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Research Article

Existence and Uniqueness Theorems of Ordered Contractive Map in Banach Lattices

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This paper presents some existence and uniqueness theorems of the fixed point for ordered contractive mapping in Banach lattices. Moreover, we prove the existence of a unique solution for first-order ordinary differential equations with initial value conditions by using the theoretical results with no need for using the condition of a lower solution or an upper solution.

1. Introduction and Preliminaries

Existence of fixed points in partial ordered complete metric spaces has been considered further recently in [1–6]. Many new fixed point theorems are proved in a metric space endowed with partial order by using monotone iterative technique, and their results are applied to problems of existence and uniqueness of solutions for some differential equation problems. In [6] the existence of a minimal and a maximal solution for a nonlinear problem is presented by constructing an iterative sequence with the condition of a lower solution or an upper solution.

In this paper, the theoretical results of fixed points are extended by using the theorem of cone and monotone iterative technique in Banach lattices. But the iterative sequences can be constructed with no need for using the condition of a lower solution or an upper solution. To demonstrate the applicability of our results, we apply them to study a problem of ordinary differential equations in the final section of the paper, and the existence and uniqueness of solution are obtained.

Let E be a Banach space and P a cone of E. We define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone $P \subset E$ is called normal if there is a constant $N > 0$, such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, for all $x, y \in E$. The least positive constant N satisfying the above inequality is called the normal constant of P.

Let E be a Riesz space equipped with a Riesz norm. We call E a Banach lattice in the partial ordering $\leq$, if E is norm complete. For arbitrary $x, y \in E$, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist. One can see [7] for the definition and the properties about the lattice.

Let $D \subset E$; the operator $A : D \to E$ is said to be an increasing operator if $x, y \in D$, $x \leq y$, implies $Ax \leq Ay$; the operator $A : D \to E$ is said to be a decreasing operator if $x, y \in D$, $x \leq y$, implies $Ay \leq Ax$.

Lemma 1.1 (see [8]). *Let P be a normal cone in a real Banach space E. Suppose that $\{x_n\}$ is a monotone sequence which has a subsequence $\{x_{n_i}\}$ converging to x^*, then $\{x_n\}$ also converges to x^*. Moreover, if $\{x_n\}$ is an increasing sequence, then $x_n \leq x^*$ $(n = 1, 2, 3, \ldots)$; if $\{x_n\}$ is a decreasing sequence, then $x^* \leq x_n$ $(n = 1, 2, 3, \ldots)$.*

Lemma 1.2 (see [9]). *Let Ω be a bounded open set in a real Banach space E such that $\theta \in \Omega$; let P be a cone of E. Let $A : P \cap \overline{\Omega} \to P$ is completely continuous. Suppose that*

$$x \nleq Ax, \quad \forall x \in P \cap \overline{\Omega}. \tag{1.1}$$

Then $i(A, P \cap \Omega, P) = 1$.

Lemma 1.3 (see [9]). *Let E be a real Banach space, and let $P \subset E$ be a cone. Assume Ω_1 and Ω_2 are two bounded open subsets of E with $\theta \in \Omega_1 \subset \Omega_2$ and $\overline{\Omega_1} \subset \Omega_2$, and let $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is completely continuous. Suppose that either*

H_1 *$x \nleq Ax$, for all $x \in P \cap \overline{\Omega_1}$ and $Ax \nleq x$, for all $x \in P \cap \overline{\Omega_2}$, or*

H_2 *$Ax \nleq x$, for all $x \in P \cap \overline{\Omega_1}$ and $x \nleq Ax$, for all $x \in P \cap \overline{\Omega_2}$.*

Then A has a fixed point in $P \cap (\Omega_2 \setminus \overline{\Omega_1})$.

2. Main Results

Theorem 2.1. *Let E be a real Banach lattice, and let $P \subset E$ be a normal cone. Suppose that $A : E \to E$ is a decreasing operator such that there exists a linear operator $L : E \to E$ with spectral radius $r(L) < 1$ and*

$$Av - Au \leq L(u - v), \quad \text{for } u, v \in E \text{ with } v \leq u. \tag{2.1}$$

Then the operator A has a unique fixed point.

Proof. For any $u_0 \in E$, since $A : E \to E$, we have $Au_0 \in E$. Now we suppose the following two cases.

Case (I). Suppose that u_0 is comparable to Au_0. Firstly, without loss of generality, suppose that $u_0 \leq Au_0$. If $Au_0 = u_0$, then the proof is finished. Suppose $Au_0 \neq u_0$. Since A is decreasing together with $u_0 \leq Au_0$, we obtain by induction that $\{A^{n+1}(u_0)\}$ and $\{A^n(u_0)\}$ are comparable, for every $n = 0, 1, 2, \ldots$. Using the contractive condition (2.1), we can obtain by induction that

$$\left\| A^{n+1}(u_0) - A^n(u_0) \right\| \leq N \| L^n (Au_0 - u_0) \|, \quad n \in N. \tag{2.2}$$

In fact, for $n = 1$, using the fact that P is normal, we have

$$\left\| A(u_0) - A^2 u_0 \right\| \le N \| L(Au_0 - u_0) \|. \tag{2.3}$$

Suppose that (2.2) is true when $n = k$ then when $n = k + 1$, we obtain

$$\begin{aligned} \left\| A^{n+2}(u_0) - A^{n+1}(u_0) \right\| &= \left\| A\left(A^{n+1}(u_0)\right) - A(A^n(u_0)) \right\| \\ &\le N \left\| L\left(A^{n+1}(u_0) - A^n(u_0)\right) \right\| \le N \left\| L^{n+1}(Au_0 - u_0) \right\|. \end{aligned} \tag{2.4}$$

For any $m, n \in N$, $m > n$, since P is normal cone, we have

$$\begin{aligned} \| A^m(u_0) - A^n(u_0) \| &= \left\| \left(A^m(u_0) - A^{m-1}(u_0)\right) + \cdots + \left(A^{n+1}(u_0) - A^n(u_0)\right) \right\| \\ &\le N \left\| \left(L^{m-1} + L^{m-2} + \cdots + L^n\right)(Au_0 - u_0) \right\| \\ &\le N r\left(\left(L^{m-1} + L^{m-2} + \cdots + L^n\right)\right) \| Au_0 - u_0 \| \\ &\le N\left(r\left(L^{m-1}\right) + r\left(L^{m-2}\right) + \cdots + r(L^n)\right) \| Au_0 - u_0 \|. \end{aligned} \tag{2.5}$$

Here N is the normal constant.

Given a α such that $r(L) < \alpha < 1$, since $\lim_{n \to +\infty} \| L^n \|^{1/n} = r(L) < \alpha < 1$, there exists a $n_0 \in N$ such that

$$\| L^n \| < \alpha^n, \quad n \ge n_0. \tag{2.6}$$

For any $m, n \in N$, $m > n \ge n_0$, since P is normal cone, we have

$$\begin{aligned} \| A^m(u_0) - A^n(u_0) \| &\le N\left(r\left(L^{m-1}\right) + r\left(L^{m-2}\right) + \cdots + r(L^n)\right) \| Au_0 - u_0 \| \\ &\le N\left(\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n\right) \| Au_0 - u_0 \| \\ &\le N\left(\frac{\alpha^n - \alpha^m}{1 - \alpha}\right) \| Au_0 - u_0 \| \le N\left(\frac{\alpha^n}{1 - \alpha}\right) \| Au_0 - u_0 \|. \end{aligned} \tag{2.7}$$

This implies that $\{A^n(u_0)\}$ is a Cauchy sequence in E. The complete character of E implies the existence of $x^* \in P$ such that

$$\lim_{n \to +\infty} A^n(u_0) = x^*. \tag{2.8}$$

Next, we prove that x^* is a fixed point of A in E. Since A is decreasing and $u_0 \le Au_0$, we can get $A^2 u_0 \le Au_0$.

So

$$Au_0 - A^2(u_0) \leq L(Au_0 - u_0), \tag{2.9}$$

then

$$\begin{aligned} A^2u_0 - u_0 &= (Au_0 - u_0) - \left(Au_0 - A^2(u_0)\right) \\ &\geq (I - L)(Au_0 - u_0) \geq \theta. \end{aligned} \tag{2.10}$$

It is easy to know that A^2 is increasing and

$$A^2(u_0) \leq A^4(u_0), \qquad A^3(u_0) \leq A(u_0). \tag{2.11}$$

By induction, we obtain that

$$u_0 \leq A^2(u_0) \leq \cdots \leq A^{2n}(u_0) \leq \cdots \leq A^{2n+1}(u_0) \leq \cdots \leq A^3(u_0) \leq Au_0. \tag{2.12}$$

Hence, the sequence $\{A^n(u_0)\}$ has an increasing Cauchy subsequence $\{A^{2n}(u_0)\}$ and a decreasing Cauchy subsequence $\{A^{2n+1}(u_0)\}$ such that

$$\lim_{n\to+\infty} A^{2n}(u_0) = u^*, \qquad \lim_{n\to+\infty} A^{2n+1}(u_0) = v^*. \tag{2.13}$$

Thus Lemma 1.1 implies that $A^{2n}(u_0) \leq u^*, v^* \leq A^{2n+1}(u_0)$.

Since $\{A^n(u_0)\}$ is a Cauchy sequence, we can get that $u^* = v^* = x^*$.

Moreover

$$\begin{aligned} \|Ax^* - x^*\| &\leq \left\|Ax^* - A\left(A^{2n}(u_0)\right)\right\| + \left\|A^{2(n+1)}(u_0) - x^*\right\| \\ &\leq N\left\|L\left(x^* - A^{2n}(u_0)\right)\right\| + \left\|A^{2(n+1)}(u_0) - x^*\right\| \\ &\leq N\alpha\left\|x^* - A^{2n}(u_0)\right\| + \left\|A^{2(n+1)}(u_0) - x^*\right\|. \end{aligned} \tag{2.14}$$

Thus $\|Ax^* - x^*\| = 0$. That is $Ax^* = x^*$. Hence x^* is a fixed point of A in E.

Case (II). On the contrary, suppose that u_0 is not comparable to Au_0.

Now, since E is a Banach lattice, there exists v_0 such that $\inf\{Au_0, u_0\} = v_0$. That is $v_0 \leq Au_0$ and $v_0 \leq u_0$. Since A is a decreasing operator, we have

$$A^2u_0 \leq Av_0, \qquad Au_0 \leq Av_0. \tag{2.15}$$

This shows that $v_0 \leq Av_0$. Similarly as the proof of case (I), we can get that A has a fixed point x^* in E.

Finally, we prove that A has a unique fixed point x^* in E. In fact, let u^* and v^* be two fixed points of A in E.

(1) If u^* is comparable to v^*, $A^n(u^*) = u^*$ is comparable to $A^n(v^*) = v^*$ for every $n = 0, 1, 2, \ldots$, and

$$\|u^* - v^*\| = \|A^n u^* - A^n v^*\| \leq N\alpha^n \|u^* - v^*\|, \tag{2.16}$$

which implies $u^* = v^*$.

(2) If u^* is not comparable to v^*, there exists either an upper or a lower bound of u^* and v^* because E is a Banach lattice, that is, there exists $z^* \in E$ such that $z^* \leq u^*, z^* \leq v^*$ or $u^* \leq z^*, u^* \leq z^*$. Monotonicity implies that $A^n(z^*)$ is comparable to $A^n(u^*)$ and $A^n(v^*)$, for all $n = 0, 1, 2, \ldots$, and

$$\begin{aligned} \|u^* - v^*\| &= \|A^n(u^*) - A^n(v^*)\| \\ &\leq \|A^n(z^*) - A^n(u^*)\| + \|A^n(z^*) - A^n(v^*)\| \\ &\leq N\alpha^n \|u^* - z^*\| + N\alpha^n \|z^* - v^*\|. \end{aligned} \tag{2.17}$$

This shows that $\|u^* - v^*\| \to 0$ when $n \to +\infty$. Hence A has a unique fixed point x^* in E. □

Theorem 2.2. *Let E be a real Banach lattice, and let $P \subset E$ be a normal cone. Suppose that $A : P \to P$ is a completely continuous and increasing operator such that there exists a linear operator $L : E \to E$ with spectral radius $r(L) < 1$ and*

$$Au - Av \leq L(u - v), \quad \text{for } u, v \in P \text{ with } v \leq u. \tag{2.18}$$

Then the operator A has a unique fixed point u^* in P.

Proof. For any $r > 0$, let $\Omega = \{x \in P : \|x\| \leq r\}$. Now we suppose the following two cases.
Case (I). Firstly, suppose that there exists $u_0 \in \partial\Omega$ such that $u_0 \leq Au_0$. If $Au_0 = u_0$, then the proof is finished. Suppose $Au_0 \neq u_0$. Since $u_0 \leq Au_0$ and A is nondecreasing, we obtain by induction that

$$u_0 \leq Au_0 \leq A^2(u_0) \leq A^3(u_0) \leq \cdots \leq A^n(u_0) \leq A^{n+1}(u_0) \leq \cdots. \tag{2.19}$$

Similarly as the proof of Theorem 2.1, we can get that $\{A^n(u_0)\}$ is a Cauchy sequence in E. Since E is complete, by Lemma 1.1, there exists $u^* \in E$, $A^n(u_0) \leq u^*$ such that

$$\lim_{n \to +\infty} A^n(u_0) = u^*. \tag{2.20}$$

Next, we prove that u^* is a fixed point of A, that is, $Au^* = u^*$. In fact

$$\begin{aligned} \|Au^* - u^*\| &\leq \|Au^* - A(A^n(u_0))\| + \left\|A^{n+1}(u_0) - u^*\right\| \\ &\leq N\|L(u^* - A^n(u_0))\| + \left\|A^{n+1}(u_0) - u^*\right\| \\ &\leq N\alpha\|u^* - A^n(u_0)\| + \left\|A^{n+1}(u_0) - u^*\right\|. \end{aligned} \tag{2.21}$$

Now, by the convergence of $\{A^n(u_0)\}$ to u^*, we can get $\|Au^* - u^*\| = 0$. This proves that u^* is a fixed point of A.

Case (II). On the contrary, suppose that $x \nleq Ax$ for all $x \in \partial\Omega$. Thus Lemma 1.2 implies the existence of a fixed point in this case also.

Finally, similarly as the proof of Theorem 2.1, we can get that A has a unique fixed point x^* in P. □

Theorem 2.3. *Let E be a real Banach lattice, and let $P \subset E$ be a normal cone. Suppose that $A : P \to P$ is a completely continuous and increasing operator which satisfies the following assumptions:*

(i) *there exists a linear operator $L : E \to E$ with spectral radius $r(L) < 1$ and*

$$Au - Av \le L(u - v), \quad \text{for } u, v \in P \text{ with } v \le u; \tag{2.22}$$

(ii) *$S = \{x \in P : Ax \le x\}$ is bounded.*

Then the operator A has a unique nonzero fixed point u^* in P.

Proof. Firstly, for any $r > 0$, let $\Omega = \{x \in P : \|x\| \le r\}$. Now we suppose the following two cases.

Case (I). Suppose that there exists $u_0 \in \partial\Omega$ such that $u_0 \le Au_0$. Similarly as proof of Theorem 2.1, we get that A has a nonzero fixed point u^* in P.

Case (II). On the contrary, suppose that $x \nleq Ax$ for all $x \in \partial\Omega$. Now, since S is bounded there exists $R > r$ such that $Ax \nleq x$ for all $x \in P$ with $\|x\| = R$. Thus Lemma 1.3 implies the existence of a nonzero fixed point in this case.

Finally, similarly as the proof of Theorem 2.1, we can get that A has a unique non-zero fixed point u^* in P. □

3. Applications

In this section, we use Theorem 2.1 to show the existence of unique solution for the first-order initial value problem

$$\begin{gathered} u'(t) = f(t, u(t)), \quad t \in I = [0, T], \\ u(0) = u_0, \end{gathered} \tag{3.1}$$

where $T > 0$ and $f : I \times R \to R$ is a continuous function.

Theorem 3.1. *Let $f : I \times R \to R$ be continuous, and suppose that there exists $0 < \mu < \lambda$, such that*

$$-\mu(y - x) \le f(t, y) + \lambda y - [f(t, x) + \lambda x] \le 0, \quad \forall y \ge x. \tag{3.2}$$

Then (3.1) *has a unique solution u^*.*

Proof. It is easy to know that $E = C(I)$ is a Banach space with maximum norm $\|\cdot\|$, and it is also a Banach lattice with maximum norm $\|\cdot\|$. Let $P = \{u \in E | u(t) \geq 0, \text{for all } t \in I\}$, and P is a normal cone in Banach lattice E. Equation (3.1) can be written as

$$u'(t) + \lambda u(t) = f(t, u(t)) + \lambda u(t), \quad t \in I = [0, T],$$
$$u(0) = u_0. \tag{3.3}$$

This problem is equivalent to the integral equation

$$u(t) = e^{-\lambda t}\left\{u_0 + \int_0^t e^{\lambda s}[f(s, u(s)) + \lambda u(s)]ds\right\}. \tag{3.4}$$

Define operator A as the following:

$$(Au)(t) = e^{-\lambda t}\left\{u_0 + \int_0^t e^{\lambda s}[f(s, u(s)) + \lambda u(s)]ds\right\}, \quad t \in I. \tag{3.5}$$

Moreover, the mapping A is decreasing in u. In fact, by hypotheses, for $u \geq v$,

$$f(t, u(t)) + \lambda u(t) \leq f(t, v(t)) + \lambda v(t) \tag{3.6}$$

implies that

$$\begin{aligned}(Au)(t) &= e^{-\lambda t}\left\{u_0 + \int_0^t e^{\lambda s}[f(s, u(s)) + \lambda u(s)]ds\right\} \\ &\leq e^{-\lambda t}\left\{u_0 + \int_0^t e^{\lambda s}[f(s, v(s)) + \lambda v(s)]ds\right\} = (Av)(t), \quad t \in I,\end{aligned} \tag{3.7}$$

so A is decreasing. Besides, for $u \geq v$,

$$\begin{aligned}A(v) - A(u) &= \int_0^t e^{\lambda(s-t)}[f(s, v(s) + \lambda v(s) - f(s, u(s)) - \lambda u(s)]\ ds \\ &\leq \int_0^t e^{\lambda(s-t)}\mu[u(s) - v(s)]ds = L(u - v),\end{aligned} \tag{3.8}$$

where $Lu = \int_0^t e^{\lambda(s-t)}\mu u(s)ds$. Since A is decreasing, then L is positive linear operator.

Now, let us prove that the spectral radius $r(L) < 1$. For $t \in I$, since $0 < e^{\lambda(s-t)} \leq 1$, we have

$$\|Lu\| = \max_{t\in I} \int_0^t e^{\lambda(s-t)} \mu u(s) ds \leq \mu \int_0^t e^{\lambda(s-t)} ds \|u\| \leq \mu t \|u\|,$$
$$\left\|L^2 u\right\| = \max_{t\in I} \int_0^t e^{\lambda(s-t)} \mu L(u(s)) ds \leq \mu^2 \int_0^t e^{\lambda(s-t)} s\, ds \|Lu\| \leq \frac{\mu^2}{2!} t^2 \|u\|. \tag{3.9}$$

By mathematical induction, for any $n \in N$, we have

$$\|L^n u\| \leq \frac{\mu^n}{n!} t^n \|u\|, \quad t \in I. \tag{3.10}$$

So

$$\|L^n\| \leq \frac{\mu^n}{n!} T^n. \tag{3.11}$$

Since $0 < \mu < \lambda$, we have

$$r(L) = \lim_{n\to+\infty} \|L^n\|^{1/n} = 0 < 1. \tag{3.12}$$

So the condition of Theorem 2.1 holds, and Theorem 3.1 is proved. □

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