tag{4.2}$$

$$[<] \leq -\rho b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2 \Big).$$

Several examples which illustrate our concept could be found in [14]. However, the following example is interesting. It is a generalization of Example 1 in [14].

Example 4.3. Let $a : [0,1] \times C^\infty([0,1]) \times C^\infty([0,1]) \to \mathbb{R}$, $x(\cdot) = (x^1(\cdot), x^2(\cdot), \ldots, x^n(\cdot))$. With $x^\circ(\cdot) \in C^\infty([0,1])$, denote

$$G(t) = \left\langle \frac{\partial a}{\partial x}(t, x^\circ(t), \dot{x}^\circ(t)) - D\frac{\partial a}{\partial \dot{x}}(t, x^\circ(t), \dot{x}^\circ(t)), \frac{\partial a}{\partial \dot{x}}(t, x^\circ(t), \dot{x}^\circ(t)) \right\rangle, \quad t \in [0,1], \quad (4.3)$$

where by D we denoted the total derivative operator.

Suppose that $G(1) - G(0) = 0$. Then, the functional

$$A(x(\cdot)) = \int_0^1 a(t, x(t), \dot{x}(t))dt \qquad (4.4)$$

is $(\rho, 1)$-quasi-invex, for $\rho \leq 0$ and any θ, at the point $x^\circ(\cdot)$, with respect to

$$\eta(\pi_x(t), \pi_{x^\circ}(t)) = (A(x(\cdot)) - A(x^\circ(\cdot)))\left(\frac{\partial a}{\partial x}(t, x^\circ(t), \dot{x}^\circ(t)) - D\frac{\partial a}{\partial \dot{x}}(t, x^\circ(t), \dot{x}^\circ(t)) \right). \qquad (4.5)$$

In their recent work [26], Pitea et al. established necessary efficiency conditions for problem (MFP). More accurately, they proved that if $x^\circ(\cdot)$ is an efficient solution of problem (MFP), then there are two vectors Λ^{10}, Λ^{20} in $\mathbb{R}^r$ and the smooth functions μ° and v°, the first from Ω_{t_0,t_1} to $\mathbb{R}^{\bar{q}sp}$, and the second from Ω_{t_0,t_1} to $\mathbb{R}^{\tilde{q}sp}$, such that

$$\left\langle \Lambda^{10}, \frac{\partial f_\alpha}{\partial x}(\pi_{x^\circ}(t)) \right\rangle - \left\langle \Lambda^{20}, \frac{\partial k_\alpha}{\partial x}(\pi_{x^\circ}(t)) \right\rangle$$

$$+ \left\langle \mu_\alpha^\circ(t), \frac{\partial g}{\partial x}(\pi_{x^\circ}(t)) \right\rangle + \left\langle v_\alpha^\circ(t), \frac{\partial h}{\partial x}(\pi_{x^\circ}(t)) \right\rangle$$

$$- D_\gamma \left(\left\langle \Lambda^{10}, \frac{\partial f_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) \right\rangle - \left\langle \Lambda^{20}, \frac{\partial k_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) \right\rangle \right. \qquad (4.6)$$

$$\left. + \left\langle \mu_\alpha^\circ(\pi_{x^\circ}(t)), \frac{\partial g}{\partial x_\gamma}(\pi_{x^\circ}(t)) \right\rangle + \left\langle v_\alpha^\circ(t), \frac{\partial h}{\partial x_\gamma}(\pi_{x^\circ}(t)) \right\rangle \right)$$

$$= 0, \quad t \in \Omega_{t_0,t_1}, \ \alpha = \overline{1,p} \qquad \text{(Euler-Lagrange PDEs)}$$

$$\langle \mu_\alpha^\circ(t), g(\pi_{x^\circ}(t)) \rangle = 0, \quad \mu_\alpha^\circ(t) \geqq 0, \ t \in \Omega_{t_0,t_1}, \ \alpha = \overline{1,p}. \qquad (4.7)$$

If $\Lambda^{10} \geq 0$ and $\Lambda^{20} \geq 0$, then $x^\circ(\cdot)$, from conditions (4.6), is called *normal efficient solution*.

Let $x^\circ(\cdot)$ be an efficient solution of primal (MFP), the scalars Λ^{10}, Λ^{20} in $\mathbb{R}^r$, and the smooth functions $\mu^\circ : \Omega_{t_0,t_1} \to \mathbb{R}^{\bar{q}sp}$, $v^\circ : \Omega_{t_0,t_1} \to \mathbb{R}^{\tilde{q}sp}$, given previously.

In order to use the idea of "grouping the resources," consider $\{P_0, P_1, \ldots, P_q\}$ and $\{Q_0, Q_1, \ldots, Q_q\}$ partitions of the sets $\{1, \ldots, \bar{q}\}$ and $\{1, \ldots, \tilde{q}\}$, respectively.

For each $\ell = \overline{1,r}$ and $\alpha = \overline{1,p}$, we denote

$$\overline{f}_\alpha^\ell(\pi_y(t)) = f_\alpha^\ell(\pi_y(t)) + \left\langle \mu_{\alpha P_0}(t), g^{P_0}(\pi_y(t)) \right\rangle + \left\langle \nu_{\alpha Q_0}(t), h^{Q_0}(\pi_y(t)) \right\rangle,$$

$$\overline{F}^\ell(y(\cdot)) = \int_{\gamma_{t_0,t_1}} \overline{f}_\alpha^\ell(\pi_y(t)) dt^\alpha. \tag{4.8}$$

Consider a function $y(\cdot) \in C^\infty(\Omega_{t_0,t_1}, M)$ and associate to (MFP) the multiobjective ratio variational problem

$$\max_{y(\cdot)} \quad \left(\frac{\overline{F}^1(y(\cdot))}{K^1(y(\cdot))}, \dots, \frac{\overline{F}^r(y(\cdot))}{K^r(y(\cdot))} \right)$$

$$\text{subject to} \quad \left\langle \Lambda^{10}, \frac{\partial f_\alpha}{\partial y}(\pi_y(t)) \right\rangle - \left\langle \Lambda^{20}, \frac{\partial k_\alpha}{\partial y}(\pi_y(t)) \right\rangle$$

$$+ \left\langle \mu_\alpha(t), \frac{\partial g}{\partial y}(\pi_y(t)) \right\rangle + \left\langle \nu_\alpha(t), \frac{\partial h}{\partial y}(\pi_y(t)) \right\rangle \tag{MFZD}$$

$$- D_\gamma \left(\left\langle \Lambda^{10}, \frac{\partial f_\alpha}{\partial y_\gamma}(\pi_y(t)) \right\rangle - \left\langle \Lambda^{20} \frac{\partial k_\alpha}{\partial y_\gamma}(\pi_y(t)) \right\rangle \right.$$

$$\left. + \left\langle \mu_\alpha(t), \frac{\partial g}{\partial y_\gamma}(\pi_y(t)) \right\rangle + \left\langle \nu_\alpha(t), \frac{\partial h}{\partial y_\gamma}(\pi_y(t)) \right\rangle \right)$$

$$= 0, \quad \alpha = \overline{1,p}, \ t \in \Omega_{t_0,t_1},$$

$$\left\langle \mu_{\alpha P_\kappa}(t), g^{P_\kappa}(\pi_y(t)) \right\rangle + \left\langle \nu_{\alpha Q_\kappa}(t), h^{Q_\kappa}(\pi_y(t)) \right\rangle \geq 0, \quad \kappa = \overline{1,q}, \ \alpha = \overline{1,p}, \ t \in \Omega_{t_0,t_1}$$

$$\Lambda^{10} \geq 0, \tag{4.9}$$

taking into account that the function $y(t)$ has to satisfy the boundary conditions $y(t_0) = x_0$, $y(t_1) = x_1$, or $y(t)|_{\partial\Omega_{t_0,t_1}} = \chi =$given.

$\varpi(x^\circ(\cdot))$ is the value of the objective function of problem (MFP) at $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$, and $\overline{\delta}(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot))$ is the maximizing functional vector of dual problem (MFZD) at the point $(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot)) \in \Delta$, where Δ is the domain of problem (MFZD).

We prove three duality results in a generalized sense of Mond-Weir-Zalmai.

Theorem 4.4 (weak duality). *Let $x^\circ(\cdot)$ be a feasible solution of problem (MFP) and let $(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot))$ be a feasible point of problem (MFZD). Assume that the following conditions are satisfied:*

(a) *$\langle \mu_{P_0}(t), g^{P_0}(\pi_y(t)) \rangle + \langle \nu_{Q_0}(t), h^{Q_0}(\pi_y(t)) \rangle \leq 0$, for all $t \in \Omega_{t_0,t_1}$;*

(b) *$\Lambda_\ell^{20} > 0$ and $\Lambda_\ell^{10} F^\ell(y(\cdot)) - \Lambda_\ell^{20} K^\ell(y(\cdot)) = 0$, for each $\ell = \overline{1,r}$;*

(c) *for each $\ell = \overline{1,r}$, the functional $\Lambda_\ell^{10} F^\ell(x(\cdot)) - \Lambda_\ell^{20} K^\ell(x(\cdot))$ is (ρ'^ℓ, b)-quasi-invex at the point $y(\cdot)$ with respect to η and θ, on $\mathcal{F}(\Omega_{t_0,t_1})$;*

(d) $\int_{\gamma_{t_0,t_1}} [\langle \mu_{\alpha P_\kappa}(t), g^{P_\kappa}(\pi_x(t)) \rangle + \langle \nu_{\alpha Q_\kappa}(t), h^{Q_\kappa}(\pi_x(t)) \rangle] dt^\alpha$ is (ρ''_κ, b)-quasi-invex at $y(\cdot)$ with respect to η and θ, for each $\kappa = \overline{1, q}$, on $F(\Omega_{t_0,t_1})$;

(e) at least one of the functionals of (c), (d) is strictly quasi-invex;

(f) $\sum_{\ell=1}^r \rho'^\ell + \sum_{\kappa=1}^q \rho''_\kappa \geq 0$.

Then, the inequality $\varpi(x^\circ(\cdot)) \leq \overline{\delta}(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot))$ is false.

Proof. By reductio ad absurdum, suppose we have

$$\frac{F^\ell(x^\circ(\cdot))}{K^\ell(x^\circ(\cdot))} \leq \frac{\overline{F}^\ell(y(\cdot))}{K^\ell(y(\cdot))}, \quad \ell = \overline{1, r}. \tag{4.10}$$

From these inequalities, it follows

$$\frac{\Lambda_\ell^{10} F^\ell(x^\circ(\cdot))}{\Lambda_\ell^{20} K^\ell(x^\circ(\cdot))} \leq 1 + \frac{1}{F^\ell(y(\cdot))} \int_{\gamma_{t_0,t_1}} \left[\left\langle \mu_{\alpha P_0}(t), g^{P_0}(\pi_y(t)) \right\rangle + \left\langle \nu_{\alpha Q_0}(t), h^{Q_0}(\pi_y(t)) \right\rangle \right] dt^\alpha, \tag{4.11}$$

and taking into account the hypotheses (a) and (b), we get

$$\Lambda_\ell^{10} F^\ell(x^\circ(\cdot)) \leq \Lambda_\ell^{20} K^\ell(x^\circ(\cdot)), \quad \forall \ell = \overline{1, r}. \tag{4.12}$$

We obtain the following implications:

$$\left(\left\langle \Lambda^{10}, F(x^\circ(\cdot)) \right\rangle - \left\langle \Lambda^{20}, K(x^\circ(\cdot)) \right\rangle \leq 0 \right)$$

$$\Longrightarrow \left(b(x^\circ(\cdot), y(\cdot)) \right.$$

$$\times \int_{\gamma_{t_0,t_1}} \left\{ \left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \left\langle \Lambda^{10}, \frac{\partial f_\alpha}{\partial y}(\pi_y(t)) \right\rangle - \left\langle \Lambda^{20}, \frac{\partial k_\alpha}{\partial y}(\pi_y(t)) \right\rangle \right\rangle \right.$$

$$+ \left\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \left\langle \Lambda^{10}, \frac{\partial f_\alpha}{\partial y_\gamma}(\pi_y(t)) \right\rangle \right\rangle$$

$$\left. - \left\langle \Lambda^{20}, \frac{\partial k_\alpha}{\partial y_\gamma}(\pi_y(t)) \right\rangle \right\rangle \right\} dt^\alpha$$

$$\left. \leq -b(x^\circ(\cdot), y(\cdot)) \| \theta(x^\circ(\cdot), y(\cdot)) \|^2 \sum_{\ell=1}^r \rho'^\ell \right). \tag{4.13}$$

Hypothesis (d) regarding the (ρ''_κ, b)-quasi-invexity property of each functional implies $(\kappa = \overline{1, q})$:

$$\left(\int_{\gamma_{t_0,t_1}} \left[\left\langle \mu_{\alpha P_\kappa}(t), g^{P_\kappa}(\pi_{x^\circ}(t)) \right\rangle + \left\langle \nu_{\alpha Q_\kappa}(t), h^{Q_\kappa}(\pi_{x^\circ}(t)) \right\rangle \right] dt^\alpha \right.$$

$$\leq \int_{\gamma_{t_0,t_1}} \left[\left\langle \mu_{\alpha P_\kappa}(t), g^{P_\kappa}(\pi_y(t)) \right\rangle + \left\langle \nu_{\alpha Q_\kappa}(t), h^{Q_\kappa}(\pi_y(t)) \right\rangle \right] dt^\alpha \right)$$

$$\Longrightarrow \left(b(x^\circ(\cdot), y(\cdot)) \right.$$

$$\times \int_{\gamma_{t_0,t_1}} \left[\left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \left\langle \mu_{\alpha P_\kappa}(t), \frac{\partial g^{P_\kappa}}{\partial y}(\pi_y(t)) \right\rangle \right\rangle \right.$$

$$+ \left\langle \nu_{\alpha Q_\kappa}(t), \frac{\partial h^{Q_\kappa}}{\partial y}(\pi_y(t)) \right\rangle$$ \hfill (4.14)

$$+ \left\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \left\langle \mu_{\alpha P_\kappa}(t), \frac{\partial g^{P_\kappa}}{\partial y_\gamma}(\pi_y(t)) \right\rangle \right\rangle$$

$$+ \left\langle \nu_{\alpha Q_\kappa}(t), \frac{\partial h^{Q_\kappa}}{\partial y_\gamma}(\pi_y(t)) \right\rangle \right] dt^\alpha$$

$$\left. \leq -\rho''_\kappa b(x^\circ(\cdot), y(\cdot)) \| \theta(x^\circ(\cdot), y(\cdot)) \|^2 \right).$$

Now, we make the sum of implications (4.13) and (4.14) side by side and from $\kappa = 1$ to $\kappa = q$. It follows

$$\left(\left\langle \Lambda^{10}, F(x^\circ(\cdot)) \right\rangle - \left\langle \Lambda^{20}, K(x^\circ(\cdot)) \right\rangle \right.$$

$$+ \int_{\gamma_{t_0,t_1}} \left[\left\langle \mu_\alpha(t), g(\pi_{x^\circ}(t)) \right\rangle + \left\langle \nu_\alpha(t), h(\pi_{x^\circ}(t)) \right\rangle \right] dt^\alpha$$

$$\left. - \int_{\gamma_{t_0,t_1}} \left[\left\langle \mu_\alpha(t), g(\pi_y(t)) \right\rangle + \left\langle \nu_\alpha(t), h(\pi_y(t)) \right\rangle \right] dt^\alpha \leq 0 \right)$$

$$\Longrightarrow \Bigg(b(x^\circ(\cdot), y(\cdot))$$

$$\times \int_{\gamma_{t_0,t_1}} \Bigg[\Bigg\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \Bigg\langle \Lambda^{10}, \frac{\partial f_\alpha}{\partial y}(\pi_y(t)) \Bigg\rangle - \Bigg\langle \Lambda^{20}, \frac{\partial k_\alpha}{\partial y}(\pi_y(t)) \Bigg\rangle \Bigg\rangle$$

$$+ \Bigg\langle \mu_\alpha(t), \frac{\partial g}{\partial y}(\pi_y(t)) \Bigg\rangle + \Bigg\langle \nu_\alpha(t), \frac{\partial h}{\partial y}(\pi_y(t)) \Bigg\rangle$$

$$+ \Bigg\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \Bigg\langle \Lambda^{10}, \frac{\partial f_\alpha}{\partial y_\gamma}(\pi_y(t)) \Bigg\rangle - \Bigg\langle \Lambda^{20}, \frac{\partial k_\alpha}{\partial y_\gamma}(\pi_y(t)) \Bigg\rangle \Bigg\rangle$$

$$+ \Bigg\langle \mu_\alpha(t), \frac{\partial g}{\partial y_\gamma}(\pi_y(t)) \Bigg\rangle + \Bigg\langle \nu_\alpha(t), \frac{\partial h}{\partial y_\gamma}(\pi_y(t)) \Bigg\rangle \Bigg\rangle \Bigg] dt^\alpha$$

$$- < b(x^\circ(\cdot), y(\cdot)) \| \theta(x^\circ(\cdot), y(\cdot)) \|^2 \Bigg(\sum_{\ell=1}^{r} \rho'_\ell + \sum_{\kappa=1}^{q} \rho''_\kappa \Bigg) \Bigg).$$

$$(4.15)$$

Since $b(x^\circ(\cdot), y(\cdot)) > 0$, we obtain

$$\int_{\gamma_{t_0,t_1}} \Bigg[\Bigg\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial V_\alpha}{\partial y} \Big(\pi_y(t), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot) \Big) \Bigg\rangle$$

$$+ \Bigg\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial V_\alpha}{\partial y_\gamma} \Big(\pi_y(t), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot) \Big) \Bigg\rangle \Bigg] dt^\alpha \qquad (4.16)$$

$$< - \| \theta(x^\circ(\cdot), y(\cdot)) \|^2 \Bigg(\sum_{\ell=1}^{r} \rho'_\ell + \sum_{\kappa=1}^{q} \rho''_\kappa \Bigg),$$

where

$$V_\alpha \Big(\pi_y(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot) \Big) = \Big\langle \Lambda^{10}, f_\alpha(\pi_y(t)) \Big\rangle - \Big\langle \Lambda^{20}, k_\alpha(\pi_y(t)) \Big\rangle$$

$$+ \langle \mu_\alpha(y(t)), g(\pi_y(t)) \rangle + \langle \nu_\alpha(t), h(\pi_y(t)) \rangle, \qquad (4.17)$$

$$t \in \Omega_{t_0,t_1}, \ \alpha = \overline{1, p}.$$

The following relation holds:

$$\left\langle D_\gamma \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial V_\alpha}{\partial y_\gamma}\left(\pi_y(t), \Lambda^{10}, \Lambda^{20}, \mu(t), \nu(t)\right) \right\rangle$$

$$= D_\gamma \left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial V_\alpha}{\partial y_\gamma}\left(\pi_y(t), \Lambda^{10}, \Lambda^{20}, \mu(t), \nu(t)\right) \right\rangle \tag{4.18}$$

$$- \left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), D_\gamma\left(\frac{\partial V_\alpha}{\partial y_\gamma}\right)\left(\pi_y(t), \Lambda^{10}, \Lambda^{20}, \mu(t), \nu(t)\right) \right\rangle.$$

By replacing relations (4.18) and by using Euler-Lagrange PDE, relation (4.16) becomes

$$\int_{\gamma_{t_0,t_1}} D_\gamma \left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial V_\alpha}{\partial y_\gamma}\left(\pi_y(t), \Lambda^{10}, \Lambda^{20}, \mu(t), \nu(t)\right) \right\rangle dt^\alpha$$

$$< -\|\theta(x^\circ(\cdot), y(\cdot))\|^2\left(\sum_{\ell=1}^{r}\rho'^\ell + \sum_{\kappa=1}^{q}\rho''_\kappa\right). \tag{4.19}$$

$\square$

For $\alpha, \gamma = \overline{1, p}$, let us denote by

$$Q_\alpha^\gamma(t) = \left\langle \eta(\pi_{x^\circ}(t), \pi_y(t)), \frac{\partial V_\alpha}{\partial y_\gamma}\left(\pi_y(t), \Lambda^{10}, \Lambda^{20}, \mu(t), \nu(t)\right) \right\rangle,$$

$$I = \int_{\gamma_{t_0,t_1}} D_\gamma Q_\alpha^\gamma(t)\ dt^\alpha. \tag{4.20}$$

According to [28], § 9, a total divergence is equal to a total derivative. Consequently, there exists $Q(t)$, with $Q(t_0) = 0$ and $Q(t_1) = 0$ such that $D_\gamma Q_\alpha^\gamma(t) = D_\alpha Q(t)$ and

$$I = \int_{\gamma_{t_0,t_1}} D_\alpha Q(t)\ dt^\alpha = Q(t_1) - Q(t_0) = 0. \tag{4.21}$$

Replacing into inequality (4.19), it follows that

$$0 < -\|\theta(x^\circ(\cdot), y(\cdot))\|^2\left(\sum_{\ell=1}^{r}\rho'^\ell + \sum_{a=1}^{q}\rho''_a\right), \tag{4.22}$$

contradicting hypothesis (f).

From relation (4.15), it follows

$$
\begin{aligned}
0 \le \; & \left\langle \Lambda^{10}, F(x^\circ(\cdot)) \right\rangle - \left\langle \Lambda^{20}, K(x^\circ(\cdot)) \right\rangle \\
& + \int_{\gamma_{t_0,t_1}} \left[\langle \mu_\alpha(t), g(\pi_{x^\circ}(t)) \rangle + \langle \nu_\alpha(t), h(\pi_{x^\circ}(t)) \rangle \right] dt^\alpha \\
& - \int_{\gamma_{t_0,t_1}} \left[\langle \mu_\alpha(t), g(\pi_y(t)) \rangle + \langle \nu_\alpha(t), h(\pi_y(t)) \rangle \right] dt^\alpha.
\end{aligned}
\tag{4.23}
$$

According to the constraints of problems (MFP) and (MFZD), the previously mentioned relation becomes $\Lambda_\ell^{10} F^\ell(x^\circ(\cdot)) - \Lambda_\ell^{20} K^\ell(x^\circ(\cdot)) > 0$, that is:

$$
K^\ell(x^\circ(\cdot)) K^\ell(y(\cdot)) \left[\frac{F^\ell(x^\circ(\cdot))}{K^\ell(x^\circ(\cdot))} - \frac{F^\ell(y(\cdot))}{K^\ell(y(\cdot))} \right] > 0.
\tag{4.24}
$$

Because $K^\ell(x^\circ(\cdot)) K^\ell(y(\cdot)) > 0$, $\ell = \overline{1,r}$, we conclude that

$$
\left(\frac{F^1(x^\circ(\cdot))}{K^1(x^\circ(\cdot))} - \frac{F^1(y(\cdot))}{K^1(y(\cdot))}, \dots, \frac{F^r(x^\circ(\cdot))}{K^r(x^\circ(\cdot))} - \frac{F^r(y(\cdot))}{K^r(y(\cdot))} \right) \nleq (0, \dots, 0),
\tag{4.25}
$$

or

$$
\left(\frac{F^1(x^\circ(\cdot))}{K^1(x^\circ(\cdot))}, \dots, \frac{F^r(x^\circ(\cdot))}{K^r(x^\circ(\cdot))} \right) \nleq \left(\frac{F^1(y(\cdot))}{K^1(y(\cdot))}, \dots, \frac{F^r(y(\cdot))}{K^r(y(\cdot))} \right).
\tag{4.26}
$$

Therefore, the inequality $\varpi(x^\circ(\cdot)) \le \overline{\delta}(y(\cdot), y_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu(\cdot), \nu(\cdot))$ contradicts relations (4.12), and this completes the proof.

Theorem 4.5 (direct duality). *Let $x^\circ(\cdot)$ be an efficient solution of primal* (MFP). *Suppose the hypotheses of Theorem 4.4 hold. Then there are Λ^{10} and Λ^{20} in $\mathbb{R}^r$, and the smooth functions $\mu^\circ : \Omega_{t_0,t_1} \to \mathbb{R}^{qsp}$ and $\nu^\circ : \Omega_{t_0,t_1} \to \mathbb{R}^{qsp}$, such that $(x^\circ(\cdot), x_\gamma^\circ(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot))$ is an efficient solution of dual program* (MFZD). *Moreover*

$$
\varpi(x^\circ(\cdot)) = \overline{\delta} \left(x^\circ(\cdot), x_\gamma^\circ(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot) \right).
\tag{4.27}
$$

Proof. $x^\circ(\cdot)$ being efficient solution of primal (MFP), there are Λ^{10} and Λ^{20} in $\mathbb{R}^r$ and the smooth functions $\mu^\circ : \Omega_{t_0,t_1} \to \mathbb{R}^{qsp}$ and $\nu^\circ : \Omega_{t_0,t_1} \to \mathbb{R}^{qsp}$, such that relations (4.6) are verified. Therefore, $\langle \mu_\alpha(t), g(\pi_{x^\circ}(t)) \rangle = 0$, $\alpha = \overline{1,p}$. It follows $(x^\circ(\cdot), x_\gamma^\circ(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot))$ is a feasible solution for program (MFZD). Obviously,

$$
\varpi(x^\circ(\cdot)) = \overline{\delta} \left(x^\circ(\cdot), x_\gamma^\circ(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), \nu^\circ(\cdot) \right).
\tag{4.28}
$$

The weak duality theorem assures the efficiency of $x^\circ(\cdot)$. $\square$

If we remark that the notion of efficient solution of problem (MFZD) is similar to those in Definition 4.1, we can state the result in the following.

Theorem 4.6 (converse duality). *Let $(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), v^\circ(\cdot))$ be an efficient solution to dual (MFZD) and suppose the following conditions are satisfied:*

(i) *$\overline{x}(\cdot)$ is a normal efficient solution of primal (MFP);*

(ii) *the hypotheses of Theorem 4.4 hold at $(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), v^\circ(\cdot))$.*

Then $x^\circ(\cdot)$ is an efficient solution to (MFP). Moreover, one has the equality

$$\varpi(x^\circ(\cdot)) = \overline{\delta}\Big(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), v^\circ(\cdot)\Big). \tag{4.29}$$

Proof. Suppose $\overline{x}(\cdot) \neq x^\circ(\cdot)$. The efficiency of $\overline{x}(\cdot)$ of primal (MFP) implies the existence of $\overline{\Lambda}^{10}$ and $\overline{\Lambda}^{20}$ in $\mathbb{R}^r$ and the smooth functions $\overline{\mu}^\circ : \Omega_{t_0,t_1} \to \mathbb{R}^{qsp}$ and $\overline{v}^\circ : \Omega_{t_0,t_1} \to \mathbb{R}^{qsp}$, such that relations (4.6) are satisfied. It follows that $(\overline{x}(\cdot), \overline{x}_\gamma(\cdot), \overline{\Lambda}^{10}, \overline{\Lambda}^{20}, \overline{\mu}^\circ(\cdot), \overline{v}^\circ(\cdot))$ is a feasible solution of (MFZD) and

$$\varpi(\overline{x}(\cdot)) = \overline{\delta}\Big(\overline{x}(\cdot), \overline{x}_\gamma(\cdot), \overline{\Lambda}^{10}, \overline{\Lambda}^{20}, \overline{\mu}^\circ(\cdot), \overline{v}^\circ(\cdot)\Big). \tag{4.30}$$

On the other hand, from the weak duality theorem,

$$\varpi(\overline{x}(\cdot)) \nleq \delta\Big(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), v^\circ(\cdot)\Big) \tag{4.31}$$

holds, that is:

$$\delta\Big(\overline{x}(\cdot), \overline{x}_\gamma(\cdot), \overline{\Lambda}^{10}, \overline{\Lambda}^{20}, \overline{\mu}^\circ(\cdot), \overline{v}^\circ(\cdot)\Big) \nleq \delta\Big(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), v^\circ(\cdot)\Big). \tag{4.32}$$

This last relation contradicts the efficiency of $(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{10}, \Lambda^{20}, \mu^\circ(\cdot), v^\circ(\cdot))$ for program (MFZD). Therefore $\overline{x}(\cdot) = x^\circ(\cdot)$, and the theorem is proved.

We would like to conclude the study in this section with the following important particular case.

In this respect, suppose that k_α^ℓ = positive constant, $\ell = \overline{1, r}$, $\alpha = \overline{1, p}$. Denote $F(x(\cdot)) = (F^1(x(\cdot)), \ldots, F^r(x(\cdot)))$. Then, program (MFP) becomes

$$\begin{aligned} \min \quad & F(x) \\ \text{subject to} \quad & x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1}), \end{aligned} \tag{MP}$$

and its dual is

$$\max \int_a^b L_e\big(\pi_y(t)\big)dt$$

subject to

$$\Lambda_\ell^{10} \frac{\partial f_\alpha^\ell}{\partial y}\big(\pi_y(t)\big) + \left\langle \mu_\alpha(t), \frac{\partial g}{\partial y}\big(\pi_y(t)\big) \right\rangle + \left\langle v_\alpha(t), \frac{\partial h}{\partial y}\big(\pi_y(t)\big) \right\rangle$$

$$- D_\gamma \left(\Lambda_\ell^{10} \frac{\partial f_\alpha^\ell}{\partial y_\gamma}\big(\pi_y(t)\big) + \left\langle \mu_\alpha(t), \frac{\partial g}{\partial y_\gamma}\big(\pi_y(t)\big) \right\rangle + \left\langle v_\alpha(t), \frac{\partial h}{\partial y_\gamma}\big(\pi_y(t)\big) \right\rangle \right) = 0,$$

$$\alpha = \overline{1,p}, \ t \in \Omega_{t_0,t_1},$$

$$\left\langle \mu_{\alpha P_\kappa}(t), g^{P_\kappa}\big(\pi_y(t)\big) \right\rangle + \left\langle v_{\alpha Q_\kappa}(t), h^{Q_\kappa}\big(\pi_y(t)\big) \right\rangle \geq 0, \quad \kappa = \overline{1,q}, \ \alpha = \overline{1,p}, \ t \in \Omega_{t_0,t_1},$$

$$\Lambda^{10} \geq 0,$$

$$\text{(GMWD)}$$

where

$$L_e\big(\pi_y(t)\big) = f\big(\pi_y(t)\big) + \left[\left\langle \mu_{P_0}(t), g^{P_0}\big(\pi_x(t)\big) \right\rangle + \left\langle v_{Q_0}(t), h^{Q_0}\big(\pi_x(t)\big) \right\rangle \right] e, \tag{4.33}$$

with $L_e = (L_1, \ldots, L_r)$.

It can be seen that we have obtained precisely the programs studied in the work [14]. Therefore, the results in this paper are stronger than the ones mentioned before. $\square$

5. Conclusion

In our recent study [14], we initiated an optimization theory on the first order jet bundle. As natural continuation, in this paper we considered a new class of multitime multiobjective variational problems of minimizing a vector of quotients of functionals of curvilinear integral type. We derived duality results for efficient solutions of multitime multiobjective ratio variational problems under the assumptions of (ρ, b)-quasi-invexity. We proved our weak duality theorem for efficient solutions, showing that the value of the objective function of the primal cannot exceed the value of the dual. Direct and converse duality theorems are stated, underlying the connections between the values of the objective functions of the primal and dual programs. As special cases, duality results of Mond-Weir-Zalmai type for a multitime multiobjective variational problem are obtained.

Having in mind the physical significance of the objective function, this study is strongly motivated by its possible applications of nonlinear optimization to mechanical engineering and economics [29].

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