Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 407351, 8 pages
doi:10.1155/2012/407351

*Research Article*

# Uniqueness Theorems on Difference Monomials of Entire Functions

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Received 5 April 2012; Accepted 11 June 2012

Academic Editor: Matti Vuorinen

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The aim of this paper is to discuss the uniqueness of the difference monomials $f^n f(z+c)$. It assumed that $f$ and $g$ are transcendental entire functions with finite order and $E_{k)}(1, f^n f(z+c)) = E_{k)}(1, g^n g(z+c))$, where $c$ is a nonzero complex constant and $n$, $k$ are integers. It is proved that if one of the following holds (i) $n \geq 6$ and $k = 3$, (ii) $n \geq 7$ and $k = 2$, and (iii) $n \geq 10$ and $k = 1$, then $fg = t_1$ or $f = t_2 g$ for some constants $t_2$ and $t_3$ which satisfy $t_2^{n+1} = 1$ and $t_3^{n+1} = 1$. It is an improvement of the result of Qi, Yang and Liu.

## 1. Introduction and Main Results

In this paper, a meromorphic (respectively entire) function always means meromorphic (respectively, analytic) in the complex plane $\mathbb{C}$. It is also assumed that the reader is familiar with the basic concepts of the Nevanlinna theory. We adopt the standard notations in the Nevanlinna value distribution theory of meromorphic functions as explained in [1, 2].

Let $f$ and $g$ be two nonconstant meromorphic functions, and let $a$ be a value in the extended plane. We say that $f$ and $g$ share the value $a$ CM, provided that $f$ and $g$ have the same $a$-pints with the same multiplicities. We say that $f$ and $g$ share the value $a$ IM, provided that $f$ and $g$ have the same $a$-points ignoring multiplicities. The order of $f$ is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}. \tag{1.1}$$

Let $f$ be a nonconstant meromorphic function on $\mathbb{C}$, let $a \in \mathbb{C}$ be a finite value, and let $k$ be a positive integer or infinity. We denote by $E(a, f)$ the set of zeros of $f - a$ and count multiplicities, while by $\overline{E}(a, f)$ the set of zeros of $f - a$ but ignore multiplicities. Also, we denote by $E_{k)}(a, f)$ the set of zeros of $f - a$ with multiplicities less than or equal to $k$ and count multiplicities. For $a \in \mathbb{C}\bigcup\{\infty\}$, we denote by $N_{k)}(r, 1/(f - a))$ the counting function corresponding to the set $E_{k)}(a, f)$ while by $N_{(k+1}(r, 1/(f - a))$ the counting function corresponding to the set $E_{(k+1}(a, f) := E(a, f) \setminus E_{k)}(a, f)$. If $E_k(a, f) = E_k(a, g)$, we say that $f$, $g$ share the value a with weight $k$.

The definition implies that if $f$ and $g$ share a value $a$ with weight $k$, then $z_0$ is a zero of $f - a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\le k)$ and $z_0$ is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where $m$ is not necessarily equal to $n$.

Also, we denote by $\overline{N}_{k)}(r, 1/(f - a))$ and $\overline{N}_{(k+1}(r, 1/(f - a))$ the reduced forms of $N_{k)}(r, 1/(f - a))$ and $N_{(k+1}(r, 1/(f - a))$, respectively. At last, we set

$$N_k(r, f) = \overline{N}(r, f) + \overline{N}_{(2}(r, f) + \cdots + \overline{N}_{(k}(r, f). \tag{1.2}$$

Hayman proposed the well-known conjecture in [3].

*Hayman Conjecture*

If an entire function $f$ satisfies $f^n f' \neq 1$ for all $n \in \mathbb{N}$, then $f$ is a constant.

In fact, Hayman has proved that the conjecture holds in the cases $n \ge 2$ in [4] while Clunie proved the cases $n = 1$ in [5], respectively. In 1997, Yang and Hua [6] studied the uniqueness theorem of the differential monomials and obtained the following result.

**Theorem A.** *Let $f$ and $g$ be nonconstant entire function, and let $n \ge 6$ be an integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$ where $c_1$, $c_2$ and $c$ are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant $t$ such that $t^{n+1} = 1$.*

In 2010, Qi et al. [7] studied the uniqueness of the difference monomials and obtained the following result.

**Theorem B.** *Let $f$ and $g$ be transcendental entire functions with finite order, $c$ a non-zero complex constant, and $n \ge 6$ an integer. If $E(1, f^n f(z + c)) = E(1, g^n g(z + c))$, then $fg = t_1$ or $f = t_2 g$ for some constants $t_1$ and $t_2$ which satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In this paper, we will obtain the following results.

**Theorem 1.1.** *Let $f$ and $g$ be transcendental entire functions with finite order, $c$ a non-zero complex constant, and $n \ge 6$ an integer. If $E_{3)}(1, f^n f(z + c)) = E_{3)}(1, g^n g(z + c))$, then the assertion of Theorem B holds.*

**Theorem 1.2.** *Let $f$ and $g$ be transcendental entire functions with finite order, $c$ a non-zero complex constant, and $n \ge 7$ an integer. If $E_{2)}(1, f^n f(z + c)) = E_{2)}(1, g^n g(z + c))$, then the assertion of Theorem B holds.*

**Theorem 1.3.** *Let $f$ and $g$ be transcendental entire functions with finite order, $c$ a non-zero complex constant, and $n \geq 10$ an integer. If $E_{1)}(1, f^n f(z+c)) = E_{1)}(1, g^n g(z+c))$, then the assertion of Theorem B holds.*

## 2. Auxiliary Results

**Lemma 2.1** (see [8, Corollary 2.5]). *Let $f(z)$ be a meromorphic function in the complex plane with finite order $\sigma = \sigma(f)$, and let $\eta$ be a fixed non-zero complex number. Then for each $\varepsilon > 0$, one has*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O\left(r^{\sigma-1+\varepsilon}\right). \tag{2.1}$$

**Lemma 2.2** (see [8, Theorem 2.1]). *Let $f(z)$ be a meromorphic function in the complex plane with finite order $\sigma = \sigma(f)$, and let $\eta$ be a fixed non-zero complex number. Then for each $\varepsilon > 0$, one has*

$$T(r, f(z+\eta)) = T(r, f(z)) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r). \tag{2.2}$$

**Lemma 2.3.** *Let $f(z)$ be an entire function with finite order $\sigma = \sigma(f)$, $c$ a fixed non-zero complex number, and*

$$P(z) = a_n f(z)^n + a_{n-1} f(z)^{n-1} + \cdots + a_1 f(z) + a_0, \tag{2.3}$$

*where $a_j (j = 0, 1, \ldots, n)$ are constants. If $F(z) = P(z) f(z+c)$, then*

$$T(r, F) = (n+1) T(r, f) + O\left(r^{\sigma(f)-1+\varepsilon}\right) + O(\log r). \tag{2.4}$$

*Proof.* Since $f(z)$ is an entire transcendental function with finite order, we can deduce from Lemma 2.1 and the standard Valiron-Mohon'ko theorem that

$$\begin{aligned}
(n+1) T(r, f(z)) &= T(r, f(z) P(z)) + O(1) \\
&= m(r, f(z) P(z)) + O(1) \\
&\leq m\left(r, \frac{f(z) P(z)}{F(z)}\right) + m(r, F(z)) + O(1) \\
&= m\left(r, \frac{f(z)}{f(z+c)}\right) + m(r, F(z)) + O(1) \\
&\leq T(r, F(z)) + O\left(r^{\sigma-1+\varepsilon}\right) + O(1).
\end{aligned} \tag{2.5}$$

Therefore

$$T(r, F) \geq (n+1) T(r, f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(1). \tag{2.6}$$

On the other hand, Lemma 2.2 implies that

$$\begin{aligned}
T(r,F(z)) &\leq T(r,P(z)) + T(r,f(z+c)) \\
&= nT(r,f(z)) + T(r,f(z)) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r) \\
&= (n+1)T(r,f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r).
\end{aligned} \tag{2.7}$$

We will obtain the conclusion of Lemma 2.3. □

*Remark 2.4.* The condition "entire" cannot be replaced by "meromorphic" in Lemma 2.3, as is shown by the following example.

*Example 2.5.* Let $f(z) = (e^z - 1)/(e^z + 1)$, $c = \pi i$, and $F(z) = f(z)f(z+c)$, we can see

$$T(r,F) \not\equiv 2T(r,f) + O\left(r^{\sigma(f)-1+\varepsilon}\right) + O(\log r) \tag{2.8}$$

for every set of $\{r_n\}$ with infinite measure.

**Lemma 2.6** (see [9, Lemma 2.1]). *Let $f$ and $g$ be two nonconstant meromorphic functions satisfying $E_{k)}(1,f) = E_{k)}(1,g)$ for some positive integer $k \in \mathbb{N}$. Define $H$ as follows:*

$$H = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right). \tag{2.9}$$

*If $H \not\equiv 0$, then*

$$\begin{aligned}
N(r,H) \leq{}& \overline{N}_{(2}(r,f) + \overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}(r,g) + \overline{N}_{(2}\left(r,\frac{1}{g}\right) + \overline{N}_0\left(r,\frac{1}{f'}\right) \\
&+ \overline{N}_0\left(r,\frac{1}{g'}\right) + \overline{N}_{(k+1}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(k+1}\left(r,\frac{1}{g-1}\right) \\
&+ S(r,f) + S(r,g),
\end{aligned} \tag{2.10}$$

*where $\overline{N}_0(r,1/f')$ denotes the counting function of zeros of $f'$ but not zeros of $f(f-1)$ and $\overline{N}_0(r,1/g')$ is similarly defined.*

**Lemma 2.7** (see [10]). *Under the condition of Lemma 2.6, one has*

$$N_{1)}\left(r,\frac{1}{f-1}\right) = N_{1)}\left(r,\frac{1}{g-1}\right) \leq N(r,H) + S(r,f) + S(r,g). \tag{2.11}$$

**Lemma 2.8** (see [10]). *Let $H$ be defined as Lemma 2.6. If $H \equiv 0$, then either $f \equiv g$ or $fg \equiv 1$ provided that*

$$\limsup_{r\to\infty, r\in I} \frac{\overline{N}(r,f) + \overline{N}(r,g) + \overline{N}(r,1/f) + \overline{N}(r,1/f)}{T(r)} < 1, \tag{2.12}$$

*where $T(r) := \max\{T(r,f), T(r,g)\}$ and $I$ is a set with infinite linear measure.*

**Lemma 2.9** (see [11, Lemma 2.2]). *Let $T : (0,+\infty) \to (0,+\infty)$ be a nondecreasing continuous function, $s > 0$, $0 < \alpha < 1$, and let $F \subset R_+$ be the set of all $r$ such that*

$$T(r) \leq \alpha T(r+s). \tag{2.13}$$

*If the logarithmic measure of $F$ is infinite, then*

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \infty. \tag{2.14}$$

## 3. Proof of Theorem 1.1

We define

$$\begin{aligned} F &:= f^n f(z+c), \\ G &:= g^n g(z+c). \end{aligned} \tag{3.1}$$

First of all, suppose that $H \not\equiv 0$. We replace $f$ and $g$ by $F$ and $G$, respectively, in Lemma 2.7 and Lemma 2.8. Thus,

$$\begin{aligned} N_{1)}\left(r, \frac{1}{F-1}\right) = N_{1)}\left(r, \frac{1}{G-1}\right) &\leq N(r,H) + S(r,f) + S(r,g) \\ &\leq \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) \\ &\quad + \overline{N}_{(4}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(4}\left(r, \frac{1}{G-1}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.2}$$

Applying the second main theorem to $F$ and $G$ jointly implies that

$$\begin{aligned} T(r,F) + T(r,G) \leq{}& \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\ & - \overline{N}_0\left(r, \frac{1}{F'}\right) - \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.3}$$

Noting that

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(4}\left(r,\frac{1}{F-1}\right)\le\frac{1}{2}N\left(r,\frac{1}{F-1}\right)\le\frac{1}{2}T(r,F),\\
&\overline{N}\left(r,\frac{1}{G-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{G-1}\right)+\overline{N}_{(4}\left(r,\frac{1}{G-1}\right)\le\frac{1}{2}N\left(r,\frac{1}{G-1}\right)\le\frac{1}{2}T(r,G).
\end{aligned}\tag{3.4}$$

According to Lemma 2.9 and (3.2)–(3.4), we can obtain that

$$\begin{aligned}
T(r,F)+T(r,G)&\le 2N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g)\\
&\le 4\overline{N}\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{g}\right)+2N\left(r,\frac{1}{f(z+c)}\right)\\
&\quad+2N\left(r,\frac{1}{g(z+c)}\right)+S(r,f)+S(r,g)\\
&\le 6N\left(r,\frac{1}{f}\right)+6N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g)\\
&\le 6T\left(r,\frac{1}{f}\right)+6T\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).
\end{aligned}\tag{3.5}$$

Lemma 2.3 shows that

$$\begin{aligned}
T(r,F)&=(n+1)T(r,f)+O\left(r^{\sigma(f)-1+\varepsilon}\right)+O(\log r),\\
T(r,G)&=(n+1)T(r,g)+O\left(r^{\sigma(g)-1+\varepsilon}\right)+O(\log r).
\end{aligned}\tag{3.6}$$

We can deduce that

$$(n-5)(T(r,f)+T(r,g))\le O\left(r^{\sigma(f)-1+\varepsilon}\right)+O\left(r^{\sigma(g)-1+\varepsilon}\right)+S(r,f)+S(r,g),\tag{3.7}$$

which is impossible since $n\ge 6$. Therefore, we have $H\equiv 0$. Noting that

$$N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)\le 3T(r,f)+3T(r,g)+S(r,f)+S(r,g)\le T(r),\tag{3.8}$$

where $T(r)=\max\{T(r,F),T(r,G)\}$. Together with Lemma 2.8, it shows that either $F\equiv G$ or $FG\equiv 1$. We will consider the following two cases.

*Case 1.* Suppose that $F(z)=G(z)$. Therefore

$$f(z)^n f(z+c)=g(z)^n g(z+c).\tag{3.9}$$

Let $h_1(z) = f(z)/g(z)$; we have

$$h_1(z)^n h_1(z+c) \equiv 1. \tag{3.10}$$

If $h_1(z)$ is not a constant, then Lemma 2.2 and (3.10) imply that

$$nT(r,h_1) = T(r,h_1(z+c)) + O(1) = T(r,h_1) + O\left(r^{\sigma(h_1)-1+\varepsilon}\right) + O(\log r), \tag{3.11}$$

which is a contraction with $n \geq 6$. Thus, $h_1(z) \equiv t_1$, where $t_1$ is a constant. From (3.10), we have $f(z) = t_1 g(z)$ and $t_1^{n+1} = 1$.

*Case 2.* Suppose that $F(z)G(z) \equiv 1$. Therefore

$$f(z)^n f(z+c) g(z)^n g(z+c) \equiv 1. \tag{3.12}$$

Let $h_2(z) = f(z)g(z)$; we have

$$h_2(z)^n h_2(z+c) \equiv 1. \tag{3.13}$$

By the same way as Case 1, we can obtain that $h_2$ is a constant. Therefore, $f(z)g(z) = t_2$ and $t_2^{n+1} = 1$.

## 4. Proof of Theorem 1.2

Noting that

$$\begin{gathered}
\overline{N}\left(r,\frac{1}{F-1}\right) - \frac{2}{5}N_{1)}\left(r,\frac{1}{F-1}\right) + \frac{4}{5}\overline{N}_{(3}\left(r,\frac{1}{F-1}\right) \leq \frac{3}{5}N\left(r,\frac{1}{F-1}\right) \leq \frac{3}{5}T(r,F), \\
\overline{N}\left(r,\frac{1}{G-1}\right) - \frac{2}{5}N_{1)}\left(r,\frac{1}{G-1}\right) + \frac{4}{5}\overline{N}_{(3}\left(r,\frac{1}{G-1}\right) \leq \frac{3}{5}N\left(r,\frac{1}{G-1}\right) \leq \frac{3}{5}T(r,G).
\end{gathered} \tag{4.1}$$

According to (3.1) and (4.1), we can obtain the conclusion of Theorem 1.2 by the same way as Section 3.

## 5. Proof of Theorem 1.3

Noting that

$$\begin{gathered}
\overline{N}\left(r,\frac{1}{F-1}\right) - \frac{1}{4}N_{1)}\left(r,\frac{1}{F-1}\right) + \frac{1}{2}\overline{N}_{(2}\left(r,\frac{1}{F-1}\right) \leq \frac{3}{4}N\left(r,\frac{1}{F-1}\right) \leq \frac{3}{4}T(r,F), \\
\overline{N}\left(r,\frac{1}{G-1}\right) - \frac{1}{4}N_{1)}\left(r,\frac{1}{G-1}\right) + \frac{1}{2}\overline{N}_{(2}\left(r,\frac{1}{G-1}\right) \leq \frac{3}{4}N\left(r,\frac{1}{G-1}\right) \leq \frac{3}{4}T(r,G).
\end{gathered} \tag{5.1}$$

According to (3.1) and (5.1), we can obtain the conclusion of Theorem 1.2 by the same way as Section 3.

## Acknowledgment

The third author was supported in part by China Scholarship Council (CSC).

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