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Research Article

The Meir-Keeler Type for Solving Variational Inequalities and Fixed Points of Nonexpansive Semigroups in Banach Spaces

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The aim of this paper is to introduce a new iterative scheme for finding common solutions of the variational inequalities for an inverse strongly accretive mapping and the solutions of fixed point problems for nonexpansive semigroups by using the modified viscosity approximation method associate with Meir-Keeler type mappings and obtain some strong convergence theorem in a Banach spaces under some parameters controlling conditions. Our results extend and improve the recent results of Li and Gu (2010), Wangkeeree and Preechasilp (2012), Yao and Maruster (2011), and many others.

1. Introduction

The theory of variational inequalities and variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years since they have wide applications in the optimization and control, economics, engineering science, physical sciences, and applied sciences. For these reasons, many existence result and iterative algorithms for various variational inclusion have been studied extensively by many authors (see, e.g., [1–8]). The important generalization of variational inequalities has been extensively studied and generalized in different directions to study a wide class of problems arising in mechanics, optimization, nonlinear programming, finance, and applied sciences (see, e.g., [2, 8]).

Let C be a nonempty closed convex subset of a real Banach space E and E^* be the dual space of E with norm $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ pairing between E and E^*. For $q>1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{ f \in E^* : \langle x, f\rangle = \|x\|^q, \left\|f\right\| = \|x\|^{q-1} \right\} \tag{1.1}$$

for all $x \in E$. In particular, if $q=2$, the mapping J_2 is called the *normalized duality mapping* and, usually, written $J_2 = J$. Further, we have the following properties of the generalized duality mapping J_q: (i) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in E$ with $x \neq 0$; (ii) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in E$ and $t \in [0,\infty)$; (iii) $J_q(-x) = -J_q(x)$ for all $x \in E$.

Recall that a mapping $A : C \to C$ is said to be (i) *Lipschitzian* with Lipschitz constant $L>0$ if $\|Ax-Ay\| \le L\|x-y\|$, for all $x,y \in C$; (ii) *contraction* if there exists a constant $\alpha \in (0,1)$ such that $\|Ax-Ay\| \le \alpha\|x-y\|$, for all $x,y \in C$; (iii) *nonexpansive* if $\|Ax-Ay\| \le \|x-y\|$, for all $x,y \in C$. An operator $A : C \to E$ is said to be

(i) *accretive* if there exists $j(x-y) \in J(x-y)$ such that

$$\left\langle Ax - Ay, j(x-y)\right\rangle \ge 0, \quad \forall x,y \in C, \tag{1.2}$$

(ii) *β-strongly accretive* if there exists a constant $\beta>0$ such that

$$\left\langle Ax - Ay, j(x-y)\right\rangle \ge \beta\left\|x-y\right\|^2, \quad \forall x,y \in C, \tag{1.3}$$

(iii) *β-inverse strongly accretive* if, for any $\beta>0$,

$$\left\langle Ax - Ay, j(x-y)\right\rangle \ge \beta\left\|Ax-Ay\right\|^2, \quad \forall x,y \in C. \tag{1.4}$$

Let D be a subset of C and $Q : C \to D$. Then Q is said to *sunny* if $Q(Qx + t(x-Qx)) = Qx$, whenever $Qx + t(x-Qx) \in C$ for $x \in C$ and $t \ge 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction Q of C onto D. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q.

A family $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$ for all $s,t \ge 0$;

(iii) $\|T(t)x - T(t)y\| \le \|x-y\|$ for all $x,y \in C$ and $t \ge 0$;

(iv) for all $x \in C, t \mapsto T(t)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common *fixed points of* $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$, that is, $F(\mathcal{S}) = \cap_{t\ge 0} F(T(t))$. It is known that $F(\mathcal{S})$ is closed and convex (see also [9, 10]).

A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an *L function* if $\psi(0) = 0$, $\psi(t) > 0$ for each $t>0$, and for every $s>0$ there exists $u>s$ such that $\psi(t) \le s$ for all $t \in [s,u]$. As a consequence, every L-function ψ satisfies $\psi(t) < t$ for each $t>0$.

Definition 1.1. Let (X, d) be a matric space. A mapping $f : X \to X$ is said to be:

(i) (ψ, L)-*contraction* if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an L-function and $d(f(x), f(y)) < \psi(d(x, y))$ for all $x, y \in X$ with $x \neq y$;

(ii) *Meir-Keeler type mapping* if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in X$ with $d(x, y) < \epsilon + \delta$ we have $d(f(x), f(y)) < \epsilon$.

Remark 1.2. From Definition 1.1, if $\psi(t) = \alpha t$, $\alpha \in [0, 1)$, $t \in \mathbb{R}^+$, then we get the usual contraction mapping with coefficient α.

At the same time, we are also interesting in the variational inequality problems for an inverse strongly accretive mappings in Banach spaces. In 2006, Aoyama et al. [11] introduced the following iteration scheme for an inverse strongly accretive operator A in Banach spaces E:

$$\begin{gathered} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n), \end{gathered} \tag{1.5}$$

for all $n \geq 1$, where $C \subset E$ and Q_C is a sunny nonexpansive retraction from E onto C. They proved a weak convergence theorem in a Banach spaces. Moreover, the sequence $\{x_n\}$ in (1.5) solved the *generalized variational inequality problem* for finding a point $x \in C$ such that

$$\langle Ax, j(y - x) \rangle \geq 0 \tag{1.6}$$

for all $y \in C$. The set of solutions of (1.6) is denoted by $VI(C, A)$.

An interesting is the proof by using a nonexpansive semigroup and Meir-Keeler type mapping, in 2010, Li and Gu [12] defined the following sequence:

$$\begin{gathered} x_1 = x \in E, \\ y_n - \alpha_n x_n + (1 - \alpha_n) T(t_n) x_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n, \quad n \geq 1. \end{gathered} \tag{1.7}$$

Wangkeeree and Preechasilp [13] introduced the following iterative scheme:

$$\begin{gathered} x_0 \in C, \\ z_n - \gamma_n x_n + (1 - \gamma_n) T(t_n) x_n, \\ y_n - \alpha_n x_n + (1 - \alpha_n) T(t_n) z_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n, \quad n \geq 0. \end{gathered} \tag{1.8}$$

In 2011, Yao and Maruster [8] proved some strong convergence theorems for finding a solution of variational inequality problem (1.6) in Banach spaces. They defined a sequence $\{x_n\}$ iteratively by given arbitrarily $x_0 \in C$ and

$$x_{n+1} = \beta_n x_n + (1-\beta_n) Q_C[(1-\alpha_n)(x_n - \lambda A x_n)], \quad \forall n \geq 0, \tag{1.9}$$

where Q_C is a sunny nonexpansive retraction from a uniformly convex and 2-uniformly smooth Banach space E, and A is an α-inverse strongly accretive operator of C into E.

Motivated and inspired by the idea of Li and Gu [12], Wangkeeree and Preechasilp [13], and Yao and Maruster [8], in this paper, we introduce a new iterative scheme for finding common solutions of the variational inequalities for an inverse strongly accretive mapping and the solutions of fixed point problems for a nonexpansive semigroup by using the modified viscosity approximation method associated with Meir-Keeler type mapping. We will prove the strong convergence theorem under some parameters controlling conditions. Our results extend and improve the recent results of Li and Gu [12], Wangkeeree and Preechasilp [13], Yao and Maruster [8], and many others.

2. Preliminaries

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to *uniformly convex* if, for any $\epsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x-y\| \geq \epsilon$ implies $\|(x+y)/2\| \leq 1-\delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0}((\|x+ty\| - \|x\|)/t)$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. The *modulus of smoothness* of E is defined by

$$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\right\}, \tag{2.1}$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0}(\rho(\tau)/\tau) = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

A Banach space E is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \to \infty)$ implies that

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|, \quad \forall y \in E \text{ with } x \neq y. \tag{2.2}$$

By [14, Theorem 1], it is well known that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition, and E is smooth.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 2.1 (see [15]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction, and let J be the normalized duality mapping on E. Then the following are equivalent:*

(i) *Q is sunny and nonexpansive;*

(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$, $\forall x, y \in E$;

(iii) $\langle x - Qx, J(y - Qx)\rangle \leq 0$, $\forall x \in E$, $y \in C$.

Proposition 2.2 (see [16]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

Lemma 2.3 (see [17]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then x is a fixed point of T.*

We need the following lemmas for proving our main results.

Lemma 2.4 (see [18]). *Let $r > 0$, and let E be a uniformly convex Banach space. Then, there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\left\|\lambda x + (1 - \lambda)y\right\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\left\|y\right\|^2 - \lambda(1 - \lambda)g(\left\|x - y\right\|) \tag{2.3}$$

for all $x, y \in B_r := \{z \in E : \|z\| \leq r\}$ and $0 \leq \lambda \leq 1$.

Lemma 2.5 (see [19]). *Let E be a real smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\left\|x - y\right\|) \leq \|x\|^2 - 2\langle x, jy\rangle + \left\|y\right\|^2, \quad \forall x, y \in B_r. \tag{2.4}$$

Lemma 2.6 (see [18]). *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inquality holds:*

$$\left\|x + y\right\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\left\|Ky\right\|^2, \quad \forall x, y \in E. \tag{2.5}$$

Lemma 2.7 (see [20]). *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$, one has*

$$\left\|x + y\right\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle \tag{2.6}$$

for all $j(x + y) \in J(x + y)$ with $x \neq y$.

Lemma 2.8 (see [21]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.9 (see [22]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \quad n \geq 0, \tag{2.7}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Theorem 2.10 (see [23]). *Let (X,d) be a complete metric space and $f : X \to X$ a Meir-Keeler type mapping. Then f has a unique fixed point.*

Theorem 2.11 (see [24]). *Let (X,d) be a metric space and $f : X \to X$ a mapping. Then the following assertions are equivalent:*

(i) *f is a Meir-Keeler type mapping;*

(ii) *there exists an L function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is a (φ, L) contraction.*

Proposition 2.12 (see [21]). *Let C be a convex subset of a Banach space E. Let $f : C \to C$ be a Meir-Keeler type mapping. Then for each $\epsilon > 0$ there exists $r \in (0,1)$ such that for each $x, y \in C$ with $\|x-y\| \geq \epsilon$, one has*

$$\|f(x) - f(y)\| \leq r\|x-y\|. \tag{2.8}$$

Proposition 2.13 (see [21]). *Let C be a convex subset of a Banach space E. Let T be a nonexpansive mapping on C, and let $f : C \to C$ be a Meir-Keeler-type mapping. Then the following holds:*

(i) *$T \circ f$ is a Meir-Keeler type mapping on C;*

(ii) *for each $\alpha \in (0,1)$ the mapping $x \mapsto \alpha f(x) + (1-\alpha)T(x)$ is a Meir-Keeler-type mapping on C.*

The following lemma is characterized by the set of solutions of variational inequality by using sunny nonexpansive retractions.

Lemma 2.14 (see [11]). *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C, and let A be an accretive operator of C into E. Then, for all $\lambda > 0$,*

$$VI(C,A) = F(Q(I-\lambda A)), \tag{2.9}$$

where $VI(C,A) = \{x^* \in C : \langle Ax^*, J(x-x^*)\rangle \geq 0, \forall x \in C\}$.

3. Strong Convergence Theorems

In this section, we suppose that the function φ from the definition of the (φ, L) contraction is continuous and strictly increasing and $\lim_{t\to\infty}\eta(t) = \infty$, where $\eta(t) = t - \varphi(t)$, $t \in \mathbb{R}^+$.

In consequence, we have that η is a bijection on $\mathbb{R}^+$ and the function ψ satisfies the assumption in Remark 1.2.

Suppose that $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, and $\{\mu_n\}, \{\lambda_n\} \subset (0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(C2) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$ and $0 < a \leq \lambda_n \leq b < \beta/K^2$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\lim_{n\to\infty}\mu_n = 0$;

(C5) $\lim_{n\to\infty}\sup_{x\in\widetilde{C}}\|T(\mu_{n+1})x - T(\mu_n)x\| = 0$, $\widetilde{C}$ bounded subset of C.

Next, we stat the main result.

Theorem 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C and $A : C \to E$ be an β-inverse-strongly accretive operator. Let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup from C into itself and f be a Meir-Keeler contraction of C into itself. Suppose that $F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$ and the conditions (C1)–(C5). For arbitrary given $x_1 \in C$, the sequences $\{x_n\}$ are generated by*

$$\begin{aligned} u_n &= Q_C(x_n - \lambda_n A x_n), \\ y_n &= Q_C[\alpha_n f(x_n) + (1-\alpha_n)u_n], \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)T(\mu_n)y_n. \end{aligned} \tag{3.1}$$

Then $\{x_n\}$ converges strongly to $x^* = Q_F f x^* \in F$ which also solves the following variational inequality:

$$\langle (f-I)x^*, J(z-x^*)\rangle \leq 0, \quad \forall z \in F. \tag{3.2}$$

Proof. First we prove that $\{x_n\}$ bounded. Let $p \in F$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|Q_C(x_n - \lambda_n A x_n) - Q_C(p - \lambda_n A p)\|^2 \\ &\leq \|(x_n - \lambda_n A x_n) - (p - \lambda_n A p)\|^2 \\ &= \|(I-\lambda_n A)x_n - (I-\lambda_n A)p\|^2 \\ &= \|(x_n - p) - \lambda_n(Ax_n - Ap)\|^2 \\ &\leq \|x_n - p\|^2 - 2\lambda_n\langle Ax_n - Ap, J(x_n - p)\rangle + 2K^2\lambda_n^2\|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - 2\lambda_n\beta\|Ax_n - Ap\|^2 + 2K^2\lambda_n^2\|Ax_n - Ap\|^2 \\ &= \|x_n - p\|^2 + 2\lambda_n\left(\lambda_n K^2 - \beta\right)\|Ax - Ay\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.3}$$

So, we get $\|u_n - p\| \le \|x_n - p\|$, for all $n \ge 1$. It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n (x_n - p) + (1-\beta_n)(T(\mu_n)y_n - p)\| \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)\|y_n - p\| \\
&= \beta_n \|x_n - p\| + (1-\beta_n)\|Q_C[\alpha_n f(x_n) + (1-\alpha_n)u_n] - Q_C p\| \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)[\alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|u_n - p\|] \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)[\alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\|] \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)[\alpha_n \psi(\|x_n - p\|) + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\|] \\
&= \|x_n - p\| + \alpha_n (1-\beta_n)[\eta(\|x_n - p\|)] + \alpha_n (1-\beta_n)\eta\left[\eta^{-1}(\|f(p) - p\|)\right] \\
&\le \max\left\{\|x_n - p\|, \eta^{-1}(\|f(p) - p\|)\right\}.
\end{aligned} \tag{3.4}$$

By induction, we conclude that

$$\|x_n - p\| \le \max\left\{\|x_n - p\|, \eta^{-1}(\|f(p) - p\|)\right\}, \quad \forall n \ge 1. \tag{3.5}$$

This implies that $\{x_n\}$ bounded, so are $\{f(x_n)\}$, $\{y_n\}$, $\{u_n\}$, $\{Ax_n\}$, and $\{T(\mu_n)y_n\}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we observe that

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|Q_C(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - Q_C(x_n - \lambda_n Ax_n)\| \\
&\le \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_n Ax_n)\| \\
&= \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n) + (\lambda_{n+1} - \lambda_n)Ax_n\| \\
&\le \|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_{n+1}A)x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\
&\le \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\|, \\
\|y_{n+1} - y_n\| &= \|Q_C[\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1})u_{n+1}] - Q_C[\alpha_n f(x_n) + (1-\alpha_n)u_n]\| \\
&\le \|[\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1})u_{n+1}] - [\alpha_n f(x_n) + (1-\alpha_n)u_n]\| \\
&\le \|u_{n+1} - u_n\| + \alpha_{n+1}(\|u_{n+1}\| + \|f(x_{n+1})\|) + \alpha_n(\|u_n\| + \|f(x_n)\|) \\
&\le \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \alpha_{n+1}(\|u_{n+1}\| + \|f(x_{n+1})\|) \\
&\quad + \alpha_n(\|u_n\| + \|f(x_n)\|).
\end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned}
\|T(\mu_{n+1})y_{n+1} - T(\mu_n)y_n\| &\le \|T(\mu_{n+1})y_{n+1} - T(\mu_{n+1})y_n\| + \|T(\mu_{n+1})y_n - T(\mu_n)y_n\| \\
&\le \|y_{n+1} - y_n\| + \|T(\mu_{n+1})y_n - T(\mu_n)y_n\| \\
&\le \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \alpha_{n+1}(\|u_{n+1}\| + \|f(x_{n+1})\|) \\
&\quad + \alpha_n(\|u_n\| + \|f(x_n)\|) + \|T(\mu_{n+1})y_n - T(\mu_n)y_n\| \\
&\le \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \alpha_{n+1}(\|u_{n+1}\| + \|f(x_{n+1})\|) \\
&\quad + \alpha_n(\|u_n\| + \|f(x_n)\|) + \sup_{y\in\{y_n\}} \|T(\mu_{n+1})y - T(\mu_n)y\|.
\end{aligned} \tag{3.7}$$

By (C1), (C2), and (C4), they imply that

$$\limsup_{n\to\infty}(\|T(\mu_{n+1})y_{n+1} - T(\mu_n)y_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{3.8}$$

Applying Lemma 2.8, we obtain

$$\lim_{n\to\infty}\|T(\mu_n)y_n - x_n\| = 0. \tag{3.9}$$

Therefore, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.10}$$

On the other hand, we consider

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\left[\alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|u_n - p\|^2\right] \\
&= \beta_n\|x_n - p\|^2 + (1-\beta_n) \\
&\quad \times \left[\alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|Q_C(x_n - \lambda_n Ax_n) - Q_C(p - \lambda_n Ap)\|^2\right] \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n) \\
&\quad \times \left[\alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\left(\|x_n - p\|^2 + 2\lambda_n\left(\lambda_n K^2 - \beta\right)\|Ax - Ay\|^2\right)\right] \\
&= [\beta_n + (1-\beta_n)(1-\alpha_n)]\|x_n - p\|^2 + \alpha_n(1-\beta_n)\|f(x_n) - p\|^2 \\
&\quad + 2\lambda_n\left(\lambda_n K^2 - \beta\right)(1-\beta_n)(1-\alpha_n)\|Ax - Ay\|^2 \\
&\le \|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 + 2\lambda_n\left(\lambda_n K^2 - \beta\right)\|Ax - Ay\|^2.
\end{aligned} \tag{3.11}$$

Then, we obtain that

$$\begin{aligned}
2\lambda_n\left(\beta-\lambda_n K^2\right)\|Ax-Ay\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|f(x_n)-p\|^2\\
&\le \|x_n-x_{n+1}\|\left(\|x_n-p\|+\|x_{n+1}-p\|\right)+\alpha_n\|f(x_n)-p\|^2.
\end{aligned}\tag{3.12}$$

By (C1), (C2), (C3), and (3.10), we get

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{3.13}$$

From Proposition 2.1 (ii) and Lemma 2.5, we also have

$$\begin{aligned}
\|u_n-p\|^2 &= \|Q_C(x_n-\lambda_n Ax_n)-Q_C(p-\lambda_n Ap)\|^2\\
&\le \left\langle (x_n-\lambda_n Ax_n)-(p-\lambda_n Ap), J(u_n-p)\right\rangle\\
&= \left\langle (x_n-p)-\lambda_n(Ax_n-Ap), J(u_n-p)\right\rangle\\
&= \left\langle x_n-p, J(u_n-p)\right\rangle-\lambda_n\left\langle Ax_n-Ap, J(u_n-p)\right\rangle\\
&\le \frac{1}{2}\left[\|x_n-p\|^2+\|u_n-p\|^2-g\|x_n-u_n\|\right]+\lambda_n\|Ax_n-Ap\|\|u_n-p\|.
\end{aligned}\tag{3.14}$$

So, we get,

$$\|u_n-p\|^2\le\|x_n-p\|^2-g\|x_n-u_n\|+2\lambda_n\|Ax_n-Ap\|\|u_n-p\|. \tag{3.15}$$

Therefore, using (3.11), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\left[\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|u_n-p\|^2\right]\\
&= \beta_n\|x_n-p\|^2+\alpha_n(1-\beta_n)\|f(x_n)-p\|^2+(1-\alpha_n)(1-\beta_n)\|u_n-p\|^2\\
&\le \beta_n\|x_n-p\|^2+\alpha_n(1-\beta_n)\|f(x_n)-p\|^2\\
&\quad+(1-\alpha_n)(1-\beta_n)\left[\|x_n-p\|^2-g(\|x_n-u_n\|)+2\lambda_n\|Ax_n-Ap\|\|u_n-p\|\right]\\
&\le \|x_n-p\|^2+\alpha_n\|f(x_n)-p\|^2-g(\|x_n-u_n\|)+2\lambda_n\|Ax_n-Ap\|\|u_n-p\|.
\end{aligned}\tag{3.16}$$

Then we get

$$
\begin{aligned}
g(\|x_n-u_n\|) &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|f(x_n)-p\|^2+2\lambda_n\|Ax_n-Ap\|\|u_n-p\| \\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+\alpha_n\|f(x_n)-p\|^2 \\
&\quad +2\lambda_n\|Ax_n-Ap\|\|u_n-p\|.
\end{aligned} \tag{3.17}
$$

By (C1), (3.10), and (3.13), we have

$$
\lim_{n\to\infty} g(\|x_n-u_n\|)=0. \tag{3.18}
$$

It follows from the property of g that

$$
\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.19}
$$

Again, we consider

$$
\begin{aligned}
\|y_n-p\|^2 &= \|Q_C[\alpha_n f(x_n)+(1-\alpha_n)u_n]-Q_Cp\|^2 \\
&\le \langle \alpha_n(f(x_n)-p)+(1-\alpha_n)(u_n-p), J(y_n-p)\rangle \\
&= \alpha_n\langle f(x_n)-p, J(y_n-p)\rangle+(1-\alpha_n)\langle u_n-p, J(y_n-p)\rangle \\
&\le \alpha_n\|f(x_n)-p\|\|y_n-p\|+\frac{1}{2}\left[\|u_n-p\|^2+\|y_n-p\|^2-g(\|u_n-y_n\|)\right] \\
&\le \alpha_n\|f(x_n)-p\|\|y_n-p\|+\frac{1}{2}\left[\|x_n-p\|^2+\|y_n-p\|^2-g(\|u_n-y_n\|)\right].
\end{aligned} \tag{3.20}
$$

It follows that

$$
\|y_n-p\|^2 \le 2\alpha_n\|f(x_n)-p\|\|y_n-p\|+\|x_n-p\|^2-g(\|u_n-y_n\|). \tag{3.21}
$$

By using (3.11), we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\left[\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|y_n-p\|^2\right] \\
&= \beta_n\|x_n-p\|^2+\alpha_n(1-\beta_n)\|f(x_n)-p\|^2+(1-\alpha_n)(1-\beta_n)\|y_n-p\|^2 \\
&\le \beta_n\|x_n-p\|^2+\alpha_n(1-\beta_n)\|f(x_n)-p\|^2 \\
&\quad +(1-\alpha_n)(1-\beta_n)\left[2\alpha_n\|f(x_n)-p\|\|y_n-p\|+\|x_n-p\|^2-g(\|u_n-y_n\|)\right] \\
&\le \|x_n-p\|^2+\alpha_n\|f(x_n)-p\|^2+2\alpha_n\|f(x_n)-p\|\|y_n-p\|-g(\|u_n-y_n\|).
\end{aligned} \tag{3.22}
$$

Therefore, we have

$$\begin{aligned} g(\|u_n - y_n\|) &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \|f(x_n) - p\|^2 + 2\alpha_n \|f(x_n) - p\| \|y_n - p\| \\ &\le \|x_n - x_{n+1}\| (\|x_n - p\| + \|x_{n+1} - p\|) + 2\alpha_n \|f(x_n) - p\|^2 \\ &\quad + 2\alpha_n \|f(x_n) - u_n\| \|y_n - u_n\|. \end{aligned} \tag{3.23}$$

By (C1) and (3.10), we have

$$\lim_{n\to\infty} g(\|u_n - y_n\|) = 0. \tag{3.24}$$

From the property of g, we get

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.25}$$

According (3.19) and (3.25), we also have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.26}$$

Since

$$\begin{aligned} \|x_n - T(\mu_n)x_n\| &\le \|x_n - T(\mu_n)y_n\| + \|T(\mu_n)y_n - T(\mu_n)x_n\| \\ &\le \|x_n - T(\mu_n)y_n\| + \|y_n - x_n\|, \end{aligned} \tag{3.27}$$

from (3.9) and (3.26), we get

$$\lim_{n\to\infty} \|T(\mu_n)x_n - x_n\| = 0. \tag{3.28}$$

Now, we show that $z \in F := F(\mathcal{S}) \cap VI(C, A)$. We can choose a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is bounded, and there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to z. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup z$.

(I) *We show that* $z \in VI(C, A)$. From the assumption, we see that control sequence $\{\lambda_{n_k}\}$ is bounded. So, there exists a subsequence $\{\lambda_{n_{k_j}}\}$ that converges to λ_0. We may assume, without loss of generality, that $\lambda_{n_k} \rightharpoonup \lambda_0$. Observe that

$$\begin{aligned} \|Q_C(x_{n_k} - \lambda_0 A x_{n_k}) - x_{n_k}\| &\le \|Q_C(x_{n_k} - \lambda_0 A x_{n_k}) - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\ &\le \|(x_{n_k} - \lambda_0 A x_{n_k}) - (x_{n_k} - \lambda_{n_k} A x_{n_k})\| + \|y_{n_k} - x_{n_k}\| \\ &\le M\|\lambda_{n_k} - \lambda_0\| + \|y_{n_k} - x_{n_k}\|, \end{aligned} \tag{3.29}$$

where M is as appropriate constant such that $M \geq \sup_{n\geq 1}\{\|Ax_n\|\}$. It follows from (3.26) and $\lambda_{n_k} \rightharpoonup \lambda_0$ that

$$\lim_{k\to\infty} \|Q_C(x_{n_k} - \lambda_0 A x_{n_k}) - x_{n_k}\| = 0. \tag{3.30}$$

We know that $Q_C(I-\lambda_0 A)$ is nonexpansive, and it follows from Lemma 2.3 that $z \in F(Q_C(I-\lambda_0 A))$. By using Lemma 2.14, we can obtain that $z \in F(Q_C(I-\lambda_0 A)) = VI(C,A)$.

(II) *Next, we show that* $z \in F(\mathcal{S})$. Let $\mu_{n_k} \geq 0$ such that

$$\mu_{n_k} \longrightarrow 0, \qquad \frac{\|T(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} \longrightarrow 0, \quad k \longrightarrow \infty. \tag{3.31}$$

Fix $t > 0$. Notice that

$$\begin{aligned}
\|x_{n_k} - T(t)z\| \leq{}& \sum_{i=0}^{[t/\mu_{n_k}]-1} \|T((i+1)\mu_{n_k})x_{n_k} - T(i\mu_{n_k})x_{n_k}\| \\
&+ \left\|T\left(\left[\frac{t}{\mu_{n_k}}\right]\mu_k\right)x_{n_k} - T\left(\left[\frac{t}{\mu_{n_k}}\right]\mu_{n_k}\right)z\right\| + \left\|T\left(\left[\frac{t}{\mu_{n_k}}\right]\mu_{n_k}\right)z - T(t)z\right\| \\
\leq{}& \left[\frac{t}{\mu_{n_k}}\right]\|T(\mu_{n_k})x_{n_k} - x_{n_k}\| + \|x_{n_k} - z\| + \left\|T\left(t - \left[\frac{t}{\mu_{n_k}}\right]\mu_{n_k}\right)z - z\right\| \\
\leq{}& t\frac{\|T(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} + \|x_{n_k} - p\| + \left\|T\left(t - \left[\frac{t}{\mu_{n_k}}\right]\mu_{n_k}\right)z - z\right\| \\
\leq{}& t\frac{\|T(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} + \|x_{n_k} - p\| + \max\{\|T(s)z - z\| : 0 \leq s \leq \mu_{n_k}\}.
\end{aligned} \tag{3.32}$$

For all $k \in \mathbb{N}$, we have

$$\limsup_{k\to\infty}\|x_{n_k} - T(t)z\| \leq \limsup_{k\to\infty}\|x_{n_k} - z\|. \tag{3.33}$$

Since a Banach space E with a weakly sequentially continuous duality mapping satisfies the Opial's condition, this implies that $T(t)z = z$. Therefore $z \in F(\mathcal{S})$, so $z \in F$.

Next, we show that $\limsup_{n\to\infty}\langle (f-I)x^*, J(y_n - x^*)\rangle \leq 0$, where $x^* = Q_F f x^*$, Q_F is a sunny nonexpansive retraction of C onto F. Since we have (3.26) and $x_{n_k} \rightharpoonup z$, then we have $y_{n_k} \rightharpoonup z$ such that

$$\begin{aligned}
\limsup_{n\to\infty}\langle (f-I)x^*, J(y_n - x^*)\rangle &= \lim_{k\to\infty}\langle (f-I)x^*, J(y_{n_k} - x^*)\rangle \\
&= \lim_{k\to\infty}\langle (f-I)x^*, J(z - x^*)\rangle \leq 0.
\end{aligned} \tag{3.34}$$

Finally, we show that $\{x_n\}$ converges strongly to x^*. Suppose that $\{x_n\}$ does not converge strongly to x^*, and then there exist $\epsilon > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|x_{n_i} - x^*\| > \epsilon$ for all $i \in \mathbb{N}$. By Proposition 2.12, for this ϵ there exists $r \in (0,1)$ such that

$$\left\| f(x_{n_i}) - f(p) \right\| \leq r \left\| x_{n_i} - p \right\|. \tag{3.35}$$

So, by Lemma 2.7, we have

$$\begin{aligned}
\left\| y_{n_i} - x^* \right\|^2 &= \left\| Q_C[\alpha_n f(x_{n_i}) + (1-\alpha_{n_i}) u_{n_i}] - Q_C x^* \right\|^2 \\
&\leq \left\langle \alpha_{n_i}(f(x_{n_i}) - x^*) + (1-\alpha_{n_i})(u_{n_i} - x^*), J(y_{n_i} - x^*) \right\rangle \\
&= \alpha_{n_i} \left\langle f(x_{n_i}) - x^*, J(y_{n_i} - x^*) \right\rangle + (1-\alpha_{n_i}) \left\langle u_{n_i} - x^*, J(y_{n_i} - x^*) \right\rangle \\
&\leq \alpha_{n_i} \left\langle f(x_{n_i}) - f(x^*), J(y_{n_i} - x^*) \right\rangle + \alpha_{n_i} \left\langle f(x^*) - x^*, J(y_{n_i} - x^*) \right\rangle \\
&\quad + (1-\alpha_{n_i}) \|u_{n_i} - x^*\| \left\| y_{n_i} - x^* \right\| \\
&\leq r\alpha_{n_i} \|x_{n_i} - x^*\| \left\| y_{n_i} - x^* \right\| + \alpha_{n_i} \left\langle f(x^*) - x^*, J(y_{n_i} - x^*) \right\rangle \\
&\quad + (1-\alpha_{n_i}) \|u_{n_i} - x^*\| \left\| y_{n_i} - x^* \right\| \\
&\leq r\alpha_{n_i} \|x_{n_i} - x^*\|^2 + \alpha_{n_i} \left\langle f(x^*) - x^*, J(y_{n_i} - x^*) \right\rangle + (1-\alpha_{n_i}) \|x_{n_i} - x^*\|^2 \\
&= (1 - (1-r)\alpha_{n_i}) \|x_{n_i} - x^*\|^2 + \alpha_{n_i} \left\langle f(x^*) - x^*, J(y_{n_i} - x^*) \right\rangle.
\end{aligned} \tag{3.36}$$

It follows from (3.11) that

$$\begin{aligned}
\left\| x_{n_i+1} - p \right\|^2 &\leq \beta_{n_i} \left\| x_{n_i} - p \right\|^2 + (1-\beta_{n_i}) \left\| y_{n_i} - p \right\|^2 \\
&\leq \beta_{n_i} \left\| x_{n_i} - p \right\|^2 + (1-\beta_{n_i}) \\
&\quad \times \left[(1 - (1-r)\alpha_{n_i}) \|x_{n_i} - x^*\|^2 + \alpha_{n_i} \left\langle f(x^*) - x^*, J(y_{n_i} - x^*) \right\rangle \right] \\
&= \left[1 - (1-r)(1-\beta_{n_i})\alpha_{n_i} \right] \|x_{n_i} - x^*\|^2 + (1-\beta_{n_i})\alpha_{n_i} \left\langle f(x^*) - x^*, J(y_{n_i} - x^*) \right\rangle.
\end{aligned} \tag{3.37}$$

Now, from (C1), (3.34) and applying Lemma 2.9 to (3.37), we get $\|x_n - x^*\| \to 0$ as $n \to \infty$. This is a contradiction, and hence the sequence $\{x_n\}$ converges strongly to $x^* \in F$. The proof is completed. □

Corollary 3.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C and $A : C \to E$ be an β-inverse strongly accretive operator. Let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup from C into it self and f be a Meir-Keeler contraction of C into* *itself. Suppose that* $F := F(\mathcal{S}) \cap VI(C, A) \neq \emptyset$. *For arbitrary given* $x_1 \in C$, *the sequences* $\{x_n\}$ *are generated by*

$$\begin{aligned} u_n &= Q_C(x_n - \lambda_n A x_n), \\ y_n &= Q_C\left[\alpha_n f(x_n) + (1-\alpha_n)u_n\right], \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds, \quad \forall n \geq 1, \end{aligned} \tag{3.38}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy the conditions (C1)–(C3) in Theorem 3.1 and assume that* $\lim_{n\to\infty}\sup_{x\in\tilde{C}}\|(1/t_{n+1})\int_0^{t_{n+1}} T(s)x ds - (1/t_n)\int_0^{t_n} T(s)x ds\| = 0$, $\tilde{C}$ *bounded subset of* C, $\lim_{n\to\infty}\mu_n = \infty$ *and* $\lim_{n\to\infty}(\mu_n/\mu_{n+1}) = 1$. *Then* $\{x_n\}$ *converges strongly to* $x^* = Q_F f x^* \in F$, *which also solves the following variational inequality:*

$$\left\langle (f-I)x^*, J(z-x^*)\right\rangle \leq 0, \quad \forall z \in F. \tag{3.39}$$

Corollary 3.3. *Let* E *be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant* K *and* C *a nonempty closed convex subset of* E. *Let* Q_C *be a sunny nonexpansive retraction from* E *onto* C. *Let* $\mathcal{S} = \{T(t) : t \geq 0\}$ *be a nonexpansive semigroup from* C *into itself and* f *be a Meir-Keeler contraction of* C *into itself. Suppose that* $F(\mathcal{S}) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, *and* $\{\mu_n\} \subset (0,\infty)$ *satisfy the conditions (C1), (C2), (C4), and (C5) in Theorem 3.1. For arbitrary given* $x_1 \in C$, *the sequences* $\{x_n\}$ *are generated by*

$$\begin{aligned} y_n &= Q_C\left[\alpha_n f(x_n) + (1-\alpha_n)x_n\right], \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)T(\mu_n)y_n, \quad \forall n \geq 1. \end{aligned} \tag{3.40}$$

Then $\{x_n\}$ *converges strongly to* $x^* = Q_F f x^* \in F(\mathcal{S})$, *which also solves the following variational inequality:*

$$\left\langle (f-I)x^*, J(z-x^*)\right\rangle \leq 0, \quad \forall z \in F. \tag{3.41}$$

Proof. Taking $A = 0$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof. □

Corollary 3.4. *Let* E *be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant* K *and* C *a nonempty closed convex subset of* E. *Let* Q_C *be a sunny nonexpansive retraction from* E *onto* C *and* $A : C \to E$ *be an* β*-inverse strongly accretive operator. Let* f *be a Meir-Keeler contraction of* C *into itself. Suppose that* $F := VI(C, A) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy the conditions (C1)–(C3) in Theorem 3.1. For arbitrary given* $x_1 \in C$, *the sequences* $\{x_n\}$ *are generated by*

$$\begin{aligned} u_n &= Q_C(x_n - \lambda_n A x_n), \\ y_n &= Q_C\left[\alpha_n f(x_n) + (1-\alpha_n)u_n\right], \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \geq 1. \end{aligned} \tag{3.42}$$

Then $\{x_n\}$ converges strongly to $x^* = Q_F f x^* \in F$, which also solves the following variational inequality:

$$\langle (f - I)x^*, J(z - x^*) \rangle \le 0, \quad \forall z \in F. \tag{3.43}$$

Proof. Taking $\mu_n = 0$ for all $n \ge 1$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof. □

Corollary 3.5. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C and $A : C \to E$ be an β strongly accretive and L-Lipschitz continuous operator. Let $\mathcal{S} = \{T(t) : t \ge 0\}$ be a nonexpansive semigroup from C into it self and f be a Meir-Keeler contraction of C into itself. Suppose that $F := F(\mathcal{S}) \cap VI(C, A) \ne \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $\{\mu_n\}, \{\lambda_n\} \subset (0, \infty)$ satisfy the conditions (C1) and (C3)–(C5) in Theorem 3.1. If the sequence $\{x_n\}$ is generated by $x_1 \in C$ and (3.1) and $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$ and $0 < a \le \lambda_n \le b < \beta/K^2L^2$, then the sequence $\{x_n\}$ converges strongly to $x^* = Q_F f x^* \in F$, which also solves the following variational inequality:*

$$\langle (f - I)x^*, J(z - x^*) \rangle \le 0, \quad \forall z \in F. \tag{3.44}$$

Proof. Since A be an β strongly accretive and L-Lipschitz continuous operator of C into E, we have

$$\langle Ax - Ay, j(x - y) \rangle \ge \beta \|x - y\|^2 \ge \frac{\beta}{L^2} \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{3.45}$$

Therefore, A is (β/L^2)-inverse strongly accretive. Using Theorem 3.1, we can obtain that $\{x_n\}$ converges strongly to x^*. This completes the proof. □

The following corollary is defined in a real Hilbert space. Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a mapping. The classical variational inequality problems are to find $x \in C$ such that

$$\langle Ax, y - x \rangle \ge 0, \tag{3.46}$$

for all $y \in C$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{3.47}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \ge \|P_C x - P_C y\|^2, \tag{3.48}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{3.49}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{3.50}$$

for all $x \in H,\ y \in C$.

It is well known in Hilbert spaces the smooth constant $K = \sqrt{2}/2$ and $J = I$ (identity mapping). From Theorem 3.1, we can obtain the following result immediately.

Corollary 3.6. *Let C be a nonempty compact convex subset of a real Hilbert space H. Let P_C be a metric projection of H onto C and $A : C \to E$ be an β-inverse strongly accretive operator. Let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup from C into itself and f be a Meir-Keeler contraction of C into itself. Suppose that $F := F(\mathcal{S}) \cap VI(C, A) \neq \emptyset$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{\mu_n\}, \{\lambda_n\} \subset (0,\infty)$ satisfy the conditions (C1)–(C5) in Theorem 3.1. For arbitrary given $x_1 \in C$, the sequences $\{x_n\}$ are generated by*

$$\begin{aligned} u_n &= P_C(x_n - \lambda_n A x_n), \\ y_n &= P_C\left[\alpha_n f(x_n) + (1-\alpha_n) u_n\right], \\ x_{n+1} &= \beta_n x_n + (1-\beta_n) T(\mu_n) y_n, \quad \forall n \geq 1. \end{aligned} \tag{3.51}$$

Then $\{x_n\}$ converges strongly to $x^* = P_F f x^* \in F$ which also solves the following variational inequality:

$$\left\langle (f - I)x^*, z - x^* \right\rangle \leq 0, \quad \forall z \in F. \tag{3.52}$$

Remark 3.7. Question and Open problems. Can we extend Theorem 3.1 to more general variational inequalities in the sense of Noor [1] on Banach spaces?

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