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Research Article

Algorithms for Solving System of Extended General Variational Inclusions and Fixed Points Problems

Narin Petrot[1,2] and Javad Balooee[3]

[1] *Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*
[2] *Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand*
[3] *Department of Mathematics, Islamic Azad University, Sari Branch, Sari, Iran*

Correspondence should be addressed to Javad Balooee, javad.balooee@gmail.com

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We introduce a new system of extended general nonlinear variational inclusions with different nonlinear operators and establish the equivalence between the aforesaid system and the fixed point problem. By using this equivalent formulation, we prove the existence and uniqueness theorem for solution of the system of extended general nonlinear variational inclusions. We suggest and analyze a new resolvent iterative algorithm to approximate the unique solution of the system of extended general nonlinear variational inclusions which is a fixed point of a nearly uniformly Lipschitzian mapping. Subsequently, the convergence analysis of the proposed iterative algorithm under some suitable conditions is considered. Furthermore, some related works to our main problem are pointed out and discussed.

1. Introduction

Variational inequalities theory, as a very effective and powerful tool of the current mathematical technology, has been widely applied to mechanics, physics, optimization and control, economics and transportation equilibrium, engineering sciences, and so forth. Up until now variational inequalities have been very effective and powerful tools of the current mathematical technology; see for example [1–4] and references therein. In 1968, Brézis [5] initiated the study of the existence theory of a class of variational inequalities, later known as variational inclusions, using proximal-point mappings due to Moreau [6]. It is well known that variational inclusions include variational inequalities, quasivariational inequalities, and variational-like inequalities as special cases. For application of variational inclusions, see, for example, [7–21]. A number of problems leading to the system of variational inclusions/inequalities arise in applications to variational problems and engineering see, for example, [22–30]. Variational inclusions can be viewed as innovative and novel extension of the variational principles. It is well known that the system of variational inclusions/inequalities can provide new insight regarding problems being studied and can stimulate new and innovative ideas for problem solving.

One of the most important and interesting problems in the theory of variational inclusions is the development of numerical methods which provide an efficient and implementable algorithm for solving variational inclusion and its generalizations. The method based on proximal-point mapping is a generalization of projection method and has been widely used to study the existence of solutions and to develop iterative algorithms for variational inclusions see, for example, [31–39]. In recent past, the methods based on different classes of proximal-point mappings have been developed to study the existence of solutions and to discuss convergence and stability analysis of proposed iterative algorithms; for various classes of variational/variational-like inclusions, see for example [34, 36, 37, 39–43].

Recently, Noor [29] introduced and studied a new system of general mixed variational inequalities involving three different operators (SGMVID). By using the resolvent operator technique, he established the equivalence between the SGMVID and the fixed point problem. He used this equivalent formulation to suggest and analyze some new iterative methods for solving the SGMVID. He also studied the convergence analysis of the proposed iterative methods under some certain conditions.

Very recently, M. A. Noor and K. I. Noor [44] introduced and considered the system of general variational inclusions involving seven different operators (SGVID). They suggested and analyzed two resolvent iterative algorithms for solving the SGVID and studied the convergence analysis of the proposed iterative schemes under some certain conditions.

On the other hand, related to the variational inequalities/inclusions, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to consider a unified approach to these two different problems. Motivated and inspired by the research going in this direction, Noor and Huang [45] considered the problem of finding the common element of the set of the solutions of variational inequalities and the set of the fixed points of the nonexpansive mappings. It is well known that every nonexpansive mapping is a Lipschitzian mapping. Lipschitzian mappings have been generalized by various authors. Sahu [46] introduced and investigated nearly uniformly Lipschitzian mappings as generalization of Lipschitzian mappings.

In this paper, we introduce and consider a new system of extended general nonlinear variational inclusions involving eight different nonlinear operators (SEGNVID). We first establish the equivalence between the SEGNVID and the fixed point problem, and then, by this equivalent formulation, we discuss the existence and uniqueness of solution of the SEGNVID. By using two nearly uniformly Lipschitzian mappings S_1 and S_2 and the aforementioned equivalent formulation, we suggest and analyze a new resolvent iterative algorithm for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $Q = (S_1, S_2)$ which is the unique solution of the SEGNVID. Finally, we consider the convergence analysis of the suggested iterative algorithms under some suitable conditions. Further, some related works, as appeared in [29, 44], are also discussed and improved.

2. Formulations and Basic Facts

Throughout this paper, we will let $\mathscr{H}$ be a real Hilbert space which is equipped with an inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$. To begin with, let us recall that a set-valued operator $A:\mathscr{H}\multimap\mathscr{H}$ is said to be *monotone* if and only if, for any $x,y\in\mathscr{H}$

$$\langle u-v,x-y\rangle\geq 0,\quad \forall u\in A(x),\quad v\in A(y). \tag{2.1}$$

A monotone set-valued operator A is called *maximal* if and only if its graph, $\mathrm{Gph}(A):=\{(x,y)\in\mathscr{H}\times\mathscr{H}:y\in A(x)\}$, is not properly contained in the graph of any other monotone operator. It is well known that A is a maximal monotone operator if and only if $(I+\lambda A)(\mathscr{H})=\mathscr{H}$, for all $\lambda>0$, where I denotes the identity operator on $\mathscr{H}$.

Definition 2.1 (see [47]). For any maximal monotone operator A, the resolvent operator associated with A of parameter λ is defined as

$$J_A^\lambda(u)=(I+\lambda A)^{-1}(u),\quad \forall u\in\mathscr{H}. \tag{2.2}$$

It is single valued and nonexpansive, that is,

$$\left\|J_A^\lambda(u)-J_A^\lambda(v)\right\|\leq\|u-v\|,\quad \forall u,v\in\mathscr{H}. \tag{2.3}$$

Let $T_i:\mathscr{H}\times\mathscr{H}\to\mathscr{H}$ and $g_i,h_i:\mathscr{H}\to\mathscr{H}(i=1,2)$ be six nonlinear single-valued operators and $A_i:\mathscr{H}\times\mathscr{H}\multimap\mathscr{H}(i=1,2)$ be two set-valued operators such that, for all $z,t\in\mathscr{H}$, $A_1(\cdot,z):\mathscr{H}\multimap\mathscr{H}$ and $A_2(\cdot,t):\mathscr{H}\multimap\mathscr{H}$ are two maximal monotone operators with $g_2(u)\in\mathrm{dom}(A_1(\cdot,z))$ and $h_2(v)\in\mathrm{dom}(A_2(\cdot,t))$ for all $u,v\in\mathscr{H}$. For any given constants $\rho,\eta>0$, we consider the problem of finding $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ such that

$$\begin{gathered}0\in g_2(x^*)-g_1(y^*)+\rho(T_1(y^*,x^*)+A_1(g_2(x^*),x^*)),\\ 0\in h_2(y^*)-h_1(x^*)+\eta(T_2(x^*,y^*)+A_2(h_2(y^*),y^*)).\end{gathered} \tag{2.4}$$

The problem (2.4) is called a system of *extended general nonlinear variational inclusions* involving eight different nonlinear operators (SEGNVID).

Some special cases of the SEGNVID (2.4) are as below.

If $T_i:\mathscr{H}\to\mathscr{H}$ and $A_i=A:\mathscr{H}\multimap\mathscr{H}(i=1,2)$ are univariate nonlinear operators, then taking $g_1=g$, $g_2=g_1$, $h_1=h$, and $h_2=h$, the system (2.4) reduces to the system of finding $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ such that

$$\begin{gathered}0\in g_1(x^*)-g(y^*)+\rho(T_1(y^*)+A(g_1(x^*))),\\ 0\in h_1(y^*)-h(x^*)+\eta(T_2(x^*)+A(h_1(y^*))),\end{gathered} \tag{2.5}$$

which is called the *system of general nonlinear variational inclusions* involving seven different nonlinear operators.

Remark 2.2. M. A. Noor and K. I. Noor [44] considered the system (2.5) where $A : \mathscr{H} \to \mathscr{H}$ is a single-valued operator. In view of the presented definitions and results in [44], we infer that the operator A in the system (2.5) (the system (1) in [44]) should be set valued not single valued and also be satisfied in the conditions Range $g_1 \bigcap \operatorname{dom} A \neq \emptyset$ and Range $h_1 \bigcap \operatorname{dom} A \neq \emptyset$. Therefore, if the mentioned corrections applied on the system (1) in [44], then the system (1) in [44] reduces to the system (2.5) which is a special case of the system (2.4).

Taking $g_1 = g$ and $h_1 = h$ in the system (2.5), the mentioned system collapses to the system of finding $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ such that

$$\begin{aligned} &0 \in T_1(y^*) + A(g(x^*)), \\ &0 \in T_2(x^*) + A(h(y^*)). \end{aligned} \tag{2.6}$$

The problem (2.6) is called the *nonlinear variational inclusions system* involving five different nonlinear operators.

If, for each $i = 1, 2$, $T_i = T$, $g_1 = h_1 = h = g$, $\rho = \eta$, and $x^* = y^* = x$, then the system (2.5) reduces to the variational inclusion problem or finding the zero of the sum of two (more) monotone operators considered in [48–51].

If, for each $i = 1, 2$, $A_i : \mathscr{H} \multimap \mathscr{H}$ is an univariate set-valued operator, $A_1(x) = \partial\varphi(x)$ and $A_2(x) = \partial\phi(x)$ for all $x \in \mathscr{H}$, where $\varphi, \phi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ are two proper, convex, and lower semi-continuous functionals, $\partial\varphi$ and $\partial\phi$ denote subdifferential operators of φ and ϕ, respectively, then the system (2.4) reduces to the following system.

Find $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ such that

$$\begin{aligned} &\langle \rho T_1(y^*, x^*) + g_2(x^*) - g_1(y^*), g_1(x) - g_2(x^*) \rangle \geq \rho\varphi(g_2(x^*)) - \rho\varphi(g_1(x)), \quad \forall x \in \mathscr{H}, \\ &\langle \eta T_2(x^*, y^*) + h_2(y^*) - h_1(x^*), h_1(x) - h_2(y^*) \rangle \geq \eta\phi(h_2(y^*)) - \eta\phi(h_1(x)), \quad \forall x \in \mathscr{H}, \end{aligned} \tag{2.7}$$

which appears to be a new system of extended general mixed nonlinear variational inequalities involving eight different operators.

If $g_1 = h_1 = g$, $g_2 = h_2 \equiv I$ (the identity operator), and $\varphi = \phi$, then the system (2.7) is equivalent to that of finding $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ such that

$$\begin{aligned} &\langle \rho T_1(y^*, x^*) + x^* - g(y^*), g(x) - x^* \rangle \geq \rho\varphi(x^*) - \rho\varphi(g(x)), \quad \forall x \in \mathscr{H}, \\ &\langle \eta T_2(x^*, y^*) + y^* - g(x^*), g(x) - y^* \rangle \geq \eta\varphi(y^*) - \eta\varphi(g(x)), \quad \forall x \in \mathscr{H}, \end{aligned} \tag{2.8}$$

which was considered and studied by Noor [29]. Also, if $T_1 = T_2 = T$, then the system (2.8) is considered and studied in [29].

If $T_i : \mathscr{H} \to \mathscr{H} (i = 1, 2)$ are univariate nonlinear operaotrs and $\varphi = \phi$, then the system (2.7) changes into that of finding $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ such that

$$\begin{aligned} &\langle \rho T_1(y^*) + g_2(x^*) - g_1(y^*), g_1(x) - g_2(x^*) \rangle \geq \rho\varphi(g_2(x^*)) - \rho\varphi(g_1(x)), \quad \forall x \in \mathscr{H}, \\ &\langle \eta T_2(x^*) + h_2(y^*) - h_1(x^*), h_1(x) - h_2(y^*) \rangle \geq \eta\varphi(h_2(y^*)) - \eta\varphi(h_1(x)), \quad \forall x \in \mathscr{H}, \end{aligned} \tag{2.9}$$

which was considered and investigated by M. A. Noor and K. I. Noor [44]. Also, if $T_1 = T_2 = T$, then the system (2.9) is considered and studied in [44]. When $g_1 = h_1 = g$ and $g_2 = h_2 \equiv I$, the system (2.9) is introduced and studied in [29].

If, in the system (2.7), $\varphi = \phi = \delta_K$ is the indicator function of a nonempty closed convex set K in $\mathscr{H}$ defined by

$$\delta_K(y) = \begin{cases} 0 & y \in K, \\ \infty & y \notin K, \end{cases} \tag{2.10}$$

then the system (2.7) reduces to the system of finding $(x^*, y^*) \in K \times K$ such that

$$\begin{aligned} &\langle \rho T_1(y^*, x^*) + g_2(x^*) - g_1(y^*), g_1(x) - g_2(x^*) \rangle \geq 0, \quad \forall x \in \mathscr{H} : g_1(x) \in K, \\ &\langle \eta T_2(x^*, y^*) + h_2(y^*) - h_1(x^*), h_1(x) - h_2(y^*) \rangle \geq 0, \quad \forall x \in \mathscr{H} : h_1(x) \in K, \end{aligned} \tag{2.11}$$

which has been introduced and considered by M. A. Noor and K. I. Noor [30].

Remark 2.3. When $g_1 = h_1 = g$ and $g_2 = h_2 \equiv I$, the system (2.11) is considered and studied by Noor [52]. Also, if, for each $i = 1, 2$, $T_i : \mathscr{H} \to \mathscr{H}$ is an univariate nonlinear operator and $g_i = h_i = g$, then the system (2.11) is considered and studied by Yang et al. [53]. If, for each $i = 1, 2$, $g_i = h_i \equiv I$, then the system (2.11) is considered and studied by Huang and Noor [28]. If for each $i = 1, 2$, $T_i = T$ and $g_i = h_i \equiv I$, then the system (2.11) introduced and studied by Chang et al. [26] and Verma [54]. If for each $i = 1, 2$, $T_i = T$, $g_1 = h_1 = g$, and $g_2 = h_2 \equiv I$, then the system (2.11) is considered and studied by Noor [52]. If for each $i = 1, 2$, $T_i = T : \mathscr{H} \to \mathscr{H}$ is an univariate nonlinear operator and $g_i = h_i \equiv I$, then the system (2.11) introduced and studied by Verma [55, 56]. If for each $i = 1, 2$, $T_i = T : \mathscr{H} \to \mathscr{H}$ is an univariate nonlinear operator, $g_i = h_i \equiv I$ and $x^* = y^*$, then the system (2.11) reduces to the classical variational inequality introduced and studied by Stampacchia [57] in 1964. Other special cases of the above systems can be found in [29, 44] and the references therein. In brief, for suitable and appropriate choice of the operators T_i, A_i, g_i, $h_i (i = 1, 2)$, and the constants ρ and η, one can obtain the various classes of variational inclusions and variational inequalities. This shows that the system of extended general nonlinear variational inclusions involving eight different operators (2.4) is more general and includes several classes of variational inclusions/inequalities and related optimization problems as special cases. For the recent applications, numerical methods and formulations of variational inequalities and variational inclusions, see [1–45, 47–62], and the references therein.

3. Existence of Solution and Uniqueness

In this section, we prove the existence and uniqueness theorem for a solution of the system of extended general nonlinear variational inclusions (2.4). For this end, we need the following lemma in which, by using resolvent operator technique, the equivalence between the system of extended general nonlinear variational inclusions (2.4) and a fixed point problem is proved.

Lemma 3.1. *Let T_i, A_i, g_i, $h_i (i = 1, 2)$, ρ, and η be the same as in the system* (2.4). *Then $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ is a solution of the system* (2.4) *if and only if*

$$
\begin{aligned}
g_2(x^*) &= J^{\rho}_{A_1(\cdot, x^*)}\left(g_1(y^*) - \rho T_1(y^*, x^*)\right),\\
h_2(y^*) &= J^{\eta}_{A_2(\cdot, y^*)}\left(h_1(x^*) - \eta T_2(x^*, y^*)\right),
\end{aligned}
\tag{3.1}
$$

where for all $z, t \in \mathscr{H}$, $J^{\rho}_{A_1(.,z)}$ is the resolvent operator associated with $A_1(., z)$ of parameter ρ and $J^{\eta}_{A_2(.,t)}$ is the resolvent operator associated with $A_2(., t)$ of parameter η.

Proof. Let $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ be a solution of the system (2.4). Then

$$
\begin{aligned}
&g_1(y^*) - \rho T_1(y^*, x^*) \in \left(I + \rho A_1(\cdot, x^*)\right)\left(g_2(x^*)\right),\\
&h_1(x^*) - \eta T_2(x^*, y^*) \in \left(I + \eta A_2(\cdot, y^*)\right)\left(h_2(y^*)\right),\\
&\Longleftrightarrow\\
&g_2(x^*) = J^{\rho}_{A_1(\cdot, x^*)}\left(g_1(y^*) - \rho T_1(y^*, x^*)\right),\\
&h_2(y^*) = J^{\eta}_{A_2(\cdot, y^*)}\left(h_1(x^*) - \eta T_2(x^*, y^*)\right),
\end{aligned}
\tag{3.2}
$$

where I is identity operator. □

Definition 3.2. A nonlinear single-valued operator $g : \mathscr{H} \to \mathscr{H}$ is said to be

(a) *monotone* if

$$\langle g(x) - g(y), x - y \rangle \geq 0, \quad \forall x, y \in \mathscr{H}; \tag{3.3}$$

(b) *κ-strongly monotone* if there exists a constant $\kappa > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \kappa \|x - y\|^2, \quad \forall x, y \in \mathscr{H}; \tag{3.4}$$

(c) *(ϱ, ν)-relaxed cocoercive* if there exist constants $\varrho, \nu > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq -\varrho \|g(x) - g(y)\|^2 + \nu \|x - y\|^2, \quad \forall x, y \in \mathscr{H}; \tag{3.5}$$

(d) *γ-Lipschitz continuous* if there exists a constant $\gamma > 0$ such that

$$\|g(x) - g(y)\| \leq \gamma \|x - y\|, \quad \forall x, y \in \mathscr{H}. \tag{3.6}$$

Definition 3.3. A nonlinear single-valued operator $T : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is called

(a) *monotone in the first variable* if for all $x, y \in \mathcal{H}$

$$\langle T(x,u) - T(y,v), x - y \rangle \geq 0, \quad \forall u, v \in \mathcal{H}; \tag{3.7}$$

(b) *r-strongly monotone in the first variable* if there exists a constant $r > 0$ such that for all $x, y \in \mathcal{H}$

$$\langle T(x,u) - T(y,v), x - y \rangle \geq r\|x - y\|^2, \quad \forall u, v \in \mathcal{H}; \tag{3.8}$$

(c) (ξ, ς)*-relaxed cocoercive in the first variable* if there exist constants $\xi, \varsigma > 0$ such that for all $x, y \in \mathcal{H}$

$$\langle T(x,u) - T(y,v), x - y \rangle \geq -\xi\|T(x,u) - T(y,v)\|^2 + \varsigma\|x - y\|^2, \quad \forall u, v \in \mathcal{H}; \tag{3.9}$$

(d) μ*-Lipschitz continuous in the first variable* if there exists a constant $\mu > 0$ such that for all $x, y \in \mathcal{H}$

$$\|T(x,u) - T(y,v)\| \leq \mu\|x - y\|, \quad \forall u, v \in \mathcal{H}. \tag{3.10}$$

Now, we present the sufficient conditions for the existence solutions of our main considered problem (2.4).

Theorem 3.4. *Let T_i, A_i, g_i, $h_i (i = 1,2)$, ρ, and η be the same as in the system* (2.4) *and suppose further that, for $i = 1,2$,*

(a) *T_i is ξ_i-strongly monotone and μ_i-Lipschitz continuous in the first variable;*

(b) *g_i is ς_i-strongly monotone and σ_i-Lipschitz continuous;*

(c) *h_i is ϱ_i-strongly monotone and δ_i-Lipschitz continuous;*

(d) *there exists constants τ_i such that*

$$\left\| J^{\rho}_{A_1(\cdot,u)}(z) - J^{\rho}_{A_1(\cdot,v)}(z) \right\| \leq \tau_1\|u - v\|, \qquad \left\| J^{\eta}_{A_2(\cdot,u)}(z) - J^{\eta}_{A_2(\cdot,v)}(z) \right\| \leq \tau_2\|u - v\|, \quad \forall u, v, z \in \mathcal{H}. \tag{3.11}$$

If two constants ρ and η satisfy the following conditions

$$\begin{gathered}
\left|\rho-\frac{\xi_1}{\mu_1^2}\right|<\frac{\sqrt{\xi_1^2-\mu_1^2\mu(2-\mu)}}{\mu_1^2}, \\
\left|\eta-\frac{\xi_2}{\mu_2^2}\right|<\frac{\sqrt{\xi_2^2-\mu_2^2\nu(2-\nu)}}{\mu_2^2}, \\
\xi_1>\mu_1\sqrt{\mu(2-\mu)}, \qquad \xi_2>\mu_2\sqrt{\nu(2-\nu)}, \\
\kappa_i=\sqrt{1-2\varsigma_i+\sigma_i^2}<1, \qquad 2\varsigma_i<1+\sigma_i^2, \\
\omega_i=\sqrt{1-2\varrho_i+\delta_i^2}<1, \qquad 2\varrho_i<1+\delta_i^2, \\
\mu=\tau_2+\omega_2+\kappa_1<1, \qquad \nu=\tau_1+\omega_1+\kappa_2<1, \\
i=1,2,
\end{gathered} \tag{3.12}$$

then the system (2.4) *admits a unique solution.*

Proof . Firstly, let us define $\psi,\phi:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ by

$$\begin{aligned}
\psi(x,y)&=x-g_2(x)+J_{A_1(\cdot,x)}^{\rho}(g_1(y)-\rho T_1(y,x)), \\
\phi(x,y)&=y-h_2(y)+J_{A_2(\cdot,y)}^{\eta}(h_1(x)-\eta T_2(x,y)),
\end{aligned} \tag{3.13}$$

for all $(x,y)\in\mathcal{H}\times\mathcal{H}$.

Also, define $F:\mathcal{H}\times\mathcal{H}\to\mathcal{H}\times\mathcal{H}$ as follows:

$$F(x,y)=(\psi(x,y),\phi(x,y)), \quad \forall(x,y)\in\mathcal{H}\times\mathcal{H}. \tag{3.14}$$

Consider a function $\|\cdot\|_*$ on $\mathcal{H}\times\mathcal{H}$ which is defined by

$$\|(x,y)\|_*=\|x\|+\|y\|, \quad \forall(x,y)\in\mathcal{H}\times\mathcal{H}. \tag{3.15}$$

It is obvious that $(\mathscr{H}\times\mathscr{H},\|\cdot\|_*)$ is a Hilbert space. Now, we establish that F is a contraction mapping on $(\mathscr{H}\times\mathscr{H},\|\cdot\|_*)$. Let $(x,y),(\hat{x},\hat{y})\in\mathscr{H}\times\mathscr{H}$ be given. Since for all $z\in\mathscr{H}$, the resolvent operator $J^{\rho}_{A_1(\cdot,z)}$ is nonexpansive, we have

$$\begin{aligned}
&\|\varphi(x,y)-\varphi(\hat{x},\hat{y})\|\\
&\quad\le\|x-\hat{x}-(g_2(x)-g_2(\hat{x}))\|+\left\|J^{\rho}_{A_1(\cdot,x)}(g_1(y)-\rho T_1(y,x))-J^{\rho}_{A_1(\cdot,\hat{x})}(g_1(\hat{y})-\rho T_1(\hat{y},\hat{x}))\right\|\\
&\quad\le\|x-\hat{x}-(g_2(x)-g_2(\hat{x}))\|+\left\|J^{\rho}_{A_1(\cdot,x)}(g_1(y)-\rho T_1(y,x))-J^{\rho}_{A_1(\cdot,\hat{x})}(g_1(y)-\rho T_1(y,x))\right\|\\
&\qquad+\left\|J^{\rho}_{A_1(\cdot,\hat{x})}(g_1(y)-\rho T_1(y,x))-J^{\rho}_{A_1(\cdot,\hat{x})}(g_1(\hat{y})-\rho T_1(\hat{y},\hat{x}))\right\|\\
&\quad\le\|x-\hat{x}-(g_2(x)-g_2(\hat{x}))\|+\tau_1\|x-\hat{x}\|+\|g_1(y)-g_1(\hat{y})-\rho(T_1(y,x)-T_1(\hat{y},\hat{x}))\|\\
&\quad\le\|x-\hat{x}-(g_2(x)-g_2(\hat{x}))\|+\tau_1\|x-\hat{x}\|+\|y-\hat{y}-(g_1(y)-g_1(\hat{y}))\|\\
&\qquad+\|y-\hat{y}-\rho(T_1(y,x)-T_1(\hat{y},\hat{x}))\|.
\end{aligned}\tag{3.16}$$

From ς_2-strongly monotonicity and σ_2-Lipschitz continuity of g_2, it follows that

$$\begin{aligned}
&\|x-\hat{x}-(g_2(x)-g_2(\hat{x}))\|^2\\
&\quad=\|x-\hat{x}\|^2-2\langle g_2(x)-g_2(\hat{x}),x-\hat{x}\rangle+\|g_2(x)-g_2(\hat{x})\|^2\\
&\quad\le\left(1-2\varsigma_2+\sigma_2^2\right)\|x-\hat{x}\|^2,
\end{aligned}\tag{3.17}$$

which leads to

$$\|x-\hat{x}-(g_2(x)-g_2(\hat{x}))\|\le\sqrt{1-2\varsigma_2+\sigma_2^2}\|x-\hat{x}\|.\tag{3.18}$$

In similar way, by using ς_1-strongly monotonicity and σ_1-Lipschitz continuity of g_1, we deduce that

$$\|y-\hat{y}-(g_1(y)-g_1(\hat{y}))\|\le\sqrt{1-2\varsigma_1+\sigma_1^2}\|y-\hat{y}\|.\tag{3.19}$$

Since T_1 is ξ_1-strongly monotone and μ_1-Lipschitz continuous in the first variable, we conclude that

$$\begin{aligned}
&\|y-\hat{y}-\rho(T_1(y,x)-T_1(\hat{y},\hat{x}))\|^2\\
&\quad=\|y-\hat{y}\|^2-2\rho\langle T_1(y,x)-T_1(\hat{y},\hat{x}),y-\hat{y}\rangle+\rho^2\|T_1(y,x)-T_1(\hat{y},\hat{x})\|^2\\
&\quad\le\left(1-2\rho\xi_1+\rho^2\mu_1^2\right)\|y-\hat{y}\|^2.
\end{aligned}\tag{3.20}$$

Substituting (3.18)–(3.20) in (3.16), we obtain

$$\left\|\psi(x,y)-\psi(\hat{x},\hat{y})\right\| \leq (\tau_1+\kappa_2)\|x-\hat{x}\|+(\kappa_1+\theta_1)\left\|y-\hat{y}\right\|, \tag{3.21}$$

where

$$\kappa_i=\sqrt{1-2\varsigma_i+\sigma_i^2}, \quad (i=1,2), \qquad \theta_1=\sqrt{1-2\rho\xi_1+\rho^2\mu_1^2}. \tag{3.22}$$

On the other hand, since for $i = 1,2$, h_i is ϱ_i-strongly monotone and δ_i-Lipschitz continuous in the first variable, T_2 is ξ_2-strongly monotone and μ_2-Lipschitz continuous in the first variable, in similar way to the proofs of (3.16)–(3.21), we can prove that

$$\left\|\phi(x,y)-\phi(\hat{x},\hat{y})\right\| \leq (\omega_1+\theta_2)\|x-\hat{x}\|+(\tau_2+\omega_2)\left\|y-\hat{y}\right\|, \tag{3.23}$$

where

$$\omega_i=\sqrt{1-2\varrho_i+\delta_i^2}, \quad (i=1,2), \qquad \theta_2=\sqrt{1-2\eta\xi_2+\eta^2\mu_2^2}. \tag{3.24}$$

It follows from (3.14), (3.21), and (3.23) that

$$\begin{aligned}\left\|F(x,y)-F(\hat{x},\hat{y})\right\|_* &= \left\|\psi(x,y)-\psi(\hat{x},\hat{y})\right\|+\left\|\phi(x,y)-\phi(\hat{x},\hat{y})\right\| \\ &\leq (\tau_1+\omega_1+\kappa_2+\theta_2)\|x-\hat{x}\|+(\tau_2+\omega_2+\kappa_1+\theta_1)\left\|y-\hat{y}\right\| \\ &\leq \vartheta\left\|(x,y)-(\hat{x},\hat{y})\right\|_*,\end{aligned} \tag{3.25}$$

where $\vartheta = \max\{\tau_1+\omega_1+\kappa_2+\theta_2, \tau_2+\omega_2+\kappa_1+\theta_1\}$. By condition (3.12), we note that $0 \leq \vartheta < 1$, and so (3.25) guarantees that F is a contraction mapping. According to Banach fixed point theorem, there exists a unique point $(x^*,y^*) \in \mathscr{H}\times\mathscr{H}$ such that $F(x^*,y^*) = (x^*,y^*)$. From (3.13) and (3.14), we conclude that $g_2(x^*) = J^{\rho}_{A_1(\cdot,x^*)}(g_1(y^*)-\rho T_1(y^*,x^*))$ and $h_2(y^*) = J^{\eta}_{A_2(\cdot,y^*)}(h_1(x^*)-\eta T_2(x^*,y^*))$. Now, Lemma 3.1 guarantees that $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ is the unique solution of the system (2.4). This completes the proof. □

4. Some New Resolvent Iterative Algorithms

In recent years, the nonexpansive mappings have been generalized and investigated by various authors. One of these generalizations is class of the nearly uniformly Lipschitzian mappings. In this section, we first recall some generalizations of the nonexpansive mappings which have been introduced in recent years, then we use two nearly uniformly Lipschitzian mappings S_1 and S_2 and by using the equivalent alternative formulation (3.1), we suggest and analyze a new resolvent iterative algorithm for finding an element of the set of the fixed points of $Q = (S_1,S_2)$ which is the unique solution of the SEGNVID (2.4).

In two next definitions, several generalizations of the nonexpansive mappings are stated.

Definition 4.1. A nonlinear mapping $T : \mathscr{H} \to \mathscr{H}$ is called as follows:

(a) *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in \mathscr{H}$;

(b) *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in \mathscr{H}; \tag{4.1}$$

(c) *generalized Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L(\|x - y\| + 1), \quad \forall x, y \in \mathscr{H}; \tag{4.2}$$

(d) *generalized* (L, M)*-Lipschitzian* [46] if there exist two constants $L, M > 0$ such that

$$\|Tx - Ty\| \le L(\|x - y\| + M), \quad \forall x, y \in \mathscr{H}; \tag{4.3}$$

(e) *asymptotically nonexpansive* [63] if there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \le k_n\|x - y\|, \quad \forall x, y \in \mathscr{H}; \tag{4.4}$$

(f) *pointwise asymptotically nonexpansive* [64] if, for each integer $n \ge 1$,

$$\|T^n x - T^n y\| \le \alpha_n(x)\|x - y\|, \quad x, y \in \mathscr{H}, \tag{4.5}$$

where $\alpha_n \to 1$ pointwise on X;

(g) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that for each $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall x, y \in \mathscr{H}. \tag{4.6}$$

Definition 4.2 (see [46]). A nonlinear mapping $T : \mathscr{H} \to \mathscr{H}$ is said to be

(a) *nearly Lipschitzian* with respect to the sequence $\{a_n\}$ if, for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \le k_n(\|x - y\| + a_n), \quad \forall x, y \in \mathscr{H}, \tag{4.7}$$

where $\{a_n\}$ is a fix sequence in $[0, \infty)$ with $a_n \to 0$, as $n \to \infty$.

For an arbitrary, but fixed $n \in \mathbb{N}$, the infimum of constants k_n in (4.7) is called *nearly Lipschitz constant* and is denoted by $\eta(T^n)$. Notice that

$$\eta(T^n) = \sup\left\{ \frac{\|T^n x - T^n y\|}{\|x - y\| + a_n} : x, y \in \mathscr{H}, x \ne y \right\}. \tag{4.8}$$

A nearly Lipschitzian mapping T with the sequence $\{(a_n, \eta(T^n))\}$ is said to be

(b) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$, that is,

$$\|T^n x - T^n y\| \leq \|x - y\| + a_n, \quad \forall x, y \in \mathscr{H}; \tag{4.9}$$

(c) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\eta(T^n) = 1$, in other words, $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$;

(d) *nearly uniformly L-Lipschitzian* if $\eta(T^n) \leq L$ for all $n \in \mathbb{N}$, in other words, $k_n = L$ for all $n \in \mathbb{N}$.

For some interesting examples to investigate relations between these mappings, introduced in Definitions 4.1 and 4.2, ones may consult [58].

Let $S_1 : \mathscr{H} \to \mathscr{H}$ be a nearly uniformly L_1-Lipschitzian mapping with the sequence $\{a_n\}_{n=1}^{\infty}$ and $S_2 : \mathscr{H} \to \mathscr{H}$ be a nearly uniformly L_2-Lipschitzian mapping with the sequence $\{b_n\}_{n=1}^{\infty}$. We define the self-mapping Q of $\mathscr{H} \times \mathscr{H}$ as follows:

$$Q(x, y) = (S_1 x, S_2 y), \quad \forall x, y \in \mathscr{H}. \tag{4.10}$$

Then $Q = (S_1, S_2) : \mathscr{H} \times \mathscr{H} \to \mathscr{H} \times \mathscr{H}$ is a nearly uniformly $\max\{L_1, L_2\}$-Lipschitzian mapping with the sequence $\{a_n + b_n\}_{n=1}^{\infty}$ with respect to the norm $\|\cdot\|_*$ in $\mathscr{H} \times \mathscr{H}$, where $\|\cdot\|_*$ is defined as (3.15). Because, for any $(x, y), (x', y') \in \mathscr{H} \times \mathscr{H}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|Q^n(x, y) - Q^n(x', y')\|_* \\
&\quad = \|(S_1^n x, S_2^n y) - (S_1^n x', S_2^n y')\|_* = \|(S_1^n x - S_1^n x', S_2^n y - S_2^n y')\|_* \\
&\quad = \|S_1^n x - S_1^n x'\| + \|S_2^n y - S_2^n y'\| \leq L_1(\|x - x'\| + a_n) + L_2(\|y - y'\| + b_n) \\
&\quad \leq \max\{L_1, L_2\}(\|x - x'\| + \|y - y'\| + a_n + b_n) \\
&\quad = \max\{L_1, L_2\}(\|(x, y) - (x', y')\|_* + a_n + b_n).
\end{aligned} \tag{4.11}$$

We denote the sets of all the fixed points of $S_i (i = 1, 2)$ and Q by $\mathrm{Fix}(S_i)$ and $\mathrm{Fix}(Q)$, respectively, and the set of all the solutions of the system (2.4) by SEGNVID$(\mathscr{H}, T_i, A_i, g_i, h_i, i = 1, 2)$. In view of (4.10), for any $(x, y) \in \mathscr{H} \times \mathscr{H}$, we see that $(x, y) \in \mathrm{Fix}(Q)$ if and only if $x \in \mathrm{Fix}(S_1)$ and $y \in \mathrm{Fix}(S_2)$. That is, $\mathrm{Fix}(Q) = \mathrm{Fix}(S_1, S_2) = \mathrm{Fix}(S_1) \times \mathrm{Fix}(S_2)$. We now characterize the following problem: if $(x^*, y^*) \in \mathrm{Fix}(Q) \cap$ SEGNVID$(\mathscr{H}, T_i, A_i, g_i, h_i, i = 1, 2)$, then by using Lemma 3.1, it is easy to see that for each $n \in \mathbb{N}$,

$$\begin{aligned}
x^* &= S_1^n x^* = x^* - g_2(x^*) + J^{\rho}_{A_1(\cdot, x^*)}(g_1(y^*) - \rho T_1(y^*, x^*)) \\
&= S_1^n\left[x^* - g_2(x^*) + J^{\rho}_{A_1(\cdot, x^*)}(g_1(y^*) - \rho T_1(y^*, x^*))\right], \\
y^* &= S_2^n y^* = y^* - h_2(y^*) + J^{\eta}_{A_2(\cdot, y^*)}(h_1(x^*) - \eta T_2(x^*, y^*)) \\
&= S_2^n\left[y^* - h_2(y^*) + J^{\eta}_{A_2(\cdot, y^*)}(h_1(x^*) - \eta T_2(x^*, y^*))\right].
\end{aligned} \tag{4.12}$$

The fixed point formulation (4.12) enables us to suggest the following iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $Q = (S_1, S_2)$ which is the unique solution of the system of extended general nonlinear variational inclusions (2.4).

Algorithm 4.3. Let T_i, A_i, g_i, $h_i (i = 1, 2)$, ρ, and η be the same as in the system (2.4). For an arbitrary chosen initial point $(x_1, y_1) \in \mathscr{H} \times \mathscr{H}$, compute the iterative sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in the following way:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n - \beta_n) x_n + \alpha_n S_1^n \left[x_n - g_2(x_n) + J^{\rho}_{A_1(\cdot, x_n)} (g_1(y_n) - \rho T_1(y_n, x_n)) \right] + \alpha_n e_n + \beta_n j_n + r_n, \\ y_{n+1} &= (1 - \alpha_n - \beta_n) y_n + \alpha_n S_2^n \left[y_n - h_2(y_n) + J^{\eta}_{A_2(\cdot, y_n)} (h_1(x_n) - \eta T_2(x_n, y_n)) \right] + \alpha_n p_n + \beta_n s_n + l_n, \end{aligned} \tag{4.13}$$

where $S_1, S_2 : \mathscr{H} \to \mathscr{H}$ are two nearly uniformly Lipschitzian mappings, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are two sequences in interval $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\alpha_n + \beta_n \leq 1$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\{e_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$, $\{j_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$, $\{l_n\}_{n=1}^{\infty}$ are six sequences in $\mathscr{H}$ to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions: $\{j_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$ are two bounded sequences in $\mathscr{H}$, and $\{e_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$, $\{l_n\}_{n=1}^{\infty}$ are four sequences in $\mathscr{H}$ such that

$$\begin{gathered} e_n = e'_n + e''_n, \qquad p_n = p'_n + p''_n, \qquad n \in \mathbb{N}, \\ \lim_{n \to \infty} \left\| (e'_n, p'_n) \right\|_* = 0, \qquad \sum_{n=1}^{\infty} \left\| (e''_n, p''_n) \right\|_* < \infty, \\ \sum_{n=1}^{\infty} \| r_n, l_n \|_* < \infty. \end{gathered} \tag{4.14}$$

Remark 4.4. If, for each $i = 1, 2$, $S_i \equiv I$, then Algorithm 4.3 reduces to the following iterative algorithm for solving the system (2.4).

Algorithm 4.5. Suppose that T_i, A_i, g_i, $h_i (i = 1, 2)$, ρ, and η are the same as in the system (2.4). For an arbitrary chosen initial point $(x_1, y_1) \in \mathscr{H} \times \mathscr{H}$, compute the iterative sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in the following way:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n - \beta_n) x_n + \alpha_n \left[x_n - g_2(x_n) + J^{\rho}_{A_1(\cdot, x_n)} (g_1(y_n) - \rho T_1(y_n, x_n)) \right] + \alpha_n e_n + \beta_n j_n + r_n, \\ y_{n+1} &= (1 - \alpha_n - \beta_n) y_n + \alpha_n \left[y_n - h_2(y_n) + J^{\eta}_{A_2(\cdot, y_n)} (h_1(x_n) - \eta T_2(x_n, y_n)) \right] + \alpha_n p_n + \beta_n s_n + l_n, \end{aligned} \tag{4.15}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{e_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$, $\{j_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$, and $\{l_n\}_{n=1}^{\infty}$ are the same as in Algorithm 4.3.

5. Convergence Analysis

In this section, under some suitable conditions, we establish the strong convergence of the sequence generated by iterative Algorithm 4.3. We need the following lemma to prove our main result.

Lemma 5.1. *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition. There exists a natural number n_0 such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0, \tag{5.1}$$

where $t_n \in [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Proof. The proof directly follows from Lemma 2 in Liu [59]. □

Theorem 5.2. *Let T_i, A_i, g_i, $h_i (i = 1,2)$, ρ and η be the same as in Theorem 3.4 and let all the conditions Theorem 3.4 hold. Assume that $S_1 : \mathscr{H} \to \mathscr{H}$ is a nearly uniformly L_1-Lipschitzian mapping with the sequence $\{b_n\}_{n=1}^{\infty}$, $S_2 : \mathscr{H} \to \mathscr{H}$ is a nearly uniformly L_2-Lipschitzian mapping with the sequence $\{c_n\}_{n=1}^{\infty}$, and the self-mapping Q of $\mathscr{H} \times \mathscr{H}$ is defined by* (4.10) *such that* $\mathrm{Fix}(Q) \cap$ *SEGNVID$(\mathscr{H}, T_i, A_i, g_i, h_i, i = 1,2) \neq \emptyset$. Furthermore, let, for each $i = 1,2$, $L_i\vartheta < 1$, where ϑ is the same as in* (3.25)*. Then the iterative sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$, generated by Algorithm 4.3, converges strongly to the only element of* $\mathrm{Fix}(Q) \cap$ *SEGNVID$(\mathscr{H}, T_i, A_i, g_i, h_i, i = 1,2)$.*

Proof. According to Theorem 3.4, the system (2.4) has a unique solution (x^*, y^*) in $\mathscr{H} \times \mathscr{H}$, and so Lemma 3.1 implies that (x^*, y^*) satisfies (3.1). Since SEGNVID$(\mathscr{H}, T_i, A_i, g_i, h_i, i = 1,2)$ is a singleton set and $\mathrm{Fix}(Q) \bigcap$ SEGNVID$(\mathscr{H}, T_i, A_i, g_i, h_i, i = 1,2) \neq \emptyset$, we conclude that $x^* \in \mathrm{Fix}(S_1)$ and $y^* \in \mathrm{Fix}(S_2)$. Therefore, for each $n \in \mathbb{N}$, we can write

$$\begin{aligned} x^* &= (1 - \alpha_n - \beta_n)x^* + \alpha_n S_1^n\Big[x^* - g_2(x^*) + J^{\rho}_{A_1(\cdot,x^*)}\big(g_1(y^*) - \rho T_1(y^*, x^*)\big)\Big] + \beta_n x^*, \\ y^* &= (1 - \alpha_n - \beta_n)y^* + \alpha_n S_2^n\Big[y^* - h_2(y^*) + J^{\eta}_{A_2(\cdot,y^*)}\big(h_1(x^*) - \eta T_2(x^*, y^*)\big)\Big] + \beta_n y^*, \end{aligned} \tag{5.2}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are the same as in Algorithm 4.3. Let $K = \sup_{n\geq 1}\{\|j_n - x^*\|, \|s_n - y^*\|\}$. Then, by using (4.13), (5.2), and the assumptions, we have

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\quad \leq (1 - \alpha_n - \beta_n)\|x_n - x^*\| \\ &\qquad + \alpha_n \Big\| S_1^n\Big[x_n - g_2(x_n) + J^{\rho}_{A_1(\cdot,x_n)}\big(g_1(y_n) - \rho T_1(y_n, x_n)\big)\Big] \\ &\qquad\qquad - S_1^n\Big[x^* - g_2(x^*) + J^{\rho}_{A_1(\cdot,x^*)}\big(g_1(y^*) - \rho T_1(y^*, x^*)\big)\Big]\Big\| + \alpha_n\|e_n\| + \beta_n\|j_n - x^*\| + \|r_n\| \\ &\quad \leq (1 - \alpha_n - \beta_n)\|x_n - x^*\| \end{aligned}$$

$$
\begin{aligned}
&+\alpha_n L_1\Big(\|x_n-x^*-(g_2(x_n)-g_2(x^*))\| \\
&\qquad +\Big\|J^{\rho}_{A_1(\cdot,x_n)}(g_1(y_n)-\rho T_1(y_n,x_n))-J^{\rho}_{A_1(\cdot,x^*)}(g_1(y^*)-\rho T_1(y^*,x^*))\Big\|+b_n\Big) \\
&+\alpha_n(\|e'_n\|+\|e''_n\|)+\|r_n\|+\beta_n K \\
&\le (1-\alpha_n-\beta_n)\|x_n-x^*\| \\
&+\alpha_n L_1\Big(\|x_n-x^*-(g_2(x_n)-g_2(x^*))\| \\
&\qquad +\Big\|J^{\rho}_{A_1(\cdot,x_n)}(g_1(y_n)-\rho T_1(y_n,x_n))-J^{\rho}_{A_1(\cdot,x^*)}(g_1(y_n)-\rho T_1(y_n,x_n))\Big\| \\
&\qquad +\Big\|J^{\rho}_{A_1(\cdot,x^*)}(g_1(y_n)-\rho T_1(y_n,x_n))-J^{\rho}_{A_1(\cdot,x^*)}(g_1(y^*)-\rho T_1(y^*,x^*))\Big\|+b_n\Big) \\
&+\alpha_n\|e'_n\|+\|e''_n\|+\|r_n\|+\beta_n K \\
&\le (1-\alpha_n-\beta_n)\|x_n-x^*\| \\
&+\alpha_n L_1(\|x_n-x^*-(g_2(x_n)-g_2(x^*))\|+\tau_1\|x_n-x^*\| \\
&\qquad +\|g_1(y_n)-g_1(y^*)-\rho(T_1(y_n,x_n)-T_1(y^*,x^*))\|+b_n) \\
&+\alpha_n\|e'_n\|+\|e''_n\|+\|r_n\|+\beta_n K \\
&\le (1-\alpha_n-\beta_n)\|x_n-x^*\|+\alpha_n L_1((\tau_1+\kappa_2)\|x_n-x^*\|+(\kappa_1+\theta_1)\|y_n-y^*\|+b_n) \\
&+\alpha_n\|e'_n\|+\|e''_n\|+\|r_n\|+\beta_n K,
\end{aligned}
\tag{5.3}
$$

where $\kappa_i (i=1,2)$ and θ_1 are the same as in (3.21).

In similar way to the proof of (5.3), one can establish that

$$
\begin{aligned}
&\|y_{n+1}-y^*\| \\
&\quad\le (1-\alpha_n-\beta_n)\|y_n-y^*\|+\alpha_n L_2((\omega_1+\theta_2)\|x_n-x^*\|+(\tau_2+\omega_2)\|y_n-y^*\|+c_n) \\
&\quad +\alpha_n\|p'_n\|+\|p''_n\|+\|l_n\|+\beta_n K,
\end{aligned}
\tag{5.4}
$$

where $\omega_i (i=1,2)$ and θ_2 are the same as in (3.23). Letting $L=\max\{L_1,L_2\}$ and applying (5.3) and (5.4), we obtain that

$$
\begin{aligned}
&\|(x_{n+1},y_{n+1})-(x^*,y^*)\|_* \\
&\quad\le (1-\alpha_n-\beta_n)\|(x_n,y_n)-(x^*,y^*)\|_*+\alpha_n L\vartheta\|(x_n,y_n)-(x^*,y^*)\|_* \\
&\quad +\alpha_n L(b_n+c_n)+\alpha_n\|(e'_n,p'_n)\|_*+\|(e''_n,p''_n)\|_*+\|(r_n,l_n)\|_*+2\beta_n K \\
&\quad\le (1-(1-L\vartheta)\alpha_n)\|(x_n,y_n)-(x^*,y^*)\|_*+\alpha_n\vartheta\frac{\|(e'_n,p'_n)\|_*+L(b_n+c_n)}{\vartheta} \\
&\quad +\|(e''_n,p''_n)\|_*+\|(r_n,l_n)\|_*+2\beta_n K,
\end{aligned}
\tag{5.5}
$$

where ϑ is the same as in (3.25). Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, in view of (4.14), it is obvious that the conditions of Lemma 5.1 are satisfied. Now, Lemma 5.1 and (5.5) guarantee that $(x_n, y_n) \to (x^*, y^*)$, as $n \to \infty$. Therefore, the sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$, generated by Algorithm 4.3, converges strongly to the only element (x^*, y^*) of $\mathrm{Fix}(Q) \cap \mathrm{SEGNVID}(\mathscr{H}, T_i, A_i, g_i, h_i, i = 1, 2)$. This completes the proof. $\square$

Like in the proof of Theorem 5.2, one can prove the strong convergence of the iterative sequence generated by Algorithm 4.5, and we omit its proof.

Theorem 5.3. *Suppose that T_i, A_i, g_i, $h_i (i = 1, 2)$, ρ, and η are the same as in Theorem 3.4 and let all the conditions Theorem 3.4 hold. Then the iterative sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$, generated by Algorithm 4.5, converges strongly to the unique solution (x^*, y^*) of the system* (2.4).

6. Some Comments on Related Works

This section is devoting to investigate and analyze the results in [29, 44]. We state some remarks on main results in [29] and also the explicit iterative forms, which are related to Algorithms 3.1 and 3.2 from [44], are constructed. The incorrectness of Theorem 4.1 from [44] is discussed. Furthermore, the correct versions of the aforesaid algorithms and theorem are presented.

Noor [29] proposed the following two-step iterative algorithm for solving the system of general mixed variational inequalities (2.8) and studied convergence analysis of the proposed iterative algorithm under some certain conditions.

Algorithm 6.1 (see [29, Algorithm 3.1]). For arbitrary chosen initial points $x_0, y_0 \in \mathscr{H}$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n J_\varphi\left[g(y_n) - \rho T_1(y_n, x_n)\right], \\ y_{n+1} &= J_\varphi\left[g(x_{n+1}) - \eta T_2(x_{n+1}, y_n)\right], \end{aligned} \tag{6.1}$$

where $a_n \in [0, 1]$ for all $n \geq 0$.

Theorem 6.2 (see [29, Theorem 3.1]). *Let x^*, y^* be the solution of SGMVID* (2.8). *Suppose that $T_1 : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian in the first variable, and $T_2 : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian in the first variable. Let g be relaxed (γ_3, r_3)-cocoercive and μ_3-Lipschitzian. If*

$$\begin{gathered} \left|\rho - \frac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \frac{\sqrt{\left(r_1 - \gamma_1\mu_1^2\right)^2 - \mu_1^2 k(2-k)}}{\mu_1^2}, \\ r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{k(2-k)}, \quad k < 1, \\ \left|\eta - \frac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \frac{\sqrt{\left(r_2 - \gamma_2\mu_2^2\right)^2 - \mu_2^2 k(2-k)}}{\mu_2^2}, \\ r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{k(2-k)}, \quad k < 1, \end{gathered} \tag{6.2}$$

where

$$k = \sqrt{1 - 2\left(r_3 - \gamma_3\mu_3^2\right) + \mu_3^2}, \tag{6.3}$$

and $a_n \in [0,1]$, $\sum_{n=0}^{\infty} a_n = \infty$, *then for arbitrarily chosen initial points* $x_0, y_0 \in \mathcal{H}$, x_n *and* y_n *obtained from Algorithm 6.1 converge strongly to* x^* *and* y^*, *respectively.*

By using Definition 2.1, we note that the condition relaxed cocoercivity of the operator T is weaker than the condition of strong monotonicity of T. In other words, the class of relaxed cocoercive operators is more general than the class of strongly monotone operators. Now, we show that, unlike claim of Noor [29], he studied the convergence analysis of the proposed iterative algorithm under the condition of strong monotonicity, not the mild condition relaxed cocoercivity.

Remark 6.3. In view of the conditions (6.2) (the conditions (4.1) and (4.2) in [29]), we have $k \in (0,1)$. The condition (6.3) (the condition (4.3) in [29]) and $k > 0$ imply that $2(r_3 - \gamma_3\mu_3^2) < 1 + \mu_3^2$. Therefore, the condition $2(r_3 - \gamma_3\mu_3^2) < 1 + \mu_3^2$ should be added to the conditions (6.2)-(6.3). On the other hand, since $k < 1$ from the condition (6.3), it follows that $r_3 > \gamma_3\mu_3^2$. The conditions $r_i > \gamma_i\mu_i^2 + \mu_i\sqrt{k(2-k)}$ $(i = 1,2)$, and $k < 1$ imply that $r_i > \gamma_i\mu_i^2$ for each $i = 1,2$. Since, for each $i = 1,2$, the operator T_i is (γ_i, r_i)-relaxed cocoercive and μ_i-Lipschitz continuous, the conditions $r_i > \gamma_i\mu_i^2 (i = 1,2)$ guarantee that, for each $i = 1,2$, the operator T_i is $(r_i - \gamma_i\mu_i^2)$ strongly monotone. Similarly, since g is (γ_3, r_3) relaxed cocoercive and μ_3-Lipschitz continuous, the condition $r_3 > \gamma_3\mu_3^2$ implies that the operator g is $(r_3 - \gamma_3\mu_3^2)$-strongly monotone.

In view of the above remark, one can rewrite Theorem 6.2 as follows.

Theorem 6.4. *Let* x^* *and* y^* *be the solution of the SGMVID* (2.8) *and suppose that* $T_1 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ *is* ξ_1*-strongly monotone and* μ_1*-Lipschitz continuous in the first variable, and* $T_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ *is relaxed* ξ_2*-strongly monotone and* μ_2*-Lipschitz continuous in the first variable. Moreover, let* g *be* ξ_3*-strongly monotone and* μ_3*-Lipschitz continuous. If two constants* ρ *and* η *satisfy the following conditions:*

$$\begin{gathered}
\left|\rho - \frac{\xi_1}{\mu_1^2}\right| < \frac{\sqrt{\xi_1^2 - \mu_1^2 k(2-k)}}{\mu_1^2}, \\
\left|\eta - \frac{\xi_2}{\mu_2^2}\right| < \frac{\sqrt{\xi_2^2 - \mu_2^2 k(2-k)}}{\mu_2^2}, \\
\xi_i > \mu_i\sqrt{k(2-k)}, \quad (i = 1,2), \\
k = \sqrt{1 - 2\xi_3 + \mu_3^2} < 1, \qquad 2\xi_3 < 1 + \mu_3^2,
\end{gathered} \tag{6.4}$$

and $\sum_{n=0}^{\infty} a_n = \infty$, *then the iterative sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by Algorithm 6.1 converge strongly to* x^* *and* y^*, *respectively.*

M. A. Noor and K. I. Noor in [44] proposed the following iterative scheme for solving the system of general variational inclusions (1) from [44].

Algorithm 6.5 (see [44, Algorithm 3.1]). For arbitrary chosen initial points $x_0, y_0 \in K$ compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n\left(x_{n+1} - g_1(x_{n+1})\right) + a_n J_A\left[g(y_n) - \rho T_1(y_n)\right], \\ y_{n+1} &= y_{n+1} - h_1(y_{n+1}) + J_A\left[h(x_{n+1}) - \eta T_2(x_{n+1})\right], \end{aligned} \tag{6.5}$$

where $a_n \in [0,1]$ for all $n \geq 0$ satisfies some suitable conditions.

Taking $g_1 = g$ and $h_1 = h$, Algorithm 6.5 reduces to the following iterative algorithm.

Algorithm 6.6 (see [44, Algorithm 3.2]). For arbitrary chosen initial points $x_0, y_0 \in K$ compute the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n\left(x_{n+1} - g(x_{n+1})\right) + a_n J_A\left[g(y_n) - \rho T_1(y_n)\right], \\ y_{n+1} &= y_{n+1} - h(y_{n+1}) + J_A\left[h(x_{n+1}) - \eta T_2(x_{n+1})\right], \end{aligned} \tag{6.6}$$

where $a_n \in [0,1]$ for all $n \geq 0$ satisfies some suitable conditions.

Remark 6.7. By analyzing two Algorithms 6.5 and 6.6, we note that the mentioned algorithms are in implicit forms. Further, in view of Remark 2.2, we should apply the system (2.5) instead of the system (1) from [44].

Next, we derive two explicit algorithms to solve the systems (2.5) and (2.6), respectively, as follows.

Algorithm 6.8. Let T_1, T_2, A, g, h, g_1, and h_1 be the same as in the system (2.5), and let h be an onto operator. For arbitrary chosen initial points $x_0, y_0 \in K$ compute the sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n\left(x_n - g_1(x_n) + J_A\left(g(y_{n+1}) - \rho T_1(y_{n+1})\right)\right), \\ h_1(y_{n+1}) &= J_A\left(h(x_n) - \eta T_2(x_n)\right) \end{aligned} \tag{6.7}$$

where $a_n \in [0,1]$ for all $n \geq 0$ satisfies some suitable conditions.

Algorithm 6.9. Let T_1, T_2, A, g, h, g_1, and h_1 be the same as in the system (2.5). For arbitrary chosen initial points $x_0, y_0 \in K$ compute the sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n\left(x_n - g_1(x_n) + J_A\left(g(y_n) - \rho T_1(y_n)\right)\right), \\ y_{n+1} &= (1-a_n)y_n + a_n\left(y_n - h_1(y_n) + J_A\left(h(x_n) - \eta T_2(x_n)\right)\right), \end{aligned} \tag{6.8}$$

where $a_n \in [0,1]$ for all $n \geq 0$ satisfies some suitable conditions.

Taking $g_1 = g$ and $h_1 = h$ in two Algorithms 6.8 and 6.9, we can obtain the following two algorithms.

Algorithm 6.10. For arbitrary chosen initial points $x_0, y_0 \in K$ compute the sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n\left(x_n - g(x_n) + J_A\left(g(y_{n+1}) - \rho T_1(y_{n+1})\right)\right), \\ h(y_{n+1}) &= J_A\left(h(x_n) - \eta T_2(x_n)\right), \end{aligned} \tag{6.9}$$

where $a_n \in [0,1]$ for all $n \geq 0$ satisfies some suitable conditions.

Algorithm 6.11. For arbitrary chosen initial points $x_0, y_0 \in K$ compute the sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n\left(x_n - g(x_n) + J_A\left(g(y_n) - \rho T_1(y_n)\right)\right), \\ y_{n+1} &= (1-a_n)y_n + a_n\left(y_n - h(y_n) + J_A\left(h(x_n) - \eta T_2(x_n)\right)\right), \end{aligned} \tag{6.10}$$

where $a_n \in [0,1]$ for all $n \geq 0$ satisfies some suitable conditions.

We now recall some facts, which has presented in [44].

Lemma 6.12 (see [44, Lemma 3.1]). *If the operator A is maximal monotone, then $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ is a solution of* (2.5) *(the correct version of the system (1) in [44]) if and only if $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ satisfies*

$$\begin{aligned} g_1(x^*) &= J_A^{\rho}\left[g(y^*) - \rho T_1(y^*)\right], \\ h_1(y^*) &= J_A^{\eta}\left[h(x^*) - \eta T_2(x^*)\right]. \end{aligned} \tag{6.11}$$

Remark 6.13. In view of Lemma 6.12, $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ is a solution of the system (2.6) if and only if

$$\begin{aligned} g(x^*) &= J_A^{\rho}\left[g(y^*) - \rho T_1(y^*)\right], \\ h(y^*) &= J_A^{\eta}\left[h(x^*) - \eta T_2(x^*)\right], \end{aligned} \tag{6.12}$$

where $\rho, \eta > 0$ are two constants.

Theorem 6.14. *Let $T_i (i = 1,2)$, g, and h be the same as in Algorithm 6.10, and let x^*, y^* be the solution of the system* (2.6). *Assume that for $i = 1,2$, the operator $T_i : \mathscr{H} \to \mathscr{H}$ is ξ_i-strongly monotone and μ_i-Lipschitz continuous. Furthermore, let g be ξ_3-strongly monotone and μ_3-Lipschitz continuous and h be ξ_4-strongly monotone and μ_4-Lipschitz continuous. If there exist two constants ρ and η such that*

$$(k+\theta_1)(k_1+\theta_2) < (1-k)(1-k_1), \tag{6.13}$$

where

$$
\begin{aligned}
k &= \sqrt{1 - 2\xi_3 + \mu_3^2} < 1, \quad 2\xi_3 < 1 + \mu_3^2, \\
k_1 &= \sqrt{1 - 2\xi_4 + \mu_4^2} < 1, \quad 2\xi_4 < 1 + \mu_4^2, \\
\theta_1 &= \sqrt{1 - 2\rho\xi_1 + \rho^2\mu_1^2} < 1, \quad 2\rho\xi_1 < 1 + \rho^2\mu_1^2, \\
\theta_2 &= \sqrt{1 - 2\eta\xi_2 + \eta^2\mu_2^2} < 1, \quad 2\eta\xi_2 < 1 + \eta^2\mu_2^2
\end{aligned}
\tag{6.14}
$$

and $\sum_{n=0}^{\infty} a_n = \infty$*, and then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by Algorithm 6.10 converge strongly to* x^* *and* y^**, respectively.*

Proof. Since $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ is a solution of the system (2.6), in view of Remark 6.13, we have

$$
g(x^*) = J_A^{\rho}(g(y^*) - \rho T_1(y^*)), \qquad h(y^*) = J_A^{\eta}(h(x^*) - \eta T_2(x^*)),
\tag{6.15}
$$

where ρ and η are two constants. For each $n \geq 0$, one can rewrite the above equations as below:

$$
\begin{aligned}
x^* &= (1 - a_n)x^* + a_n\Big(x^* - g(x^*) + J_A^{\rho}(g(y^*) - \rho T_1(y^*))\Big), \\
h(y^*) &= J_A^{\eta}(h(x^*) - \eta T_2(x^*)),
\end{aligned}
\tag{6.16}
$$

where the sequence $\{a_n\}$ is the same as in Algorithm 6.10. It follows from (6.9), (6.16), and the nonexpansivity property of the resolvent operator J_A^{ρ}, that

$$
\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\leq (1 - a_n)\|x_n - x^*\| \\
&\quad + a_n\Big(\|x_n - x^* - (g(x_n) - g(x^*))\| + \Big\|J_A^{\rho}(g(y_{n+1}) - \rho T_1(y_{n+1})) - J_A^{\rho}(g(y^*) - \rho T_1(y^*))\Big\|\Big) \\
&\leq (1 - a_n)\|x_n - x^*\| \\
&\quad + a_n(\|x_n - x^* - (g(x_n) - g(x^*))\| + \|g(y_{n+1}) - g(y^*) - \rho(T_1(y_{n+1}) - T_1(y^*))\|) \\
&\leq (1 - a_n)\|x_n - x^*\| \\
&\quad + a_n(\|x_n - x^* - (g(x_n) - g(x^*))\| + \|y_{n+1} - y^* - (g(y_{n+1}) - g(y^*))\| \\
&\qquad + \|y_{n+1} - y^* - \rho(T_1(y_{n+1}) - T_1(y^*))\|).
\end{aligned}
\tag{6.17}
$$

Since T_1 is ξ_1-strongly monotone and μ_1-Lipschitz continuous, we have

$$\begin{aligned}
&\left\| y_{n+1} - y^* - \rho\left(T_1(y_{n+1}) - T_1(y^*)\right)\right\|^2 \\
&\quad = \left\| y_{n+1} - y^*\right\|^2 - 2\rho\left\langle T_1(y_{n+1}) - T_1(y^*), y_{n+1} - y^*\right\rangle \\
&\quad\quad + \rho^2\left\| T_1(y_{n+1}) - T_1(y^*)\right\|^2 \\
&\quad \leq \left(1 - 2\rho\xi_1 + \rho^2\mu_1^2\right)\left\| y_{n+1} - y^*\right\|^2
\end{aligned} \tag{6.18}$$

which leads to

$$\left\| y_{n+1} - y^* - \rho\left(T_1(y_{n+1}) - T_1(y^*)\right)\right\| \leq \sqrt{1 - 2\rho\xi_1 + \rho^2\mu_1^2}\left\| y_{n+1} - y^*\right\|. \tag{6.19}$$

Since g is ξ_3-strongly monotone and μ_3-Lipschitz continuous, we get

$$\left\| x_n - x^* - \left(g(x_n) - g(x^*)\right)\right\| \leq \sqrt{1 - 2\xi_3 + \mu_3^2}\| x_n - x^*\|, \tag{6.20}$$

$$\left\| y_{n+1} - y^* - \left(g(y_{n+1}) - g(y^*)\right)\right\| \leq \sqrt{1 - 2\xi_3 + \mu_3^2}\left\| y_{n+1} - y^*\right\|. \tag{6.21}$$

Combining (6.17)–(6.21), we get the following:

$$\| x_{n+1} - x^*\| \leq (1 - a_n)\| x_n - x^*\| + a_n k\| x_n - x^*\| + a_n(k + \theta_1)\left\| y_{n+1} - y^*\right\|, \tag{6.22}$$

where k and θ_1 are the same as in (6.14). Now, we find an estimation for $\| y_{n+1} - y^*\|$. Using (6.9), (6.16), and the nonexpansivity property of the resolvent operator J_A^η, we obtain that

$$\begin{aligned}
\left\| y_{n+1} - y^*\right\| &\leq \left\| y_{n+1} - y^* - \left(h(y_{n+1}) - h(y^*)\right)\right\| + \left\| h(y_{n+1}) - h(y^*)\right\| \\
&\leq \left\| y_{n+1} - y^* - \left(h(y_{n+1}) - h(y^*)\right)\right\| \\
&\quad + \left\| J_A^\eta\left(h(x_n) - \eta T_2(x_n)\right) - J_A^\eta\left(h(x^*) - \eta T_2(x^*)\right)\right\| \\
&\leq \left\| y_{n+1} - y^* - \left(h(y_{n+1}) - h(y^*)\right)\right\| \\
&\quad + \left\| h(x_n) - h(x^*) - \eta(T_2(x_n) - T_2(x^*))\right\| \\
&\leq \left\| y_{n+1} - y^* - \left(h(y_{n+1}) - h(y^*)\right)\right\| \\
&\quad + \| x_n - x^* - (h(x_n) - h(x^*))\| \\
&\quad + \left\| x_n - x^* - \eta(T_2(x_n) - T_2(x^*))\right\|.
\end{aligned} \tag{6.23}$$

Since T_2 is ξ_2-strongly monotone and μ_2-Lipschitz continuous, and h is ξ_4-strongly monotone and μ_4-Lipschitz continuous, in similar way to the proofs of (6.19)–(6.21), we can establish that

$$\begin{gathered}\|x_n - x^* - \eta(T_2(x_n) - T_2(x^*))\| \le \sqrt{1 - 2\eta\xi_2 + \eta^2\mu_2^2}\|x_n - x^*\|, \\ \|x_n - x^* - (h(x_n) - h(x^*))\| \le \sqrt{1 - 2\xi_4 + \mu_4^2}\|x_n - x^*\|, \\ \|y_{n+1} - y^* - (h(y_{n+1}) - h(y^*))\| \le \sqrt{1 - 2\xi_4 + \mu_4^2}\|y_{n+1} - y^*\|.\end{gathered} \tag{6.24}$$

Substituting (6.24) in (6.23), we obtain that

$$\|y_{n+1} - y^*\| \le k_1\|y_{n+1} - y^*\| + (k_1 + \theta_2)\|x_n - x^*\|, \tag{6.25}$$

where k_1 and θ_2 are the same as in (6.14). From (6.25), we conclude that

$$\|y_{n+1} - y^*\| \le \frac{k_1 + \theta_2}{1 - k_1}\|x_n - x^*\|. \tag{6.26}$$

It follows from (6.22) and (6.26) that

$$\|x_{n+1} - x^*\| \le \left(1 - a_n\left(1 - k - \frac{(k + \theta_1)(k_1 + \theta_2)}{1 - k_1}\right)\right)\|x_n - x^*\|. \tag{6.27}$$

Letting $\iota = 1 - k - ((k + \theta_1)(k_1 + \theta_2))/(1 - k_1)$, the condition (6.13) implies that $\iota \in (0, 1)$. Since $\sum_{n=0}^{\infty} a_n = \infty$, setting $b_n = c_n = 0$, for all $n \ge 0$, we note that all the conditions of Lemma 5.1 are satisfied. Now, Lemma 5.1 and (6.27) guarantee that $\|x_n - x^*\|_* \to 0$, as $n \to \infty$. The inequality (6.26) implies that $\|y_n - y^*\|_* \to 0$, as $n \to \infty$, and so the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 6.10 converge strongly to x^* and y^*, respectively. This completes the proof. □

Theorem 6.15. *Let $T_i (i = 1, 2)$, g and h be the same as in Algorithm 6.11, and let $x^*$and y^* be the solution of the system* (2.6)*. Suppose that, for $i = 1, 2$, the operator $T_i : \mathscr{H} \to \mathscr{H}$ is ξ_i-strongly monotone and μ_i-Lipschitz continuous. Moreover, let g be ξ_3-strongly monotone and μ_3-Lipschitz continuous and h be ξ_4-strongly monotone and μ_4-Lipschitz continuous. If there exist two constant ρ and η such that*

$$\theta_1 + \theta_2 < 1 - 2(k + k_1), \tag{6.28}$$

where k, k_1, θ_1, and θ_2 are the same as in (6.14) *and $\sum_{n=0}^{\infty} a_n = \infty$, and then the iterative sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 6.11 converge strongly to x^* and y^*, respectively.*

Proof. Since $(x^*, y^*) \in \mathscr{H} \times \mathscr{H}$ is a solution of the system (2.6), in view of Remark 6.13, we have

$$g(x^*) = J_A^{\rho}(g(y^*) - \rho T_1(y^*)), \qquad h(y^*) = J_A^{\eta}(h(x^*) - \eta T_2(x^*)), \tag{6.29}$$

where $\rho, \eta > 0$ are two constants. For each $n \geq 0$, one can rewrite the above equations as follows:

$$\begin{aligned} x^* &= (1 - a_n)x^* + a_n\Big(x^* - g(x^*) + J_A^{\rho}\big(g(y^*) - \rho T_1(y^*)\big)\Big), \\ y^* &= (1 - a_n)y^* + a_n\Big(y^* - h(y^*) + J_A^{\eta}\big(h(x^*) - \eta T_2(x^*)\big)\Big), \end{aligned} \tag{6.30}$$

where the sequence $\{a_n\}$ is the same as in Algorithm 6.11. From (6.10), (6.30), and the nonexpansivity property of the resolvent operator J_A^{ρ}, it follows that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\quad \leq (1 - a_n)\|x_n - x^*\| \\ &\qquad + a_n\Big(\|x_n - x^* - (g(x_n) - g(x^*))\| + \Big\|J_A^{\rho}\big(g(y_n) - \rho T_1(y_n)\big) - J_A^{\rho}\big(g(y^*) - \rho T_1(y^*)\big)\Big\|\Big) \\ &\quad \leq (1 - a_n)\|x_n - x^*\| \\ &\qquad + a_n\big(\|x_n - x^* - (g(x_n) - g(x^*))\| + \|g(y_n) - g(y^*) - \rho(T_1(y_n) - T_1(y^*))\|\big) \\ &\quad \leq (1 - a_n)\|x_n - x^*\| \\ &\qquad + a_n\big(\|x_n - x^* - (g(x_n) - g(x^*))\| + \|y_n - y^* - (g(y_n) - g(y^*))\| \\ &\qquad\qquad + \|y_n - y^* - \rho(T_1(y_n) - T_1(y^*))\|\big). \end{aligned} \tag{6.31}$$

Since T_1 is ξ_1-strongly monotone and μ_1-Lipschitz continuous, and g is ξ_3-strongly monotone and μ_3-Lipschitz continuous, one can prove that

$$\begin{aligned} \big\|y_n - y^* - \rho(T_1(y_n) - T_1(y^*))\big\| &\leq \sqrt{1 - 2\rho\xi_1 + \rho^2\mu_1^2}\,\|y_n - y^*\|, \\ \big\|x_n - x^* - (g(x_n) - g(x^*))\big\| &\leq \sqrt{1 - 2\xi_3 + \mu_3^2}\,\|x_n - x^*\|, \\ \big\|y_n - y^* - (g(y_n) - g(y^*))\big\| &\leq \sqrt{1 - 2\xi_3 + \mu_3^2}\,\|y_n - y^*\|. \end{aligned} \tag{6.32}$$

Combining (6.31) and (6.32), we obtain that

$$\|x_{n+1} - x^*\| \leq (1 - a_n)\|x_n - x^*\| + a_n k\|x_n - x^*\| + a_n(k + \theta_1)\big\|y_n - y^*\big\|, \tag{6.33}$$

where k and θ_1 are the same as in (6.14). Because T_2 is ξ_2-strongly monotone and μ_2-Lipschitz continuous, and h is ξ_4-strongly monotone and μ_4-Lipschitz continuous, in similar way to the proofs of (6.31)–(6.33), we can verify that

$$\big\|y_{n+1} - y^*\big\| \leq (1 - a_n)\big\|y_n - y^*\big\| + a_n k_1\big\|y_n - y^*\big\| + a_n(k_1 + \theta_2)\|x_n - x^*\|, \tag{6.34}$$

where k_1 and θ_2 are the same as in (6.14). From (6.33) and (6.34), it follows that

$$\begin{aligned}
&\left\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\right\|_* \\
&\quad\le (1 - a_n)\left\|(x_n, y_n) - (x^*, y^*)\right\|_* + a_n(k + k_1)\left\|(x_n, y_n) - (x^*, y^*)\right\|_* \\
&\qquad + a_n(k + k_1 + \theta_1 + \theta_2)\left\|(x_n, y_n) - (x^*, y^*)\right\|_* \\
&\quad= (1 - a_n(1 - (2(k + k_1) + \theta_1 + \theta_2)))\left\|(x_n, y_n) - (x^*, y^*)\right\|_*.
\end{aligned} \tag{6.35}$$

Letting $\varpi = 2(k+k_1)+\theta_1+\theta_2$, the condition (6.28) guarantees that $\varpi \in (0,1)$. Since $\sum_{n=0}^{\infty} a_n = \infty$, setting $b_n = c_n = 0$, for all $n \ge 0$, we infer that all the conditions of Lemma 5.1 are satisfied. Now, Lemma 5.1 and (6.35) guarantee that $\|(x_n, y_n) - (x^*, y^*)\|_* \to 0$, as $n \to \infty$, and so the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 6.11 converge strongly to x^* and y^*, respectively. This completes the proof. □

Remark 6.16. M. A. Noor and K. I. Noor [44] established the strong convergence of the sequences generated by iterative Algorithm 6.6. We would like to notice that, as we have made an observation in Remark 6.3, some assumptions should be added to ([44], Theorem 4.1).

7. Conclusions

In this paper, we have introduced and considered a new system of extended general nonlinear variational inclusions involving eight different nonlinear operators (SEGNVID). We have verified the equivalence between the SEGNVID and the fixed point problem and then by this equivalent formulation, and we have discussed the existence and uniqueness theorem for a solution of the SEGNVID. This equivalence and two nearly uniformly Lipschitzian mappings $S_i (i = 1, 2)$ are used to suggest and analyze a new resolvent iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $Q = (S_1, S_2)$ which is the unique solution of the SEGNVID. In the final section, comments on some related works are presented. It is expected that the results proved in this paper may simulate further research regarding the numerical methods and their applications in various fields of pure and applied sciences.

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