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Research Article

Coupled Coincidence Point Results for (ψ, α, β)-Weak Contractions in Partially Ordered Metric Spaces

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In this paper coupled coincidence points of mappings satisfying a nonlinear contractive condition in the framework of partially ordered metric spaces are obtained. Our results extend the results of Harjani et al. (2011). Moreover, an example of the main result is given. Finally, some coupled coincidence point results for mappings satisfying some contraction conditions of integral type in partially ordered complete metric spaces are deduced.

1. Introduction and Mathematical Preliminaries

The existence of fixed points for certain mappings in ordered metric spaces has been studied and applied by Ran and Reurings [1] and then by Nieto and Rodríguez-López [2]. So far, many researchers have obtained fixed point and common fixed point results for mappings under various contractive conditions in different metric spaces (see, e.g., [3–8]).

Existence of coupled fixed points in partially ordered metric spaces was first investigated in 2006 by Bhaskar and Lakshmikantham [9] and then by Lakshmikantham and Ćirić [10]. Further results in this direction under weak contraction conditions in different metric spaces were proved in, for example, [4, 5, 10–15].

Bhaskar and Lakshmikantham [9] introduced the following definitions.

Definition 1.1 (see [9]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a self-map. One can say that F has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for any $y \in X$, and for all $y_1, y_2 \in X$, $y_1 \succeq y_2$ implies $F(x, y_1) \preceq F(x, y_2)$ for any $x \in X$.

Definition 1.2 (see [9]). An element $(x, y) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

The main results of Bhaskar and Lakshmikantham in [9] are the following coupled fixed point theorems.

Theorem 1.3 (see [9]). *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x,y), F(u,v)) \le \frac{k}{2}[d(x,u) + d(y,v)], \tag{1.1}$$

for all $x \preceq u$ and $y \succeq v$. If there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Theorem 1.4 (see [9]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d in X such that (X, d) is a complete metric space. Assume that X has the following properties:*

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$, for all n;*

(ii) *if a nonincreasing sequence $y_n \to y$, then $y \preceq y_n$, for all n.*

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d(F(x,y), F(u,v)) \le \frac{k}{2}(d(x,u) + d(y,v)), \tag{1.2}$$

for all $x \preceq u$ and $y \succeq v$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Recently, Abbas et al. [11] have introduced the concept of w-compatible mappings to obtain coupled coincidence point for nonlinear contractive mappings in a cone metric space.

Definition 1.5 (see [11]). The mappings $F : X \times X \to X$ and $g : X \to X$ are called w-compatible if $g(F(x, y)) = F(gx, gy)$, whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Ćirić et al. [3] have presented the concepts of a mixed g-monotone mapping, coupled coincidence point, and commutative mapping. They proved some coupled coincidence and coupled common fixed point theorems for mixed g-monotone nonlinear contractive mappings in partially ordered complete metric spaces. The results of Lakshmikantham and Ćirić are generalizations of Theorems 1.3 and 1.4.

Definition 1.6 (see [10]). An element $(x, y) \in X \times X$ is called

(1) a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$,

(2) a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 1.7 (see [3]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two self-mappings. F has the mixed g-monotone property if F is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, that is, for all $x_1, x_2 \in X$, $gx_1 \preceq gx_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for any $y \in X$, and for all $y_1, y_2 \in X$, $gy_1 \preceq gy_2$ implies $F(x, y_1) \succeq F(x, y_2)$ for any $x \in X$.

Definition 1.8 (see [3]). Let X be a nonempty set. One can say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $g(F(x, y)) = F(gx, gy)$, for all $x, y \in X$.

Theorem 1.9 (Corollary 2.1[3]). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(gx, gu) + d(gy, gv)], \tag{1.3}$$

for all $x, y, u, v \in X$ for which $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) *F is continuous, or,*

(b) *X has the following properties,*

  (i) *if a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,*

  (ii) *if a nonincreasing sequence $y_n \to y$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g have a coupled coincidence point.

Harjani et al. [7] obtained the following theorem for mappings with the mixed monotone property.

Theorem 1.10 (see [7]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X and continuous such that*

$$\psi(d(F(x, y), F(u, v))) \leq \psi(\max\{d(x, u), d(y, v)\}) - \varphi(\max\{d(x, u), d(y, v)\}), \tag{1.4}$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$, where ψ and φ are altering distance functions. If there exist $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Also, they proved that the above theorem is still valid for F not necessarily continuous, assuming the following hypothesis.

If $\{x_n\}$ is a nondecreasing sequence with $x_n \to x$, then $x_n \preceq x$, for all $n \in \mathbb{N}$.

If $\{y_n\}$ is a nonincreasing sequence with $y_n \to y$, then $y_n \succeq y$, for all $n \in \mathbb{N}$.

Theorem 1.11 (see [7]). *If in Theorem 1.10 one substitutes the continuity of F by the condition mentioned above one also obtains the existence of a coupled fixed point for F.*

The aim of this paper is to study necessary conditions for the existence of coupled coincidence and common coupled fixed points of (ψ, α, β)-weak contractions in ordered metric spaces. For more details on (ψ, α, β)-weakly contractive mappings we refer the reader to [16].

2. Main Results

The notion of an altering distance function was introduced by Khan et al. [17] as follows.

Definition 2.1. The function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied:

(1) ψ is continuous and nondecreasing,

(2) $\psi(t) = 0$ if and only if $t = 0$.

Now, we establish an existence theorem for coupled coincidence point of mappings satisfying (ψ, α, β)-weak contraction condition in the setup of partially ordered metric spaces. Note that (ψ, α, β)-weak contraction condition was first appeared in [16].

Theorem 2.2. *Let $(X, \preceq, d)$ be a partially ordered complete metric space and let $F : X^2 \to X$ and $g : X \to X$ be such that $F(X^2) \subseteq g(X)$ and F is continuous. Assume that*

$$\begin{aligned} \psi(d(F(x, y), F(u, v))) &\leq \alpha(\max\{d(gx, gu), d(gy, gv)\}) \\ &\quad - \beta(\max\{d(gx, gu), d(gy, gv)\}), \end{aligned} \tag{2.1}$$

for every $x, y, u, v \in X$ with $gx \preceq gu$ and $gy \succeq gv$, where $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ are such that, ψ is an altering distance function, α is continuous, β is lower semicontinuous, $\alpha(0) = \beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$.

Assume that

(1) *F has the mixed g-monotone property,*

(2) *g is continuous and commutes with F.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point in X.

Proof. Let $x_0, y_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Define $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$ and in this way, we construct the sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$\begin{aligned} a_n &= gx_n = F(x_{n-1}, y_{n-1}), \\ b_n &= gy_n = F(y_{n-1}, x_{n-1}), \end{aligned} \tag{2.2}$$

for all $n \geq 0$.

We will do the proof in two steps.

Step I. We will show that $\{a_n\}$ and $\{b_n\}$ are Cauchy. Let

$$\delta_n = \max\{d(a_{n-1}, a_n), d(b_{n-1}, b_n)\}. \tag{2.3}$$

As $gx_{n-1} \preceq gx_n$ and $gy_{n-1} \succeq gy_n$, using (2.1) we obtain that

$$\begin{aligned}\psi(d(a_n, a_{n+1})) &= \psi(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\ &\leq \alpha(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}) \\ &\quad - \beta(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}) \\ &= \alpha(\max\{d(a_{n-1}, a_n), d(b_{n-1}, b_n)\}) \\ &\quad - \beta(\max\{d(a_{n-1}, a_n), d(b_{n-1}, b_n)\}).\end{aligned} \tag{2.4}$$

In a similar way, since $gy_n \preceq gy_{n-1}$ and $gx_n \succeq gx_{n-1}$, we have

$$\begin{aligned}\psi(d(b_{n+1}, b_n)) &= \psi(d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))) \\ &\leq \alpha(\max\{d(gy_n, gy_{n-1}), d(gx_n, gx_{n-1})\}) \\ &\quad - \beta(\max\{d(gy_n, gy_{n-1}), d(gx_n, gx_{n-1})\}) \\ &= \alpha(\max\{d(a_{n-1}, a_n), d(b_{n-1}, b_n)\}) \\ &\quad - \beta(\max\{d(a_{n-1}, a_n), d(b_{n-1}, b_n)\}).\end{aligned} \tag{2.5}$$

If for an $n \geq 1$, $\delta_n = 0$, then the conclusion of the theorem follows. So, we assume that

$$\delta_n \neq 0, \tag{2.6}$$

for all $n \geq 1$.

Let, for some n, $\delta_{n-1} < \delta_n$. So, from (2.4) and (2.5) as ψ is nondecreasing, we have

$$\begin{aligned}\psi(\max\{d(a_{n-1}, a_n), d(b_{n-1}, b_n)\}) &< \psi(\max\{d(a_n, a_{n+1}), d(b_n, b_{n+1})\}) \\ &= \max\{\psi(d(a_n, a_{n+1})), \psi(d(b_n, b_{n+1}))\} \\ &\leq \alpha(\max\{d(a_{n-1}, a_n), d(b_{n-1}, b_n)\}) \\ &\quad - \beta(\max\{d(a_{n-1}, a_n), d(b_{n-1}, b_n)\}),\end{aligned} \tag{2.7}$$

that is, $\psi(\delta_n) - \alpha(\delta_n) + \beta(\delta_n) \leq 0$. By our assumptions, we have $\delta_n = 0$, which contradicts (2.6). Therefore, for all $n \geq 1$ we deduce that

$$\delta_{n+1} \leq \delta_n, \tag{2.8}$$

that is, $\{\delta_n\}$ is a nonincreasing sequence of nonnegative real numbers. Thus, there exists an $r \geq 0$ such that $\lim_{n\to\infty}\delta_n = r$.

Taking $n \to \infty$ in (2.7) and using the lower semicontinuity of β and the continuity of ψ and α, we obtain $\psi(r) \leq \alpha(r) - \beta(r)$, which further implies that $r = 0$, from our assumptions about ψ, α, and β. Therefore,

$$\lim_{n \to \infty} \max\{d(a_{n-1}, a_n), d(b_{n-1}, b_n)\} = 0. \tag{2.9}$$

Next, we claim that $\{a_n\}$ and $\{b_n\}$ are Cauchy.

We will show that for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that if $m, n \geq k$,

$$\max\{d(a_n, a_m), d(b_n, b_m)\} < \varepsilon. \tag{2.10}$$

Suppose the above statement is false.

Then, there exists an $\varepsilon > 0$ for which we can find subsequences $\{a_{m(k)}\}$ and $\{a_{n(k)}\}$ of $\{a_n\}$ and $\{b_{m(k)}\}$ and $\{b_{n(k)}\}$ of $\{b_n\}$ such that $n(k) > m(k) > k$ and

$$\max\{d(a_{m(k)}, a_{n(k)}), d(b_{m(k)}, b_{n(k)})\} \geq \varepsilon, \tag{2.11}$$

where $n(k)$ is the smallest index with this property, that is,

$$\max\{d(a_{m(k)}, a_{n(k)-1}), d(b_{m(k)}, b_{n(k)-1})\} < \varepsilon. \tag{2.12}$$

From triangle inequality,

$$d(a_{m(k)}, a_{n(k)}) \leq d(a_{m(k)}, a_{n(k)-1}) + d(a_{n(k)-1}, a_{n(k)}). \tag{2.13}$$

Similarly,

$$d(b_{m(k)}, b_{n(k)}) \leq d(b_{m(k)}, b_{n(k)-1}) + d(b_{n(k)-1}, b_{n(k)}). \tag{2.14}$$

So,

$$\begin{aligned}\max\{d(a_{m(k)}, a_{n(k)}), d(b_{m(k)}, b_{n(k)})\} &\leq \max\{d(a_{m(k)}, a_{n(k)-1}), d(b_{m(k)}, b_{n(k)-1})\} \\ &\quad + \max\{d(a_{n(k)-1}, a_{n(k)}), d(b_{n(k)-1}, b_{n(k)})\}.\end{aligned} \tag{2.15}$$

Letting $k \to \infty$, as $\lim_{n \to \infty} \delta_n = 0$, from (2.11) and (2.12), we conclude that

$$\lim_{k \to \infty} \max\{d(a_{m(k)}, a_{n(k)}), d(b_{m(k)}, b_{n(k)})\} = \varepsilon. \tag{2.16}$$

Since

$$\begin{aligned}d(a_{n(k)+1}, a_{m(k)+1}) &\leq d(a_{n(k)+1}, a_{n(k)}) + d(a_{n(k)}, a_{m(k)}) + d(a_{m(k)}, a_{m(k)+1}), \\ d(b_{n(k)+1}, b_{m(k)+1}) &\leq d(b_{n(k)+1}, b_{n(k)}) + d(b_{n(k)}, b_{m(k)}) + d(a_{m(k)}, a_{m(k)+1}),\end{aligned} \tag{2.17}$$

we obtain that

$$\begin{aligned}\max\{d(a_{n(k)+1}, a_{m(k)+1}), d(b_{n(k)+1}, b_{m(k)+1})\} \leq\ &\max\{d(a_{n(k)+1}, a_{n(k)}), d(b_{n(k)+1}, b_{n(k)})\}\\ &+\max\{d(a_{n(k)}, a_{m(k)}), d(b_{n(k)}, b_{m(k)})\}\\ &+\max\{d(a_{m(k)}, a_{m(k)+1}), d(b_{m(k)}, b_{m(k)+1})\}.\end{aligned} \tag{2.18}$$

If in the above inequality, $k \to \infty$, as $\lim_{n\to\infty}\delta_n = 0$, from (2.16) we have

$$\lim_{k\to\infty} \max\{d(a_{n(k)+1}, a_{m(k)+1}), d(b_{n(k)+1}, b_{m(k)+1})\} \leq \varepsilon. \tag{2.19}$$

Again, since

$$\begin{aligned}d(a_{n(k)}, a_{m(k)}) &\leq d(a_{n(k)}, a_{n(k)+1}) + d(a_{n(k)+1}, a_{m(k)+1}) + d(a_{m(k)+1}, a_{m(k)}),\\ d(b_{n(k)}, b_{m(k)}) &\leq d(b_{n(k)}, b_{n(k)+1}) + d(b_{n(k)+1}, a_{m(k)+1}) + d(b_{m(k)+1}, b_{m(k)}),\end{aligned} \tag{2.20}$$

we have

$$\begin{aligned}\max\{d(a_{n(k)}, a_{m(k)}), d(b_{n(k)}, b_{m(k)})\} \leq &\max\{d(a_{n(k)}, a_{n(k)+1}), d(b_{n(k)}, b_{n(k)+1})\}\\ &+\max\{d(a_{n(k)+1}, a_{m(k)+1}), d(b_{n(k)+1}, b_{m(k)+1})\}\\ &+\max\{d(a_{m(k)+1}, a_{m(k)}), d(b_{m(k)+1}, b_{m(k)})\}.\end{aligned} \tag{2.21}$$

Letting $k \to \infty$, we have

$$\varepsilon \leq \lim_{k\to\infty} \max\{d(a_{n(k)+1}, a_{m(k)+1}), d(b_{n(k)+1}, b_{m(k)+1})\}. \tag{2.22}$$

Now, from (2.19) and (2.22), we have

$$\lim_{k\to\infty} \max\{d(a_{n(k)+1}, a_{m(k)+1}), d(b_{n(k)+1}, b_{m(k)+1})\} = \varepsilon. \tag{2.23}$$

As $n(k) > m(k)$, we have $gx_{m(k)} \preceq gx_{n(k)}$ and $gy_{m(k)} \succeq gy_{n(k)}$. Putting $x = x_{m(k)}$, $y = y_{m(k)}$, $u = x_{n(k)}$, and $v = y_{n(k)}$ in (2.1), we have

$$\begin{aligned}\psi(d(a_{m(k)+1}, a_{n(k)+1})) &= \psi(d(F(x_{m(k)}, y_{m(k)}), F(x_{n(k)}, y_{n(k)})))\\ &\leq \alpha(\max\{d(a_{m(k)}, a_{n(k)}), d(b_{m(k)}, b_{n(k)})\})\\ &\quad - \beta(\max\{d(a_{m(k)}, a_{n(k)}), d(b_{m(k)}, b_{n(k)})\}).\end{aligned} \tag{2.24}$$

Also, we have

$$\begin{aligned}\psi(d(b_{n(k)+1}, b_{m(k)+1})) &= \psi(d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))) \\ &\leq \alpha(\max\{d(b_{n(k)}, b_{m(k)}), d(a_{n(k)}, a_{m(k)})\}) \\ &\quad - \beta(\max\{d(b_{n(k)}, b_{m(k)}), d(a_{n(k)}, a_{m(k)})\}).\end{aligned} \tag{2.25}$$

Therefore,

$$\begin{aligned}&\psi(\max\{d(a_{m(k)+1}, a_{n(k)+1}), d(b_{m(k)+1}, b_{n(k)+1})\}) \\ &\quad\leq \alpha(\max\{d(a_{m(k)}, a_{n(k)}), d(b_{m(k)}, b_{n(k)})\}) \\ &\qquad - \beta(\max\{d(a_{m(k)}, a_{n(k)}), d(b_{m(k)}, b_{n(k)})\}).\end{aligned} \tag{2.26}$$

Then, in (2.26), if $k \to \infty$, from (2.16) and (2.23), we have $\psi(\varepsilon) \leq \alpha(\varepsilon) - \beta(\varepsilon)$. Thus, $\psi(\varepsilon) - \alpha(\varepsilon) + \beta(\varepsilon) \leq 0$, and hence $\varepsilon = 0$, which is a contradiction. Consequently, $\{a_n\}$ and $\{b_n\}$ are Cauchy.

Completeness of (X, d) implies that $\{a_n\}$ and $\{b_n\}$ converge to some $x, y \in X$, respectively.

Step II. We will show that F and g have a coupled coincidence point.

From the above step, we have

$$\begin{aligned}\lim_{n\to\infty} F(x_n, y_n) &= \lim_{n\to\infty} gx_n = \lim_{n\to\infty} a_n = x, \\ \lim_{n\to\infty} F(y_n, x_n) &= \lim_{n\to\infty} gy_n = \lim_{n\to\infty} b_n = y.\end{aligned} \tag{2.27}$$

Since g is continuous, by (2.27), we have

$$\lim_{n\to\infty} g(gx_n) = gx, \qquad \lim_{n\to\infty} g(gy_n) = gy. \tag{2.28}$$

Commutativity of F and g yields that

$$\begin{aligned}g(gx_{n+1}) &= g(F(x_n, y_n)) = F(gx_n, gy_n), \\ g(gy_{n+1}) &= g(F(y_n, x_n)) = F(gy_n, gx_n).\end{aligned} \tag{2.29}$$

From the continuity of F, $\{g(gx_{n+1})\}$ is convergent to $F(x, y)$ and $\{g(gy_{n+1})\}$ convergent to $F(y, x)$. From (2.28) and by uniqueness of the limit, we have $F(x, y) = gx$ and $F(y, x) = gy$, that is, g and F have a coupled coincidence point.

This completes the proof of the theorem. □

In the following theorem we omit the continuity assumption of F and g.

Theorem 2.3. *Let $(X, \preceq, d)$ be a partially ordered complete metric space and let $F : X^2 \to X$ and $g : X \to X$ be such that $F(X^2) \subseteq g(X)$. Assume that F and g satisfy* (2.1) *for every $x, y, u, v \in X$*

with $gx \preceq gu$ and $gy \succeq gv$, where $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ are such that, ψ is an altering distance function, α is continuous, β is lower semicontinuous, $\alpha(0) = \beta(0) = 0$, and $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$.

Assume that

(1) *F has the mixed g-monotone property,*

(2) *$g(X)$ is a closed subset of X.*

Also, suppose that

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$, for all $n \in \mathbb{N}$;*

(ii) *if a nonincreasing sequence $y_n \to y$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point in X.

Proof. Following the proof of the previous theorem, since $g(X)$ is closed and $\{a_n\} = \{gx_n\} \subseteq g(X)$, there exists $u \in X$ such that

$$\lim_{n \to \infty} gx_n = gu = x. \tag{2.30}$$

Similarly, there exists $v \in X$ such that

$$\lim_{n \to \infty} gy_n = gv = y. \tag{2.31}$$

From (i) and (ii), we have $gx_n \preceq gu$ and $gy_n \succeq gv$.

Now, we prove that $F(u, v) = gu$ and $F(v, u) = gv$. Using (2.1), we have

$$\begin{aligned} \psi(d(gx_{n+1}, F(u, v))) &= \psi(d(F(x_n, y_n), F(u, v))) \\ &\leq \alpha(\max\{d(gx_n, gu), d(gy_n, gv)\}) \\ &\quad - \beta(\max\{d(gx_n, gu), d(gy_n, gv)\}). \end{aligned} \tag{2.32}$$

In the above inequality, if $n \to \infty$, from properties of ψ, α, and β,

$$\begin{aligned} \psi(d(gu, F(u, v))) &\leq \alpha(\max\{d(gu, gu), d(gv, gv)\}) \\ &\quad - \beta(\max\{d(gu, gu), d(gv, gv)\}) \\ &= \alpha(0) - \beta(0) = 0. \end{aligned} \tag{2.33}$$

Hence, $d(gu, F(u, v)) = 0$, that is, $gu = F(u, v)$. Analogously, we can show that $gv = F(v, u)$. □

Theorem 2.4. *Under the hypotheses of Theorem 2.3, suppose that $gy_0 \preceq gx_0$. Then, it follows that $gu = F(u, v) = F(v, u) = gv$. Moreover, if F and g be w-compatible, then F and g have a coupled coincidence point of the form (t, t).*

Proof. If $gy_0 \preceq gx_0$, then $gv \preceq gy_n \preceq gy_0 \preceq gx_0 \preceq gx_n \preceq gu$ for all $n \in \mathbb{N}$. Thus, if $gu \neq gv$, by inequality (2.1), we have

$$\begin{aligned}\psi(d(gv, gu)) &= \psi(d(F(v, u), F(u, v))) \\ &\leq \alpha(\max\{d(gv, gu), d(gu, gv)\}) \\ &\quad - \beta(\max\{d(gv, gu), d(gu, gv)\}) \\ &= \psi(d(gu, gv)) - \beta(d(gu, gv)).\end{aligned} \tag{2.34}$$

Thus, from properties of functions ψ, α, β we obtain $d(gu, gv) = 0$, a contradiction. Hence, $gu = gv$, that is, $gu = F(u, v) = F(v, u) = gv$. Now, let $t = gu = gv$. Since F and g are w-compatible, then $gt = g(gu) = g(F(u, v)) = F(gu, gv) = F(t, t)$. Thus, F and g have a coupled coincidence point of the form (t, t). □

Remark 2.5. In Theorems 2.2 and 2.3, we extend the results of Harjani et al. (Theorems 1.10 and 1.11), if we take $\alpha(t) = \psi(t)$, for all $t \in [0, \infty)$ and $g(x) = I_X$ (the identity mapping on X).

The following theorem can be deduced from our previous obtained results.

Theorem 2.6. *Let $(X, \preceq, d)$ be a partially ordered complete metric space and let $F : X^2 \to X$ be a mapping having the mixed monotone property. Assume that*

$$\psi(d(F(x, y), F(u, v))) \leq \frac{d(x, u) + d(y, v)}{2} - \beta(\max\{d(x, u), d(y, v)\}), \tag{2.35}$$

for every $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$, where $\psi, \beta : [0, \infty) \to [0, \infty)$ are such that ψ is an altering distance function, β is lower semicontinuous, $\beta(0) = 0$, and $\psi(t) - t + \beta(t) > 0$ for all $t > 0$.

Also, suppose that

(a) *F is continuous, or,*

(b) *X has the following properties:*

  (i) *if a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$, for all n;*

  (ii) *if a nonincreasing sequence $y_n \to y$, then $y_n \succeq y$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point in X.

Proof. If F satisfies (2.35), then F satisfies (2.1) with $g(x) = I_X$ (the identity mapping on X) and $\alpha(t) = t$, for all $t \in [0, \infty)$. Then, the result follows from Theorems 2.2 and 2.3. □

Note that if $(X, \preceq)$ is a partially ordered set, then we can endow $X \times X$ with the following partial order relation:

$$(x, y) \preceq (u, v) \Longleftrightarrow x \preceq u, \quad y \succeq v, \tag{2.36}$$

for all $(x, y), (u, v) \in X \times X$ (see [3]).

In the following theorem, we give a sufficient condition for the uniqueness of the common coupled fixed point. A similar proof can be found in Theorem 2.2 of [3], Theorem 2.4 of [12], and Theorem 2.3 of [13].

Theorem 2.7. *In addition to the hypotheses of Theorem 2.2 suppose that for every (x, y) and $(x^*, y^*) \in X \times X$, there exists $(u, v) \in X^2$, such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then, F and g have a unique common coupled fixed point.*

Proof. From Theorem 2.2 the set of coupled coincidence points of F and g is nonempty. We will show that if (x, y) and (x^*, y^*) are coupled coincidence points, that is,

$$\begin{aligned} g(x) &= F(x, y), \qquad g(y) = F(y, x), \\ g(x^*) &= F(x^*, y^*), \qquad g(y^*) = F(y^*, x^*), \end{aligned} \tag{2.37}$$

then, $gx = gx^*$ and $gy = gy^*$.

Choose an element $(u, v) \in X^2$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$.

Let $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$. Then, similarly as in the proof of Theorem 2.2, we can inductively define sequences $\{gu_n\}$ and $\{gv_n\}$ such that $gu_{n+1} = F(u_n, v_n)$ and $gv_{n+1} = F(v_n, u_n)$. Since $(gx, gy) = (F(x, y), F(y, x))$ and $(F(u, v), F(v, u)) = (gu_1, gv_1)$ are comparable, we may assume that $(gx, gy) \preceq (gu_1, gv_1)$. Then, $gx \preceq gu_1$ and $gy \succeq gv_1$. Using the mathematical induction, it is easy to prove that $gx \preceq gu_n$ and $gy \succeq gv_n$, for all $n \in \mathbb{N}$.

Let $\gamma_n = \max\{d(gx, gu_n), d(gy, gv_n)\}$. We will show that $\lim_{n\to\infty}\gamma_n = 0$. First, assume that $\gamma_n = 0$, for an $n \geq 1$.

Applying (2.1), as $gx \preceq gu_n$ and $gy \succeq gv_n$ one obtains that

$$\begin{aligned} \psi(d(gx, gu_{n+1})) &= \psi(d(F(x, y), F(u_n, v_n))) \\ &\leq \alpha(\max\{d(gx, gu_n), d(gy, gv_n)\}) \\ &\quad - \beta(\max\{d(gx, gu_n), d(gy, gv_n)\}). \end{aligned} \tag{2.38}$$

Similarly, we have

$$\begin{aligned} \psi(d(gy, gv_{n+1})) &= \psi(d(F(y, x), F(v_n, u_n))) \\ &\leq \alpha(\max\{d(gy, gv_n), d(gx, gu_n)\}) \\ &\quad - \beta(\max\{d(gy, gv_n), d(gx, gu_n)\}). \end{aligned} \tag{2.39}$$

From (2.38) and (2.39), we have

$$\begin{aligned}\psi(\gamma_{n+1}) &= \psi(\max\{d(gu_{n+1}, gx), d(gv_{n+1}, gy)\}) \\ &= \max\{\psi(d(gu_{n+1}, gx)), \psi(d(gv_{n+1}, gy))\} \\ &\leq \alpha(\max\{d(gx, gu_n), d(gy, gv_n)\}) \\ &\quad - \beta(\max\{d(gx, gu_n), d(gy, gv_n)\}) \\ &= \alpha(\gamma_n) - \beta(\gamma_n) \\ &= \alpha(0) - \beta(0) = 0.\end{aligned} \tag{2.40}$$

So, from properties of ψ, α, and β, we deduce $\gamma_{n+1} = 0$. Repeating this process, we can show that $\gamma_m = 0$, for all $m \geq n$. So, $\lim_{n\to\infty}\gamma_n = 0$.

Now, let $\gamma_n \neq 0$, for all n and let $\gamma_n < \gamma_{n+1}$, for some n.

As ψ is an altering distance function, from (2.40)

$$\begin{aligned}\psi(\gamma_n) &= \psi(\max\{d(gu_n, gx), d(gv_n, gy)\}) \\ &< \psi(\gamma_{n+1}) \\ &= \psi(\max\{d(gu_{n+1}, gx), d(gv_{n+1}, gy)\}) \\ &= \max\{\psi(d(gu_{n+1}, gx)), \psi(d(gv_{n+1}, gy))\} \\ &\leq \alpha(\max\{d(gx, gu_n), d(gy, gv_n)\}) \\ &\quad - \beta(\max\{d(gx, gu_n), d(gy, gv_n)\}) \\ &= \alpha(\gamma_n) - \beta(\gamma_n).\end{aligned} \tag{2.41}$$

This implies that $\gamma_n = 0$, which is a contradiction.

Hence, $\gamma_{n+1} \leq \gamma_n$, for all $n \geq 1$. Now, if we proceed as in Theorem 2.2, we can show that

$$\lim_{n\to\infty} \max\{d(gu_n, gx), d(gv_n, gy)\} = 0. \tag{2.42}$$

So, $\{gu_n\} \to gx$ and $\{gv_n\} \to gy$.

Similarly, we can show that

$$\lim_{n\to\infty} \max\{d(gu_n, gx^*), d(gv_n, gy^*)\} = 0, \tag{2.43}$$

that is, $\{gu_n\} \to gx^*$ and $\{gv_n\} \to gy^*$. Finally, since the limit is unique, $gx = gx^*$ and $gy = gy^*$.

Since $gx = F(x, y)$ and $gy = F(y, x)$, by commutativity of F and g, we have $g(gx) = g(F(x, y)) = F(gx, gy)$ and $g(gy) = g(F(y, x)) = F(gy, gx)$. Let $gx = a$ and $gy = b$. Then, $ga = F(a, b)$ and $gb = F(b, a)$. Thus, (a, b) is another coupled coincidence point of F and g. Then, $a = gx = ga$ and $b = gy = gb$. Therefore, (a, b) is a coupled common fixed point of F and g.

To prove the uniqueness of coupled common fixed point, assume that (p,q) is another coupled common fixed point of F and g. Then, $p = gp = F(p,q)$ and $q = gq = F(q,p)$. Since (p,q) is a coupled coincidence point of F and g, we have $gp = ga$ and $gq = gb$. Thus, $p = gp = ga = a$ and $q = gq = gb = b$. Hence, the coupled common fixed point is unique. □

Theorem 2.8. *Under the hypotheses of Theorem 2.3, suppose in addition that for every (x,y) and (x^*,y^*) in X^2, there exists $(u,v) \in X^2$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. If F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (t,t).*

Proof. By Theorem 2.3, the set of coupled coincidence points of F and g is nonempty. Let (x,y) and (x^*,y^*) be coupled coincidence points of F and g. Following the proof of Theorem 2.7, we can prove that $gx = gx^*$ and $gy = gy^*$. Note that if (x,y) is a coupled coicidence point of F and g, then (y,x) is also a coupled coincidence point of F and g. Thus, we have $gx = gy$. Put $t = gx = gy$. Since $gx = F(x,y)$ and $gy = F(y,x)$ and F and g are w-compatible, we have $gt = g(gx) = g(F(x,y)) = F(gx,gy) = F(t,t)$. Thus, (t,t) is a coupled coincidence point of F and g. So, $gt = gx = gy = t$ and hence we have $t = gt = F(t,t)$. Therefore, (t,t) is a common coupled fixed point of F and g.

To prove the uniqueness of the coupled common fixed point of F and g, let (v,w) be another coupled fixed point of F and g, that is, $v = gv = F(v,w)$ and $w = gw = F(w,v)$. Clearly, we have $gt = gv$ and $gt = gw$. Therefore, $t = v = w$. Thus, F and g have a unique common coupled fixed point of the form (t,t). □

Remark 2.9. Note that Theorems 2.4 and 2.8 have been established and proved according to Theorems 2.3 and 2.5 of [12].

The following simple example guarantees that our results are proper generalizations of the results of Harjani et al. (Theorems 1.10 and 1.11).

Example 2.10. Let $X = [0,+\infty)$. We define a partial order "$\preceq$" on X as $x \preceq y$ if and only if $x \leq y$ for all $x,y \in X$. Let a metric d on X be defined by $d(x,y) = 0$, if and only if $x = y$, and $d(x,y) = x + y$, if $x \neq y$. Then (X,d) is a complete metric space.

Define $F : X \times X \to X$ as follows:

$$F(x,y) = \left|\frac{x}{4} - \frac{y}{4}\right|, \tag{2.44}$$

for all $x,y \in X$ and $g : X \to X$ with $g(x) = x$ for all $x \in X$.

Let $\psi,\alpha,\beta : [0,\infty) \to [0,\infty)$ be defined by $\psi(t) = 4t$, $\alpha(t) = 7t$, and $\beta(t) = (7/2)t$. Clearly, ψ is an altering distance function, α is continuous, β is lower semicontinuous, $\alpha(0) = \beta(0) = 0$, and $\psi(t) - \alpha(t) + \beta(t) = t/2 > 0$ for all $t > 0$.

Now, let $x \preceq u$ and $y \succeq v$. So, we have

$$\begin{aligned} \psi(d(F(x,y),F(u,v))) &= 4\left(\left|\frac{x}{4} - \frac{y}{4}\right| + \left|\frac{u}{4} - \frac{v}{4}\right|\right) \\ &\leq (x+u) + (y+v) \\ &\leq 2\max\{(x+u),(y+v)\}. \end{aligned} \tag{2.45}$$

Hence,

$$\begin{aligned}\psi(d(F(x,y),F(u,v))) &\leq 2\max\{d(gx,gu),d(gy,gv)\}\\ &\leq 7\max\{d(gx,gu),d(gy,gv)\} - \frac{7}{2}\max\{d(gx,gu),d(gy,gv)\}\\ &= \alpha(\max\{d(gx,gu),d(gy,gv)\}) - \beta(\max\{d(gx,gu),d(gy,gv)\}).\end{aligned} \tag{2.46}$$

Therefore, all of the conditions of Theorem 2.2 are satisfied. Moreover, $(0,0)$ is the unique coupled coincidence point of F and g.

However, inequality (1.4) in Theorem 1.10 is not satisfied. Indeed, let $(x,y) = (0,1)$ and $(u,v) = (0,0)$. Then,

$$\begin{aligned}\psi(d(F(x,y),F(u,v))) &= 1\\ &\not\leq \psi(\max\{d(gx,gu),d(gy,gv)\}) - \beta(\max\{d(gx,gu),d(gy,gv)\})\\ &= 4 - \frac{7}{2} = \frac{1}{2}.\end{aligned} \tag{2.47}$$

Example 2.11. Let $X = [0,\infty)$ be endowed with the euclidian metric and the usual ordering.

Define $F : X \times X \to X$ as follows:

$$F(x,y) = \begin{cases} \dfrac{x-y}{4}, & \text{if } x \geq y\\ 0, & \text{if } y > x, \end{cases} \tag{2.48}$$

for all $x, y \in X$ and $g : X \to X$ with $g(x) = x$ for all $x \in X$.

Let $\psi : [0,\infty) \to [0,\infty)$ be the identity mapping and $\alpha, \beta : [0,\infty) \to [0,\infty)$ be defined by $\alpha(t) = 2t$ and $\beta(t) = (3/2)t$.

Let $x, y, u, v \in X$ are such that $x \leq u$ and $y \geq v$. Now, we have

Case 1 ($y > x$ and $v > u$). Then,

$$\begin{aligned}\psi(d(F(x,y),F(u,v))) = 0 \leq \alpha(\max\{d(gx,gu),d(gy,gv)\})\\ - \beta(\max\{d(gx,gu),d(gy,gv)\}).\end{aligned} \tag{2.49}$$

Case 2 ($y > x$ and $u \geq v$).

$$\begin{aligned}
\psi(d(F(x,y),F(u,v))) &= \frac{1}{4}(u-v) \\
&\leq \frac{1}{2}\left(\frac{1}{2}(u-x+y-v)\right) \\
&= \frac{1}{2}\left(\frac{|x-u|+|y-v|}{2}\right) \\
&\leq \frac{1}{2}\max\{|x-u|,|y-v|\} \\
&= 2\max\{|x-u|,|y-v|\} - \frac{3}{2}\max\{|x-u|,|y-v|\} \\
&= \alpha(\max\{d(gx,gu),d(gy,gv)\}) - \beta(\max\{d(gx,gu),d(gy,gv)\}).
\end{aligned} \tag{2.50}$$

Case 3 ($x \geq y$ and $u \geq v$). As $x \leq u$ and $y \geq v$, we have $u \geq x \geq y \geq v$. Hence,

$$\begin{aligned}
\psi(d(F(x,y),F(u,v))) &= \left|\frac{x}{4} - \frac{y}{4} - \left(\frac{u}{4} - \frac{v}{4}\right)\right| \\
&\leq \frac{1}{2}\left[\frac{|x-u|+|y-v|}{2}\right] \\
&\leq \frac{1}{2}\max\{|x-u|,|y-v|\} \\
&= 2\max\{|x-u|,|y-v|\} - \frac{3}{2}\max\{|x-u|,|y-v|\} \\
&= \alpha(\max\{d(gx,gu),d(gy,gv)\}) - \beta(\max\{d(gx,gu),d(gy,gv)\}).
\end{aligned} \tag{2.51}$$

Case 4 ($x \geq y$ and $v > u$). As $x \leq u$, and $y \geq v$, we have $x = y = u = v$. Hence,

$$\begin{aligned}
\psi(d(F(x,y),F(u,v))) &= 0 = \alpha(0) - \beta(0) \\
&= \alpha(\max\{d(gx,gu),d(gy,gv)\}) \\
&\quad - \beta(\max\{d(gx,gu),d(gy,gv)\}).
\end{aligned} \tag{2.52}$$

Hence, all of the conditions of Theorem 2.2 are satisfied. Moreover, $(0,0)$ is the coupled coincidence point of F and g.

In what follows, we obtain some coupled coincidence point theorems for mappings satisfying some contraction conditions of integral type in an ordered complete metric space.

In [18], Branciari obtained a fixed point result for a single mapping satisfying an integral type inequality. Then, Altun et al. [19] established a fixed point theorem for weakly compatible maps satisfying a general contractive inequality of integral type.

Denote by Λ the set of all functions $\mu : [0, +\infty) \to [0, +\infty)$ verifying the following conditions:

(I) μ is a positive Lebesgue integrable mapping on each compact subset of $[0, +\infty)$,

(II) for all $\varepsilon > 0$, $\int_0^{\varepsilon} \mu(t)dt > 0$.

Corollary 2.12. *Replace the contractive condition* (2.1) *of Theorem 2.2 by the following condition. There exists a* $\mu \in \Lambda$ *such that*

$$\int_0^{\psi(d(F(x,y),F(u,v)))} \mu(t)dt \le \int_0^{\alpha(\max\{d(gx,gu),d(gy,gv)\})} \mu(t)dt - \int_0^{\beta(\max\{d(gx,gu),d(gy,gv)\})} \mu(t)dt. \tag{2.53}$$

If other conditions of Theorem 2.2 hold, then F and g have a coupled coincidence point.

Proof. Consider the function $\Gamma(x) = \int_0^x \mu(t)dt$. Then (2.53) becomes

$$\Gamma(\psi(d(F(x,y),F(u,v)))) \le \Gamma(\alpha(\max\{d(gx,gu),d(gy,gv)\})) - \Gamma(\beta(\max\{d(gx,gu),d(gy,gv)\})). \tag{2.54}$$

Taking $\psi_1 = \Gamma o \psi$, $\alpha_1 = \Gamma o \alpha$ and $\beta_1 = \Gamma o \beta$ and applying Theorem 2.2, we obtain the proof. □

Corollary 2.13. *Substitute the contractive condition* (2.1) *of Theorem 2.2 by the following condition. There exists a* $\mu \in \Lambda$ *such that*

$$\psi\left(\int_0^{d(F(x,y),F(u,v))} \mu(t)dt\right) \le \alpha\left(\int_0^{\max\{d(gx,gu),d(gy,gv)\}} \mu(t)dt\right) - \beta\left(\int_0^{\max\{d(gx,gu),d(gy,gv)\}} \mu(t)dt\right). \tag{2.55}$$

Then F and g have a coupled coincidence point, if other conditions of Theorem 2.2 hold.

Proof. Again, as in Corollary 2.12, define the function $\Gamma(x) = \int_0^x \phi(t)dt$. Then (2.55) changes to

$$\psi(\Gamma(d(F(x,y),F(u,v)))) \le \alpha(\Gamma(\max\{d(gx,gu),d(gy,gv)\})) - \beta(\Gamma(\max\{d(gx,gu),d(gy,gv)\})). \tag{2.56}$$

Now, if we define $\psi_1 = \psi o \Gamma$, $\alpha_1 = \alpha o \Gamma$ and $\beta_1 = \varphi o \Gamma$, and applying Theorem 2.2, then the proof is obtained. □

As in [20], let $n \in \mathbb{N}^*$ be fixed. Let $\{\mu_i\}_{1\leq i\leq N}$ be a family of N functions which belong to Λ. For all $t \geq 0$, we define

$$\begin{aligned}
I_1(t) &= \int_0^t \mu_1(s)ds, \\
I_2(t) &= \int_0^{I_1 t} \mu_2(s)ds = \int_0^{\int_0^t \mu_1(s)ds} \mu_2(s)ds, \\
I_3(t) &= \int_0^{I_2 t} \mu_3(s)ds = \int_0^{\int_0^{\int_0^t \mu_1(s)ds} \mu_2(s)ds} \mu_3(s)ds, \\
&\vdots \\
I_N(t) &= \int_0^{I_{(N-1)} t} \mu_N(s)ds.
\end{aligned} \tag{2.57}$$

We have the following result.

Corollary 2.14. *Replace the inequality* (2.1) *of Theorem 2.2 by the following condition:*

$$\begin{aligned}
\psi\left(\int_0^{I_{(N-1)}(d(F(x,y),F(u,v)))} \mu_N(s)ds\right) &\leq \alpha\left(\int_0^{I_{(N-1)}(\max\{d(gx,gu),d(gy,gv)\})} \mu_N(s)ds\right) \\
&\quad - \beta\left(\int_0^{I_{(N-1)}(\max\{d(gx,gu),d(gy,gv)\})} \mu_N(s)ds\right).
\end{aligned} \tag{2.58}$$

Assume further that all other conditions of Theorem 2.2 are also satisfied, then F and g have a coupled coincidence point.

Proof. Consider $\widehat{\varphi} = \psi o I_N$, $\widehat{\alpha} = \alpha o I_N$, and $\widehat{\beta} = \beta o I_N$. Then the above inequality becomes

$$\begin{aligned}
\widehat{\varphi}(d(F(x,y),F(u,v))) &\leq \widehat{\alpha}(\max\{d(gx,gu),d(gy,gv)\}) \\
&\quad - \widehat{\beta}(\max\{d(gx,gu),d(gy,gv)\}).
\end{aligned} \tag{2.59}$$

Applying Theorem 2.2, we obtain the desired result. □

Other consequence of our theorems is the following result.

Corollary 2.15. *Replace the contractive condition* (2.1) *of Theorem 2.2 by the following condition.*

There exist $\mu_1, \mu_2, \mu_3 \in \Lambda$ *such that*

$$\begin{aligned}\int_0^{d(F(x,y),F(u,v))} \mu_1(t)dt \le \int_0^{\max\{d(gx,gu),d(gy,gv)\}} \mu_2(t)dt \\ - \int_0^{\max\{d(gx,gu),d(gy,gv)\}} \mu_3(t)dt.\end{aligned} \tag{2.60}$$

Let other conditions of Theorem 2.2 *are satisfied, then* F *and* g *have a coupled coincidence point.*

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