Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 193085, 14 pages
doi:10.1155/2012/193085

Research Article

Best Proximity Point Results for MK-Proximal Contractions

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Received 10 October 2012; Accepted 15 December 2012

Academic Editor: Qamrul Hasan Ansari

Let A and B be nonempty subsets of a metric space with the distance function d, and $T : A \to B$ is a given non-self-mapping. The purpose of this paper is to solve the nonlinear programming problem that consists in minimizing the real-valued function $x \mapsto d(x, Tx)$, where T belongs to a new class of contractive mappings. We provide also an iterative algorithm to find a solution of such optimization problems.

1. Introduction

Let A and B be nonempty subsets of a metric space (X, d). Because the functional equation $Tx = x$ $(x \in A)$, where $T : A \to B$ is a given non-self-mapping, does not necessarily have a solution, it is desirable in this case to find an optimal approximate solution to the equation $Tx = x$ when the equation has no solution. In view of the fact that $d(A, B)$ is a lower bound for $d(x, Tx)$, where

$$d(A,B) = \inf\{d(a,b) : \ a \in A, \ b \in B\}, \tag{1.1}$$

an approximate solution $x^* \in A$ to the equation $Tx = x$ produces the least possible error if $d(x^*, Tx^*) = d(A, B)$. Such a solution is called a best proximity point of the mapping $T : A \to B$. Due to the fact that $d(x, Tx) \geq d(A, B)$ for all $x \in A$, a best proximity point provides the global minimum of the nonlinear programming problem $\min_{x\in A} d(x, Tx)$. The results that provide sufficient conditions that ensure the existence of a best proximity point are known as best proximity point theorems. Best proximity point theorems for various classes of non-self-mapppings have been established in [1–32].

This work focuses on best proximity point theorems for a new family of non-self-mapppings known as MK-proximal contractions. An iterative algorithm is presented to compute an optimal approximate solution to some fixed point equations. The presented theorems extend and generalize several existing results in the literature including the well-known result of Meir and Keeler [33].

2. Preliminaries

We present in this section some notations and notions that will be used later.

Let (X,d) be a metric space; A and B are two nonempty subsets of X. We consider the following notations:

$$\begin{aligned} d(A,B) &:= \inf\{d(a,b) : a \in A, b \in B\}; \\ a \in A, d(a,B) &:= \inf\{d(a,b) : b \in B\}; \\ A_0 &:= \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\}; \\ B_0 &:= \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\}. \end{aligned} \tag{2.1}$$

Definition 2.1 (see [3]). B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ of B satisfying the condition that $d(x,y_n) \to d(x,B)$ for some x in A has a convergent subsequence.

Definition 2.2. An element $x^* \in A$ is said to be a best proximity point of the non-self-mappping $T : A \to B$ if it satisfies the condition that

$$d(x^*, Tx^*) = d(A,B). \tag{2.2}$$

Because of the fact that $d(x,Tx) \geq d(A,B)$ for all $x \in A$, the global minimum of the mapping $x \mapsto d(x,Tx)$ is attained at a best proximity point. Moreover, if the underlying mapping is a self-mapping, then it can be observed that a best proximity point is essentially a fixed point.

Definition 2.3. One says that $g : A \to A$ is an isometry if for any $x, y \in A$, one has

$$d(gx, gy) = d(x,y). \tag{2.3}$$

Definition 2.4 (see [6]). Given a mapping $T : A \to B$ and an isometry $g : A \to A$, the mapping T is said to preserve the isometric distance with respect to g if for any $x, y \in A$, one has

$$d(T(gx), T(gy)) = d(Tx, Ty). \tag{2.4}$$

Definition 2.5 (see [27]). Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property if and only if

$$\left.\begin{array}{l} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{array}\right\} \Longrightarrow d(x_1,x_2) = d(y_1,y_2), \tag{2.5}$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

We introduce the concept of the weakly P-property as follows.

Definition 2.6. Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the weakly P-property if and only if

$$\left.\begin{array}{l} d(x_1,y) = d(A,B) \\ d(x_2,y) = d(A,B) \end{array}\right\} \Longrightarrow x_1 = x_2, \tag{2.6}$$

where $x_1, x_2 \in A$ and $y \in B$.

Note that if the pair (A,B) has the P-property, then it has the weakly P-property. The following example shows that the converse is not true in general.

Example 2.7. Consider the Euclidean space $\mathbb{R}^2$ endowed with the Euclidean metric d. Let us define the sets

$$A := \{(a,0) : a \geq 0\}, \qquad B := \{(b,1) : b \geq 0\} \cup \{(b,-1) : b \geq 0\}. \tag{2.7}$$

Clearly, we have $d(A,B) = 1$. On the other hand, we have $d((0,0),(0,1)) = d((2,0),(2,-1)) = d(A,B)$ but $d((0,0),(2,0)) = 2 \neq 4 = d((0,1),(2,-1))$. Thus (A,B) does not satisfy the P-property. Now, suppose that $d((a,0),(b,1)) = 1 = d((a',0),(b,1))$ for some $a,b,a' \geq 0$. This implies immediately that $a = a'$, that is, $(a,0) = (a',0)$. Similarly, if $d((a,0),(b,-1)) = 1 = d((a',0),(b,-1))$ for some $a,b,a' \geq 0$, we get that $(a,0) = (a',0)$. This implies that (A,B) has the weakly P-property.

Definition 2.8. A self-mapping $T : A \to A$ is said to be an MK-contraction of the first kind if, for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$x,y \in A, \quad \varepsilon \leq d(x,y) < \varepsilon + \delta(\varepsilon) \Longrightarrow d(Tx,Ty) < \varepsilon. \tag{2.8}$$

The class of MK-contractions of the first kind was introduced by Meir and Keeler in [33]. It is easy to show that every contraction is an MK-contraction of the first kind.

Definition 2.9. A self-mapping $T : A \to A$ is said to be an MK-contraction of the second kind if, for all $\varepsilon > 0$, there exists $\gamma(\varepsilon) > 0$ such that

$$x,y \in A, \quad \varepsilon \leq d(Tx,Ty) < \varepsilon + \gamma(\varepsilon) \Longrightarrow d\left(T^2x,T^2y\right) < \varepsilon. \tag{2.9}$$

Clearly every MK-contraction of the first kind is an MK-contraction of the second kind.

Definition 2.10. A non-self-mappping $T : A \to B$ is said to be an MK-proximal contraction of the first kind if, for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\begin{gathered} x, y, u, v \in A, \quad d(u, Tx) = d(v, Ty) = d(A, B), \\ \varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon) \Longrightarrow d(u, v) < \varepsilon. \end{gathered} \tag{2.10}$$

Clearly, if $B = A$, an MK-proximal contraction of the first kind is an MK-contraction of the first kind.

Lemma 2.11. *Let $T : A \to B$ be an MK-proximal contraction of the first kind. Suppose that the pair (A, B) has the weakly P-property. Then,*

$$x, y, u, v \in A, \quad d(u, Tx) = d(v, Ty) = d(A, B) \Longrightarrow d(u, v) \leq d(x, y). \tag{2.11}$$

Proof. Let $x, y, u, v \in A$ such that $d(u, Tx) = d(v, Ty) = d(A, B)$ and $x \neq y$. Let $\varepsilon = d(x, y) > 0$. Since $\varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon)$, we have $d(u, v) < \varepsilon = d(x, y)$. If $x = y$, since (A, B) has the weakly P-property, we get that $u = v$, that is, $d(Tx, Ty) = d(u, v)$. □

Definition 2.12. A non-self-mappping $T : A \to B$ is said to be an MK-proximal contraction of the second kind if, for all $\varepsilon > 0$, there exists $\gamma(\varepsilon) > 0$ such that

$$\begin{gathered} x, y, u, v \in A, \quad d(u, Tx) = d(v, Ty) = d(A, B), \\ \varepsilon \leq d(Tx, Ty) < \varepsilon + \gamma(\varepsilon) \Longrightarrow d(Tu, Tv) < \varepsilon. \end{gathered} \tag{2.12}$$

If $B = A$, an MK-proximal contraction of the second kind is an MK-contraction of the second kind.

Similarly, we have the following.

Lemma 2.13. *Let $T : A \to B$ be an MK-proximal contraction of the second kind. Suppose that the pair (A, B) has the weakly P-property. Then,*

$$x, y, u, v \in A, \quad d(u, Tx) = d(v, Ty) = d(A, B) \Longrightarrow d(Tu, Tv) \leq d(Tx, Ty). \tag{2.13}$$

3. Main Results

We have the following best proximity point result for MK-proximal contractions.

Theorem 3.1. *Let A and B be closed subsets of a complete metric space (X, d) such that A_0 is nonvoid and the pair (A, B) satisfies the weakly P-property. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions:*

(a) *T is an MK-proximal contraction of the first and second kinds;*

(b) *$T(A_0) \subseteq B_0$;*

(c) *g is an isometry;*

(d) $A_0 \subseteq g(A_0)$;

(e) T *preserves the isometric distance with respect to* g.

Then, there exists a unique element $x^* \in A$ *such that*

$$d(gx^*, Tx^*) = d(A, B). \tag{3.1}$$

Further, for any fixed element $x_0 \in A_0$, *the iterative sequence* $\{x_n\}$, *defined by*

$$d(gx_{n+1}, Tx_n) = d(A, B), \tag{3.2}$$

converges to x^*.

Proof. Let $x_0 \in A_0$ (such a point exists since $A_0 \neq \emptyset$). From conditions (b) and (d), there exists $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B). \tag{3.3}$$

Again, from conditions (b) and (d), there exists $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B). \tag{3.4}$$

Continuing this process, we can construct a sequence $\{x_n\} \subset A_0$ such that

$$d(gx_n, Tx_{n-1}) = d(A, B), \quad \forall n \in \mathbb{N}. \tag{3.5}$$

Since T is an MK-proximal contraction of the first kind and g is an isometry, it follows from (3.5) and Lemma 2.11 that

$$d(x_{n+1}, x_{n+2}) = d(gx_{n+1}, gx_{n+2}) \leq d(x_n, x_{n+1}), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.6}$$

Further, since T is an MK-proximal contraction of the second kind and preserves isometric distance with respect to g, it follows from Lemma 2.13 that

$$d(Tx_{n+1}, Tx_{n+2}) = d(T(gx_{n+1}), T(gx_{n+2})) \leq d(Tx_n, Tx_{n+1}), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.7}$$

Claim 1. We claim that $\{x_n\}$ is a Cauchy sequence.

Let $t_n = d(x_n, x_{n+1})$. From (3.6), $\{t_n\}$ is a nonnegative, bounded below and decreasing sequence of real numbers and hence converges to some nonnegative real number t ($t_n \to t^+$). Let us suppose that $t > 0$. This implies that there exists $p \in \mathbb{N}$ such that $t \leq d(x_{p-1}, x_p) < t+\delta(t)$. Since T is an MK-proximal contraction of the first kind and g is an isometry, this implies that $d(x_p, x_{p+1}) = d(gx_p, gx_{p+1}) < t$, which is a contradiction. Thus we have $t = 0$, that is,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

Fix $\varepsilon > 0$. Without restriction of the generality, we can suppose that $\delta(\varepsilon) \leq \varepsilon$. Then, there exists $N \in \mathbb{N}$ such that

$$d(x_N, x_{N+1}) < \delta(\varepsilon). \tag{3.9}$$

Let us denote by $A(x_N, \varepsilon)$ the subset of A defined by

$$A(x_N, \varepsilon) := \{x \in A : d(x_N, x) < \varepsilon + \delta(\varepsilon)\}. \tag{3.10}$$

We will prove that

$$x \in A(x_N, \varepsilon), \quad d(gu, Tx) = d(A, B), \quad u \in A \Longrightarrow u \in A(x_N, \varepsilon). \tag{3.11}$$

Let $x \in A(x_N, \varepsilon)$ such that $d(gu, Tx) = d(A, B)$ for some $u \in A$. We distinguish two cases.

Case 1. If $d(x_N, x) \leq \varepsilon$, we have

$$\begin{aligned}
d(u, x_N) &\leq d(u, x_{N+1}) + d(x_{N+1}, x_N), \\
(\text{since } g \text{ is an isometry}) &= d(gu, gx_{N+1}) + d(x_{N+1}, x_N), \\
(\text{from (3.9)}) &< d(gu, gx_{N+1}) + \delta(\varepsilon), \\
(\text{from Lemma 2.11}) &\leq d(x, x_N) + \delta(\varepsilon) \\
&\leq \varepsilon + \delta(\varepsilon).
\end{aligned} \tag{3.12}$$

Case 2. If $\varepsilon < d(x_N, x) < \varepsilon + \delta(\varepsilon)$, we have

$$\begin{aligned}
&d(u, x_N) \leq d(u, x_{N+1}) + d(x_{N+1}, x_N), \\
&(\text{since } g \text{ is an isometry}) = d(gu, gx_{N+1}) + d(x_{N+1}, x_N), \\
&(\text{from (3.9)}) < d(gu, gx_{N+1}) + \delta(\varepsilon), \\
&(\text{since } T \text{ is an MK-proximal contraction of the first kind}) < \varepsilon + \delta(\varepsilon).
\end{aligned} \tag{3.13}$$

Thus, in all cases, we have $u \in A(x_N, \varepsilon)$, and (3.11) is proved.

Now, we shall prove that

$$x_n \in A(x_N, \varepsilon), \quad \forall n \geq N. \tag{3.14}$$

Clearly, for $n = N$, (3.14) is satisfied. Moreover, from (3.9), (3.14) is satisfied for $n = N + 1$. Now, from (3.5), we have $d(gx_{N+2}, Tx_{N+1}) = d(A, B)$. Since we have also $x_{N+1} \in A(x_N, \varepsilon)$, from (3.11), we get that $x_{N+2} \in A(x_N, \varepsilon)$. Continuing this process, by induction, we get (3.14).

Finally, for all $n, m \geq N$, from (3.14), we have

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_m, x_N) < 2(\varepsilon + \delta(\varepsilon)) \leq 4\varepsilon, \tag{3.15}$$

which implies that $\{x_n\}$ is a Cauchy sequence. Thus, Claim 1 is proved.

Claim 2. We claim that $\{Tx_n\}$ is a Cauchy sequence.

Let $s_n = d(Tx_n, Tx_{n+1})$. From (3.7), $\{s_n\}$ is a nonnegative, bounded below and decreasing sequence of real numbers and hence converges to some nonnegative real number s. Let us suppose that $s > 0$. This implies that there exists $p \in \mathbb{N}$ such that $s \leq d(Tx_{p-1}, Tx_p) < s + \gamma(s)$. Since T is an MK-proximal contraction of the second kind and T preserves the isometric distance with resect to g, we get

$$d(Tx_p, Tx_{p+1}) = d(T(gx_p), T(gx_{p+1})) < s, \tag{3.16}$$

which is a contradiction. Thus we have $s = 0$, that is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0. \tag{3.17}$$

Fix $\varepsilon > 0$. Without restriction of the generality, we can suppose that $\gamma(\varepsilon) \leq \varepsilon$. Then, there exists $N \in \mathbb{N}$ such that

$$d(Tx_N, Tx_{N+1}) < \gamma(\varepsilon). \tag{3.18}$$

Let us denote by $\widetilde{A}(x_N, \varepsilon)$ the subset of A defined by

$$\widetilde{A}(x_N, \varepsilon) := \{x \in A : d(Tx_N, Tx) < \varepsilon + \gamma(\varepsilon)\}. \tag{3.19}$$

We shall prove that

$$x \in \widetilde{A}(x_N, \varepsilon), \quad d(gu, Tx) = d(A, B), \quad u \in A \Longrightarrow u \in \widetilde{A}(x_N, \varepsilon). \tag{3.20}$$

Let $x \in \widetilde{A}(x_N, \varepsilon)$ such that $d(gu, Tx) = d(A, B)$ for some $u \in A$. We distinguish two cases.

Case 1. If $d(Tx_N, Tx) \leq \varepsilon$, we have

$$\begin{aligned} d(Tu, Tx_N) &\leq d(Tu, Tx_{N+1}) + d(Tx_{N+1}, Tx_N), \\ (\text{from (e)}) &= d(T(gu), T(gx_{N+1})) + d(Tx_{N+1}, Tx_N), \\ (\text{from (3.18)}) &< d(T(gu), T(gx_{N+1})) + \gamma(\varepsilon), \\ (\text{from Lemma 2.13}) &\leq d(Tx, Tx_N) + \gamma(\varepsilon) \\ &\leq \varepsilon + \gamma(\varepsilon). \end{aligned} \tag{3.21}$$

Case 2. If $\varepsilon < d(Tx_N, Tx) < \varepsilon + \gamma(\varepsilon)$, we have

$$\begin{aligned}
&d(Tu, Tx_N) \le d(Tu, Tx_{N+1}) + d(Tx_{N+1}, Tx_N), \\
&(\text{from (e)}) = d(T(gu), T(gx_{N+1})) + d(x_{N+1}, x_N), \\
&(\text{from (3.18)}) < d(T(gu), T(gx_{N+1})) + \gamma(\varepsilon), \\
&(\text{since } T \text{ is an MK-proximal contraction of the second kind}) < \varepsilon + \gamma(\varepsilon).
\end{aligned} \tag{3.22}$$

Thus, in all cases, we have $u \in \widetilde{A}(x_N, \varepsilon)$, and (3.20) is proved.

Now, we shall prove that

$$x_n \in \widetilde{A}(x_N, \varepsilon), \quad \forall n \ge N. \tag{3.23}$$

Clearly, for $n = N$, (3.23) is satisfied. Moreover, from (3.18), (3.23) is satisfied for $n = N + 1$. Since $d(gx_{N+2}, Tx_{N+1}) = d(A, B)$ and $x_{N+1} \in \widetilde{A}(x_N, \varepsilon)$, from (3.20), we have $x_{N+2} \in \widetilde{A}(x_N, \varepsilon)$. Continuing this process, by induction, we get (3.23).

Finally, for all $n, m \ge N$, from (3.23), we have

$$d(Tx_n, Tx_m) \le d(Tx_n, Tx_N) + d(Tx_m, Tx_N) < 2(\gamma(\varepsilon) + \varepsilon) \le 4\varepsilon, \tag{3.24}$$

which implies that $\{Tx_n\}$ is a Cauchy sequence. Thus, Claim 2 is proved.

Now, since (X, d) is complete and A is closed, there exists $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$. Similarly, since B is closed, there exists $y^* \in B$ such that $Tx_n \to y^*$ as $n \to \infty$. Therefore, we have

$$d(gx^*, y^*) = \lim_{n \to \infty} d(gx_{n+1}, Tx_n) = d(A, B). \tag{3.25}$$

This implies that $gx^* \in A_0$. From condition (d), there exists $x_* \in A_0$ such that $gx^* = gx_*$. On the other hand, since g is an isometry, we get that $d(x^*, x_*) = d(gx^*, gx_*) = 0$, which implies that $x^* = x_* \in A_0$. Now, from condition (b), we have $Tx^* \in B_0$; that is, there exists $\overline{x} \in A_0$ such that $d(\overline{x}, Tx^*) = d(A, B)$. Since T is an MK-proximal contraction of the first kind, using Lemma 2.11, we get that

$$d(\overline{x}, gx_{n+1}) \le d(x^*, x_n). \tag{3.26}$$

Letting $n \to \infty$, it follows that that $\overline{x} = gx^*$. Thus, it can be concluded that

$$d(gx^*, Tx^*) = d(A, B). \tag{3.27}$$

To assert the uniqueness, let us assume that z^* is another element in A such that $d(gz^*, Tz^*) = d(A, B)$. Due to the fact that T is an MK-proximal contraction of the first kind and g is an isometry, using Lemma 2.11, we get that

$$d(x^*, z^*) = d(gx^*, gz^*) < d(x^*, z^*), \tag{3.28}$$

which is a contradiction. Then x^* and z^* are identical. This completes the proof. □

If g is the identity mapping, then Theorem 3.1 yields the following result.

Corollary 3.2. *Let A and B be closed subsets of a complete metric space (X, d) such that A_0 is nonvoid and the pair (A, B) satisfies the weakly P-property. Suppose that the mapping $T : A \to B$ satisfies the following conditions:*

(a) *T is an MK-proximal contraction of the first and second kinds;*

(b) $T(A_0) \subseteq B_0$.

Then, there exists a unique element $x^* \in A$ such that

$$d(x^*, Tx^*) = d(A, B). \tag{3.29}$$

Further, for any fixed element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B), \tag{3.30}$$

converges to x^*.

Example 3.3. We endow $X = \mathbb{R}^4$ with the standard metric:

$$d((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - x_4|, \tag{3.31}$$

for all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. Consider the sets $A, B \subset X$ defined by

$$\begin{aligned} A &:= \left\{ \left(0, 0, \frac{1}{n}, \frac{2}{n}\right) : n \in \mathbb{N} \right\} \bigcup \{(0,0,0,0)\}, \\ B &:= \left\{ \left(1, 2, \frac{1}{n}, \frac{2}{n}\right) : n \in \mathbb{N} \right\} \bigcup \{(1,2,0,0)\}. \end{aligned} \tag{3.32}$$

Then A and B are nonempty closed subsets of $\mathbb{R}^4$ with $d(A, B) = 3$. It is easy to show that in this case we have $A_0 = A$ and $B_0 = B$. On the other hand, for all $(x, y) \in A \times B$, we have

$$\begin{aligned} &d(x, y) = 3 \\ &\quad \Longleftrightarrow (x, y) \in \left\{ \left(\left(0, 0, \frac{1}{n}, \frac{2}{n}\right), \left(1, 2, \frac{1}{n}, \frac{2}{n}\right) \right), ((0,0,0,0), (1,2,0,0)) : n \in \mathbb{N} \right\} := \Omega. \end{aligned} \tag{3.33}$$

Clearly, for all $x_1, x_2 \in A$, $y \in B$,

$$(x_1, y), (x_2, y) \in \Omega \Longrightarrow x_1 = x_2. \tag{3.34}$$

Thus the pair (A, B) satisfies the weakly P-property.

Let $T : A \to B$ be the mapping defined by

$$T\left(0,0,\frac{1}{n},\frac{2}{n}\right) = \left(1,2,\frac{1}{3n},\frac{2}{3n}\right), \quad \forall\, n \in \mathbb{N},$$
$$T(0,0,0,0) = (1,2,0,0). \tag{3.35}$$

Claim 1. T is an MK-proximal contraction of the first kind.

Let $\varepsilon > 0$ be fixed and $\delta(\varepsilon) = 2\varepsilon$. Let $x, y, u, v \in A$ such that

$$d(u,Tx) = d(v,Ty) = d(A,B) = 3, \qquad \varepsilon \le d(x,y) < \varepsilon + \delta(\varepsilon) = 3\varepsilon. \tag{3.36}$$

We consider three cases.

Case 1. There exist $p, k \in \mathbb{N}$ such that

$$u = \left(0,0,\frac{1}{3p},\frac{2}{3p}\right), \qquad x = \left(0,0,\frac{1}{p},\frac{2}{p}\right),$$
$$v = \left(0,0,\frac{1}{3k},\frac{2}{3k}\right), \qquad y = \left(0,0,\frac{1}{k},\frac{2}{k}\right). \tag{3.37}$$

In this case, from (3.36), we have

$$d(u,v) = \left|\frac{1}{p} - \frac{1}{k}\right| < \varepsilon. \tag{3.38}$$

Case 2. There exists $p \in \mathbb{N}$ such that

$$u = \left(0,0,\frac{1}{3p},\frac{2}{3p}\right), \qquad x = \left(0,0,\frac{1}{p},\frac{2}{p}\right), \qquad v = y = (0,0,0,0). \tag{3.39}$$

In this case, from (3.36), we have

$$d(u,v) = \frac{1}{p} < \varepsilon. \tag{3.40}$$

Case 3. There exists $k \in \mathbb{N}$ such that

$$u = (0,0,0,0) = x, \qquad v = \left(0,0,\frac{1}{3k},\frac{2}{3k}\right), \qquad y = \left(0,0,\frac{1}{k},\frac{2}{k}\right). \tag{3.41}$$

In this case, from (3.36), we have

$$d(u,v) = \frac{1}{k} < \varepsilon. \tag{3.42}$$

Thus, in all cases we have $d(u,v) < \varepsilon$. Then Claim 1 holds.

Claim 2. T is an MK-proximal contraction of the second kind.

Let $\varepsilon > 0$ be fixed and $\gamma(\varepsilon) = 2\varepsilon$. Let $x, y, u, v \in A$ such that

$$d(u, Tx) = d(v, Ty) = d(A, B) = 3, \qquad \varepsilon \le d(Tx, Ty) < \varepsilon + \gamma(\varepsilon) = 3\varepsilon. \tag{3.43}$$

We consider also three cases.

Case 1. There exist $p, k \in \mathbb{N}$ such that

$$\begin{aligned} u &= \left(0, 0, \frac{1}{3p}, \frac{2}{3p}\right), \qquad x = \left(0, 0, \frac{1}{p}, \frac{2}{p}\right), \\ v &= \left(0, 0, \frac{1}{3k}, \frac{2}{3k}\right), \qquad y = \left(0, 0, \frac{1}{k}, \frac{2}{k}\right). \end{aligned} \tag{3.44}$$

In this case, from (3.43), we have

$$d(Tu, Tv) = \frac{3}{9}\left|\frac{1}{p} - \frac{1}{k}\right| < \varepsilon. \tag{3.45}$$

Case 2. There exists $p \in \mathbb{N}$ such that

$$u = \left(0, 0, \frac{1}{3p}, \frac{2}{3p}\right), \qquad x = \left(0, 0, \frac{1}{p}, \frac{2}{p}\right), \qquad v = y = (0, 0, 0, 0). \tag{3.46}$$

In this case, from (3.43), we have

$$d(Tu, Tv) = \frac{1}{3p} < \varepsilon. \tag{3.47}$$

Case 3. There exists $k \in \mathbb{N}$ such that

$$u = (0, 0, 0, 0) = x, \qquad v = \left(0, 0, \frac{1}{3k}, \frac{2}{3k}\right), \qquad y = \left(0, 0, \frac{1}{k}, \frac{2}{k}\right). \tag{3.48}$$

In this case, from (3.43), we have

$$d(Tu, Tv) = \frac{1}{3k} < \varepsilon. \tag{3.49}$$

Thus, in all cases we have $d(Tu, Tv) < \varepsilon$. Then Claim 2 holds.

Finally, from Corollary 3.2, there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B) = 3$. In this example, we have $x^* = (0, 0, 0, 0)$.

The preceding best proximity point result (see Corollary 3.2) gives rise to the following fixed point theorem, due to Meir and Keeler [33], which in turn extends the famous Banach contraction principle [34].

Corollary 3.4 (Meir-Keeler [33]). *Let (X,d) be a complete metric space and $T : X \to X$ be an MK-contraction of the first kind. Then T has a unique fixed point $x^* \in X$, and for each $x \in X$, $\lim_{n\to\infty} T^n x = x^*$.*

The following result furnishes another best proximity point theorem for MK-proximal contractions.

Theorem 3.5. *Let A and B be closed subsets of a complete metric space (X,d) such that A_0 is nonvoid, the pair (A,B) satisfies the weakly P-property, and B is approximatively compact with respect to A. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions:*

(a) *T is an MK-proximal contraction of the first kind;*

(b) *$T(A_0) \subseteq B_0$;*

(c) *g is an isometry;*

(d) *$A_0 \subseteq g(A_0)$.*

Then, there exists a unique element $x^* \in A$ such that

$$d(gx^*, Tx^*) = d(A,B). \tag{3.50}$$

Further, for any fixed element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Tx_n) = d(A,B), \tag{3.51}$$

converges to x^*.

Proof. Proceeding as in Theorem 3.1, it can be shown that there is a sequence $\{x_n\}$ of elements in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A,B), \tag{3.52}$$

and any sequence satisfying the above condition must converge to some element $x^* \in A$. On the other hand, we have

$$\begin{aligned} d(gx^*, B) &\leq d(gx^*, Tx_n) \\ &\leq d(gx^*, gx_{n+1}) + d(gx_{n+1}, Tx_n) \\ &= d(x^*, x_{n+1}) + d(A,B) \\ &\leq d(x^*, x_{n+1}) + d(gx^*, B). \end{aligned} \tag{3.53}$$

Letting $n \to \infty$ in the above inequality, we get $d(gx^*, Tx_n) \to d(gx^*, B)$ as $n \to \infty$. Since B is approximatively compact with respect to A, it follows that the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n(k)}\}$ converging to some element $y^* \in B$. Thus we have

$$d(gx^*, y^*) = \lim_{k\to\infty} d(gx_{n(k)+1}, Tx_{n(k)}) = d(A,B). \tag{3.54}$$

The rest part of the proof follows as in Theorem 3.1. □

If g is the identity mapping, the preceding best proximity point theorem yields the following special case.

Corollary 3.6. *Let A and B be closed subsets of a complete metric space (X,d) such that A_0 is nonvoid, the pair (A,B) satisfies the weakly P-property, and B is approximatively compact with respect to A. Suppose that the mapping $T : A \to B$ satisfies the following conditions:*

(a) *T is an MK-proximal contraction of the first kind;*

(b) $T(A_0) \subseteq B_0$.

Then, there exists a unique element $x^* \in A$ such that

$$d(x^*, Tx^*) = d(A,B). \tag{3.55}$$

Further, for any fixed element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A,B), \tag{3.56}$$

converges to x^*.

Acknowledgment

The first and third authors are supported by the Research Center, College of Science, King Saud University.

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