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Research Article

A Note on the Stability of the Integral-Differential Equation of the Parabolic Type in a Banach Space

Maksat Ashyraliyev

Department of Mathematics and Computer Sciences, Bahcesehir University, Besiktas, 34353 Istanbul, Turkey

Correspondence should be addressed to Maksat Ashyraliyev, mashyr@yahoo.com

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The integral-differential equation of the parabolic type in a Banach space is considered. The unique solvability of this equation is established. The stability estimates for the solution of this equation are obtained. The difference scheme approximately solving this equation is presented. The stability estimates for the solution of this difference scheme are obtained.

1. Introduction

We consider the integral-differential equation

$$\frac{du(t)}{dt} + \operatorname{sgn}(t) A u(t) = \int_{-t}^{t} B(s) u(s) ds + f(t), \quad -1 \le t \le 1 \tag{1.1}$$

in an arbitrary Banach space E with unbounded linear operators A and $B(t)$ in E with dense domain $D(A) \subset D(B(t))$ and

$$\left\| B(t) A^{-1} \right\|_{E \to E} \le M, \quad -1 \le t \le 1. \tag{1.2}$$

A function $u(t)$ is called a solution of (1.1) if the following conditions are satisfied:

(i) $u(t)$ is continuously differentiable on $[-1, 1]$. The derivatives at the endpoints are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [-1,1]$, and the functions $Au(t)$ and $B(t)u(t)$ are continuous on $[-1,1]$.

(iii) $u(t)$ satisfies (1.1).

A solution of (1.1) defined in this manner will from now on be referred to as a solution of (1.1) in the space $C(E) = C([-1,1],E)$ of all continuous functions $\varphi(t)$ defined on $[-1,1]$ with values in E equipped with the norm

$$\|\varphi\|_{C(E)} = \max_{-1\leq t\leq 1}\|\varphi(t)\|_{E}. \tag{1.3}$$

We consider (1.1) under the assumption that the operator $-A$ generates an analytic semigroup $\exp\{-tA\}(t \geq 0)$, that is, the following estimates hold:

$$\left\|e^{-tA}\right\|_{E\to E} \leq M, \qquad \left\|tAe^{-tA}\right\|_{E\to E} \leq M, \quad 0 \leq t \leq 1. \tag{1.4}$$

Integral inequalities play a significant role in the theory of differential and integral-differential equations. They are useful to investigate some properties of the solutions of equations, such as existence, uniqueness and stability, see for instance [1–11].

Mathematical modelling of real-life phenomena is widely used in various applied fields of science. This is based on the mathematical description of real-life processes and the subsequent solving of the appropriate mathematical problems on the computer. The mathematical models of many real-life problems lead to already known or new differential and integral-differential equations. In most of the cases it is difficult to find the exact solutions of the differential and integral-differential equations. For this reason discrete methods play a significant role, especially with the appearance of highly efficient computers. A well-known and widely applied method of approximate solutions for differential and integral-differential equations is the method of difference schemes. Modern computers allow us to implement highly accurate difference schemes. Hence, the task is to construct and investigate highly accurate difference schemes for various types of differential and integral-differential equations. The investigation of stability and convergence of these difference schemes is based on the discrete analogues of integral inequalities.

Gronwall in 1919 showed the following result [12].

Lemma 1.1. *If $M = const > 0$, $\delta = const > 0$, and continuous function $x(t) \geq 0$ satisfies the inequalities*

$$x(t) \leq \delta + M\int_0^t x(s)ds, \quad 0 \leq t \leq T, \tag{1.5}$$

then

$$x(t) \leq \delta \exp[Mt], \quad 0 \leq t \leq T. \tag{1.6}$$

A number of different generalizations of Gronwall's integral inequality with one and two dependent limits have been obtained, see for instance [13, 14].

In numerical analysis literature, see for instance [15, 16], one can find the following discrete analogue of Lemma 1.1.

Lemma 1.2. *If x_j, $j = 0, \dots, N$ is a sequence of real numbers with*

$$|x_i| \le \delta + hM \sum_{j=0}^{i-1} |x_j|, \quad i = 1, \dots, N, \tag{1.7}$$

where $M = const > 0$ and $\delta = const > 0$, then

$$|x_i| \le (hM|x_0| + \delta) \exp[Mih], \quad i = 1, \dots, N. \tag{1.8}$$

In the current paper, we will derive the discrete analogue of generalization of the Gronwall's integral inequality. It is used to obtain the generalization of Gronwall's integral inequality with two dependent limits. We will consider the applications of these inequalities to the integral-differential equation (1.1) of the parabolic type with two dependent limits in a Banach space E. The unique solvability of this equation is established. The stability estimates for the solution of this equation are obtained. The difference scheme approximately solving this equation is presented. The stability estimates for the solution of this difference scheme are obtained.

2. Gronwall's Type Integral Inequality with Two Dependent Limits and Its Discrete Analogue

First of all, let us obtain the theorems on the Gronwall's type integral inequalities with two dependent limits and their discrete analogues. We will use these results in the remaining part of the paper.

Theorem 2.1. *Assume that $v_i \ge 0$, $a_i \ge 0$, $\delta_i \ge 0$, $i = -N, \dots, N + 2M$ are the sequences of real numbers and the inequalities*

$$v_i \le \delta_i + h\left(\sum_{j=-|i-M|+M+1}^{|i-M|+M-1} a_j v_j - a_M v_M \right), \quad i = -N, \dots, N + 2M \tag{2.1}$$

hold. Then for v_i the inequalities

$$v_{M-1} \le \delta_{M-1}, \qquad v_{M+1} \le \delta_{M+1}, \qquad v_M \le \delta_M + h(a_{M-1}\delta_{M-1} + a_{M+1}\delta_{M+1}), \tag{2.2}$$

$$v_i \le \delta_i + h \sum_{j=M+1}^{|i-M|+M-1} \left(a_j \delta_j + a_{2M-j}\delta_{2M-j} \right) B_{|i-M|+M-1,j}, \quad i = -N, \dots, M-2, M+2, \dots, N+2M \tag{2.3}$$

are satisfied, where

$$B_{k,j} = \begin{cases} \prod_{n=j+1}^{k} [1 + h(a_n + a_{2M-n})], & \text{if } j = M+1, \dots, k-1, \\ 1, & \text{if } j = k. \end{cases} \tag{2.4}$$

Proof. By putting $i = M-1, M+1, M$ directly in (2.1), we obtain the inequalities (2.2), correspondingly. Let us prove (2.3). We denote

$$y_i = h\left(\sum_{j=-|i-M|+M+1}^{|i-M|+M-1} a_j v_j - a_M v_M\right), \quad i = -N, \dots, N+2M. \tag{2.5}$$

Then (2.1) gets the form

$$v_i \le \delta_i + y_i, \quad i = -N, \dots, N+2M. \tag{2.6}$$

Moreover, we have

$$y_{2M-i} = h\left(\sum_{j=-|M-i|+M+1}^{|M-i|+M-1} a_j v_j - a_M v_M\right) = y_i, \quad i = -N, \dots, N+2M. \tag{2.7}$$

Then, using (2.5)–(2.7) for $i = M+1, \dots, N+2\mathrm{M}-1$, we obtain

$$\begin{aligned} y_{i+1} - y_i &= h\left(\sum_{j=2M-i}^{i} a_j v_j - a_M v_M\right) - h\left(\sum_{j=2M-i+1}^{i-1} a_j v_j - a_M v_M\right) \\ &= h(a_i v_i + a_{2M-i} v_{2M-i}) \\ &\le h a_i (y_i + \delta_i) + h a_{2M-i}(y_{2M-i} + \delta_{2M-i}) \\ &= h(a_i + a_{2M-i}) y_i + h(a_i \delta_i + a_{2M-i}\delta_{2M-i}). \end{aligned} \tag{2.8}$$

So,

$$y_{i+1} \le [1 + h(a_i + a_{2M-i})] y_i + h(a_i\delta_i + a_{2M-i}\delta_{2M-i}), \quad i = M+1, \dots, N+2M-1. \tag{2.9}$$

Then by induction we can prove that

$$y_i \le \prod_{n=1}^{i-M-1} [1 + h(a_{M+n} + a_{M-n})] y_{M+1} + \sum_{j=M+1}^{i-1} h(a_j\delta_j + a_{2M-j}\delta_{2M-j}) B_{i-1,j} \tag{2.10}$$

hold for $i = M+2, \dots, N+2M$. Since $y_{M+1} = 0$, using (2.6), we obtain (2.3) for $i = M + 2, \dots, N+2M$.

Let us prove (2.3) for $i = -N, \ldots, M-2$. Using (2.5)–(2.7) for $i = -N+1, \ldots, M-1$, we have

$$\begin{aligned} y_{i-1} - y_i &= h\left(\sum_{j=i}^{2M-i} a_j v_j - a_M v_M\right) - h\left(\sum_{j=i+1}^{2M-i-1} a_j v_j - a_M v_M\right) \\ &= h\left(a_i v_i + a_{2M-i} v_{2M-i}\right) \\ &\le h a_i\left(y_i + \delta_i\right) + h a_{2M-i}\left(y_{2M-i} + \delta_{2M-i}\right) \\ &= h\left(a_i + a_{2M-i}\right) y_i + h\left(a_i \delta_i + a_{2M-i}\delta_{2M-i}\right). \end{aligned} \tag{2.11}$$

So,

$$y_{i-1} \le \left[1 + h\left(a_i + a_{2M-i}\right)\right] y_i + h\left(a_i\delta_i + a_{2M-i}\delta_{2M-i}\right), \quad i = -N+1, \ldots, M-1. \tag{2.12}$$

Then by induction we can prove that

$$y_i \le \prod_{n=1}^{M-i-1}\left[1 + h\left(a_{M+n} + a_{M-n}\right)\right] y_{M-1} + \sum_{j=M+1}^{2M-i-1} h\left(a_j\delta_j + a_{2M-j}\delta_{2M-j}\right) B_{2M-i-1,j} \tag{2.13}$$

hold for $i = -N, \ldots, M-2$. Since $y_{M-1} = 0$, using (2.6), we obtain (2.3) for $i = -N, \ldots, M-2$. The proof of Theorem 2.1 is complete. □

By putting $M = 0$, $\delta_i \equiv \text{const}$, $a_i \equiv \text{const}$, $i = -N, \ldots, N$, and using the inequality $1 + x < \exp[x]$ for $x > 0$ in the Theorem 2.1, we get the following result.

Theorem 2.2. *Assume that $v_i \ge 0$, $i = -N, \ldots, N$ is the sequence of real numbers and the inequalities*

$$v_i \le \delta + Lh\left(\sum_{j=-|i|+1}^{|i|-1} v_j - v_0\right), \quad i = -N, \ldots, N \tag{2.14}$$

hold. Then for v_i the inequalities

$$v_0 \le \delta \exp[2Lh], \qquad v_i \le \delta \exp[2Lh(|i| - 1)], \quad i = -N, \ldots, -1, 1, \ldots, N \tag{2.15}$$

are satisfied.

By putting $Nh = 1$, $2Mh = T$ and passing to limit $h \to 0$ in the Theorem 2.1, we obtain the following generalization of Gronwall's integral inequality with two dependent limits.

Theorem 2.3. *Assume that* $v(t) \geq 0$, $\delta(t) \geq 0$ *are the continuous functions on* $[-1, 1+T]$ *and* $a(t) \geq 0$ *is an integrable function on* $[-1, 1+T]$ *and the inequalities*

$$v(t) \leq \delta(t) + \operatorname{sgn}\left(t - \frac{T}{2}\right) \int_{T-t}^{t} a(s)v(s)ds, \quad -1 \leq t \leq 1+T \tag{2.16}$$

hold. Then for $v(t)$ *the inequalities*

$$\begin{aligned} &v(t) \\ &\leq \delta(t) + \int_{T/2}^{t} (a(s)\delta(s) + a(T-s)\delta(T-s)) \exp\left[\int_{s}^{t} (a(\tau) + a(T-\tau))d\tau\right] ds, \quad \frac{T}{2} \leq t \leq 1+T, \\ &v(t) \\ &\leq \delta(t) + \int_{T/2}^{T-t} (a(s)\delta(s) + a(T-s)\delta(T-s)) \exp\left[\int_{s}^{T-t} (a(\tau) + a(T-\tau))d\tau\right] ds, \quad -1 \leq t < \frac{T}{2} \end{aligned} \tag{2.17}$$

are satisfied.

Finally, by putting $\delta(t) \equiv \text{const}$, $a(t) \equiv \text{const}$, $-1 \leq t \leq 1$, and $T = 0$ in the Theorem 2.3, we get the following result.

Theorem 2.4. *Assume that* $v(t) \geq 0$ *is a continuous function on* $[-1, 1]$ *and the inequalities*

$$v(t) \leq C + L \operatorname{sgn}(t) \int_{-t}^{t} v(s)ds, \quad -1 \leq t \leq 1 \tag{2.18}$$

hold, where $C = \mathit{const} \geq 0$ *and* $L = \mathit{const} \geq 0$. *Then for* $v(t)$ *the inequalities*

$$v(t) \leq C \exp(2L|t|), \quad -1 \leq t \leq 1 \tag{2.19}$$

are satisfied.

3. The Integral-Differential Equation of the Parabolic Type

Now, we consider the application of the generalizations of Gronwall's integral inequality with two dependent limits and their discrete analogues to the integral-differential equation (1.1) of the parabolic type with two dependent limits in a Banach space E.

First of all, let us give one theorem that will be needed below.

Theorem 3.1. *Suppose that $F(t) \in C([-1,1],E)$, $K(t,s) \in C([-1,1],E)$. Then there is a unique solution of the integral equation*

$$z(t) = F(t) + \operatorname{sgn}(t) \int_{-t}^{t} K(t,s) z(s) ds, \quad -1 \le t \le 1. \tag{3.1}$$

Proof. The proof of this theorem is based on a fixed-point theorem. It is easy to see that the operator

$$Bz(t) = F(t) + \operatorname{sgn}(t) \int_{-t}^{t} K(t,s) z(s) ds, \quad -1 \le t \le 1 \tag{3.2}$$

maps $C([-1,1],E)$ into $C([-1,1],E)$. By using a special value of λ in the norm

$$\|v\|_{C^*([-1,1],E)} = \max_{-1 \le t \le 1} e^{-\lambda|t|} \|v(t)\|_E, \tag{3.3}$$

we can prove that A is the contracting operator on $C^*([-1,1],E)$. Indeed, we have

$$\begin{aligned}
e^{-\lambda|t|} \|Bz(t) - Bu(t)\|_E &\le \int_{-|t|}^{|t|} \|K(t,s)\|_{E \to E} e^{-\lambda(|t|-|s|)} e^{-\lambda|s|} \|z(s) - u(s)\|_E ds \\
&\le \max_{-1 \le s,t \le 1} \|K(t,s)\|_{E \to E} \int_{-|t|}^{|t|} e^{-\lambda(|t|-|s|)} \|z-u\|_{C^*([-1,1],E)} ds \\
&= 2 \max_{-1 \le s,t \le 1} \|K(t,s)\|_{E \to E} \int_{0}^{|t|} e^{-\lambda(|t|-s)} ds \|z-u\|_{C^*([-1,1],E)} \\
&= 2 \max_{-1 \le s,t \le 1} \|K(t,s)\|_{E \to E} \|z-u\|_{C^*([-1,1],E)} \frac{1 - e^{-\lambda|t|}}{\lambda} \\
&\le \|z-u\|_{C^*([-1,1],E)} \frac{2(1 - e^{-\lambda})}{\lambda} \max_{-1 \le s,t \le 1} \|K(t,s)\|_{E \to E}
\end{aligned} \tag{3.4}$$

for any $t \in [-1,1]$. So,

$$\|Bz - Bu\|_{C^*([-1,1],E)} \le \|z-u\|_{C^*([-1,1],E)} \alpha_\lambda, \tag{3.5}$$

where $\alpha_\lambda = (2(1-e^{-\lambda})/\lambda) \max_{-1 \le s,t \le 1} \|K(t,s)\|_{E \to E}$ and $\alpha_\lambda \to 0$ when $\lambda \to \infty$. Finally, we note that the norms

$$\begin{aligned}
\|v\|_{C^*([-1,1],E)} &= \max_{-1 \le t \le 1} e^{-\lambda|t|} \|v(t)\|_E, \\
\|v\|_{C([-1,1],E)} &= \max_{-1 \le t \le 1} \|v(t)\|_E
\end{aligned} \tag{3.6}$$

are equivalent in $C([-1,1],E)$. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Suppose that assumptions* (1.2) *and* (1.4) *for the operators A and $B(t)$ hold. Assume that $f(t)$ is continuously differentiable on* $[-1,1]$ *function. Then there is a unique solution of* (1.1) *and stability inequality*

$$\max_{-1\le t\le 1}\left\|\frac{du(t)}{dt}\right\|_E + \max_{-1\le t\le 1}\|Au(t)\|_E \le M^*\left[\|f(0)\|_E + \int_{-1}^{1}\|f'(s)\|_E ds\right] \tag{3.7}$$

holds, where M^* does not depend on $f(t)$ and t.

Proof. The proof of the existence and uniqueness of the solution of (1.1) is based on the following formula:

$$\begin{aligned} u(t) = {} & \operatorname{sgn}(t)A^{-1}f(t) - \operatorname{sgn}(t)e^{-|t|A}A^{-1}f(0) - \operatorname{sgn}(t)\int_0^t e^{-(|t|-|s|)A}A^{-1}f'(s)ds \\ & + \operatorname{sgn}(t)\int_{-t}^{t}\left[I - e^{-(|t|-|s|)A}\right]A^{-1}B(s)u(s)ds, \quad -1\le t\le 1 \end{aligned} \tag{3.8}$$

and the Theorem 3.1.

First, we note that the solution of (1.1) satisfies $u(0) = 0$. Indeed, assume that $u(t)$ is the solution of (1.1) with $B \equiv 0$. Then

$$\begin{aligned} u'(t) + Au(t) &= f(t), \quad 0 < t \le 1, \\ u'(t) - Au(t) &= f(t), \quad -1 \le t < 0, \end{aligned} \tag{3.9}$$

and from the continuity of f, $u'(t)$, and Au at $t = 0$ we get

$$\begin{aligned} u'(0) + Au(0) &= f(0), \\ u'(0) - Au(0) &= f(0). \end{aligned} \tag{3.10}$$

This leads to $2Au(0) = 0$, and it follows that $u(0) = 0$.

Let us now prove (3.8). First, we consider the case when $0 \le t \le 1$. It is well known that the Cauchy problem

$$\begin{gathered} \frac{du(t)}{dt} + Au(t) = F(t), \quad 0 \le t \le 1, \\ u(0) = 0 \end{gathered} \tag{3.11}$$

for differential equations in an arbitrary Banach space E with positive operator A has the unique solution

$$u(t) = \int_0^t e^{-(t-s)A}F(s)ds, \quad 0 \le t \le 1 \tag{3.12}$$

for smooth $F(t)$. By putting

$$F(t) = \int_{-t}^{t} B(s)u(s)ds + f(t), \tag{3.13}$$

we have

$$u(t) = \int_0^t e^{-(t-s)A} f(s)ds + \int_0^t e^{-(t-s)A} \int_{-s}^{s} B(\tau)u(\tau)d\tau\, ds, \quad 0 \le t \le 1. \tag{3.14}$$

Since

$$\begin{aligned}
\int_0^t \int_{-s}^{s} e^{-(t-s)A} B(\tau)u(\tau)d\tau\, ds &= \int_0^t \int_{\tau}^{t} e^{-(t-s)A} B(\tau)u(\tau)ds\, d\tau \\
&\quad + \int_{-t}^{0} \int_{-\tau}^{t} e^{-(t-s)A} B(\tau)u(\tau)ds\, d\tau \\
&= \int_0^t \left(I - e^{-(t-\tau)A}\right) A^{-1} B(\tau)u(\tau)d\tau \\
&\quad + \int_{-t}^{0} \left(I - e^{-(t+\tau)A}\right) A^{-1} B(\tau)u(\tau)d\tau \\
&= \int_{-t}^{t} \left(I - e^{-(t-|s|)A}\right) A^{-1} B(s)u(s)ds, \\
\int_0^t e^{-(t-s)A} f(s)ds = A^{-1} f(t) - e^{-tA} A^{-1} f(0) &- \int_0^t e^{-(t-s)A} A^{-1} f'(s)ds,
\end{aligned} \tag{3.15}$$

we obtain (3.8) for $0 \le t \le 1$.

Now, let $-1 \le t \le 0$. Then we consider the problem

$$\begin{gathered}
\frac{du(t)}{dt} - Au(t) = F(t), \quad -1 \le t \le 0, \\
u(0) = 0
\end{gathered} \tag{3.16}$$

for differential equations in an arbitrary Banach space E with positive operator A, which has the unique solution

$$u(t) = -\int_t^0 e^{(t-s)A} F(s)ds, \quad -1 \le t \le 0. \tag{3.17}$$

By putting

$$F(t) = \int_{-t}^{t} B(s)u(s)ds + f(t), \tag{3.18}$$

we have

$$u(t) = -\int_t^0 e^{(t-s)A} f(s)ds + \int_t^0 e^{(t-s)A} \int_s^{-s} B(\tau)u(\tau)d\tau\, ds, \quad -1 \le t \le 0. \tag{3.19}$$

Since

$$\begin{aligned}
\int_t^0 \int_s^{-s} e^{(t-s)A} B(\tau)u(\tau)d\tau\, ds &= \int_0^{-t} \int_t^{-\tau} e^{(t-s)A} B(\tau)u(\tau)ds\, d\tau \\
&\quad + \int_t^0 \int_t^{\tau} e^{(t-s)A} B(\tau)u(\tau)ds\, d\tau \\
&= \int_0^{-t} \left(I - e^{(t+\tau)A}\right) A^{-1} B(\tau)u(\tau)d\tau \\
&\quad + \int_t^0 \left(I - e^{(t-\tau)A}\right) A^{-1} B(\tau)u(\tau)d\tau \\
&= \int_t^{-t} \left(I - e^{-(-t-|s|)A}\right) A^{-1} B(s)u(s)ds, \\
\int_t^0 e^{(t-s)A} f(s)ds = A^{-1} f(t) - e^{tA} A^{-1} f(0) &+ \int_t^0 e^{-(-t+s)A} A^{-1} f'(s)ds,
\end{aligned} \tag{3.20}$$

we obtain (3.8) for $-1 \le t \le 0$.

From (3.8) it follows that

$$\begin{aligned}
Au(t) = {} & \operatorname{sgn}(t) f(t) - \operatorname{sgn}(t) e^{-|t|A} f(0) - \operatorname{sgn}(t) \int_0^t e^{-(|t|-|s|)A} f'(s)ds \\
& + \operatorname{sgn}(t) \int_{-t}^t \left[I - e^{-(|t|-|s|)A}\right] B(s)u(s)ds, \quad -1 \le t \le 1.
\end{aligned} \tag{3.21}$$

Applying the triangle inequality and assumptions (1.2) and (1.4), we get

$$\begin{aligned}
\|Au(t)\|_E \le {} & \|f(t)\|_E + \left\|e^{-|t|A}\right\|_{E\to E} \|f(0)\|_E + \int_{-|t|}^{|t|} \left\|e^{-(|t|-|s|)A}\right\|_{E\to E} \|f'(s)\|_E ds \\
& + \int_{-|t|}^{|t|} \left[1 + \left\|e^{-(|t|-|s|)A}\right\|_{E\to E}\right] \left\|B(s)A^{-1}\right\|_{E\to E} \|Au(s)\|_E ds \\
\le {} & (M+1)\left[\|f(0)\|_E + \int_{-1}^1 \|f'(s)\|_E ds\right] \\
& + \operatorname{sgn}(t) M(M+1) \int_{-t}^t \|Au(s)\|_E ds, \quad -1 \le t \le 1.
\end{aligned} \tag{3.22}$$

Then, using the Theorem 2.4, we have

$$\begin{aligned}\|Au(t)\|_E &\le (M+1)\left[\|f(0)\|_E+\int_{-1}^{1}\|f'(s)\|_E ds\right]e^{2M(M+1)|t|}\\ &\le (M+1)e^{2M(M+1)}\left[\|f(0)\|_E+\int_{-1}^{1}\|f'(s)\|_E ds\right],\quad -1\le t\le 1.\end{aligned}\tag{3.23}$$

So,

$$\max_{-1\le t\le 1}\|Au(t)\|_E \le (M+1)e^{2M(M+1)}\left[\|f(0)\|_E+\int_{-1}^{1}\|f'(s)\|_E ds\right].\tag{3.24}$$

By applying the triangle inequality in (1.1) and assumptions (1.2) and (1.4), we obtain

$$\begin{aligned}\left\|\frac{du(t)}{dt}\right\|_E &\le \|Au(t)\|_E+\int_{-|t|}^{|t|}\left\|B(s)A^{-1}\right\|_{E\to E}\|Au(s)\|_E ds+\|f(t)\|_E\\ &\le (2M+1)\max_{-1\le t\le 1}\|Au(t)\|_E+\|f(0)\|_E+\int_{-1}^{1}\|f'(s)\|_E ds,\quad -1\le t\le 1.\end{aligned}\tag{3.25}$$

So,

$$\max_{-1\le t\le 1}\left\|\frac{du(t)}{dt}\right\|_E \le (2M+1)\max_{-1\le t\le 1}\|Au(t)\|_E+\|f(0)\|_E+\int_{-1}^{1}\|f'(s)\|_E ds.\tag{3.26}$$

Then using (3.24), we have

$$\begin{aligned}\max_{-1\le t\le 1}\left\|\frac{du(t)}{dt}\right\|_E+\max_{-1\le t\le 1}\|Au(t)\|_E &\le 2(M+1)\max_{-1\le t\le 1}\|Au(t)\|_E+\|f(0)\|_E+\int_{-1}^{1}\|f'(s)\|_E ds\\ &\le \left(2(M+1)^2e^{2M(M+1)}+1\right)\left[\|f(0)\|_E+\int_{-1}^{1}\|f'(s)\|_E ds\right].\end{aligned}\tag{3.27}$$

So, stability inequality (3.7) holds with $M^*=2(M+1)^2e^{2M(M+1)}+1$. The proof of Theorem 3.2 is complete. □

Note that it does not hold, generally speaking

$$\max_{-1\le t\le 1}\left\|\frac{du(t)}{dt}\right\|_E+\max_{-1\le t\le 1}\|Au(t)\|_E\le M^*\max_{-1\le t\le 1}\|f(t)\|_E\tag{3.28}$$

in an arbitrary Banach space E for the general strong positive operator A, see [17, Section 1.5, Chapter 1]. Nevertheless, we can establish the following theorem.

Theorem 3.3. *Suppose that assumptions* (1.4) *for the operator A hold and*

$$\left\|B(t)A^{-1}\right\|_{E_\alpha \to E_\alpha} \leq M, \quad -1 \leq t \leq 1. \tag{3.29}$$

Assume that $f(t)$ is a continuous on $[-1,1]$ function. Then there is a unique solution of (1.1) *and stability inequality*

$$\max_{-1\leq t\leq 1}\left\|\frac{du(t)}{dt}\right\|_{E_\alpha} + \max_{-1\leq t\leq 1}\|Au(t)\|_{E_\alpha} \leq M^*(\alpha)\max_{-1\leq t\leq 1}\|f(t)\|_{E_\alpha} \tag{3.30}$$

holds, where $M^*(\alpha)$ does not depend on $f(t)$ and t. Here the fractional spaces $E_\alpha = E_\alpha(E, A)$ $(0 < \alpha < 1)$, consisting of all $v \in E$ for which the following norms are finite:

$$\|v\|_{E_\alpha} = \sup_{0<z} z^{1-\alpha}\|A\ \exp\{-zA\}v\|_E. \tag{3.31}$$

Proof. First, we rewrite (3.8) as

$$u(t) = \int_0^t e^{-(|t|-|s|)A} f(s)ds + \operatorname{sgn}(t)\int_{-t}^t \left[I - e^{-(|t|-|s|)A}\right]A^{-1}B(s)u(s)ds, \quad -1 \leq t \leq 1. \tag{3.32}$$

The proof of the existence and uniqueness of the solution of (1.1) is based on the formula (3.32) and an analogue of the Theorem 3.1. Let us prove (3.30). From (3.32) it follows that

$$Au(t) = \int_0^t Ae^{-(|t|-|s|)A} f(s)ds + \operatorname{sgn}(t)\int_{-t}^t \left[I - e^{-(|t|-|s|)A}\right]B(s)u(s)ds, \quad -1 \leq t \leq 1. \tag{3.33}$$

Applying the triangle inequality, the definition of the norm of the space E_α and assumptions (1.4) and (3.29), we obtain

$$\begin{aligned} \|Au(t)\|_{E_\alpha} \leq{}& \left\|\int_{-|t|}^{|t|} Ae^{-(|t|-|s|)A} f(s)ds\right\|_{E_\alpha} \\ &+ \int_{-|t|}^{|t|}\left[1 + \left\|e^{-(|t|-|s|)A}\right\|_{E\to E}\right]\left\|B(s)A^{-1}\right\|_{E_\alpha\to E_\alpha}\|Au(s)\|_{E_\alpha}ds \\ \leq{}& \left\|\int_{-|t|}^{|t|} Ae^{-(|t|-|s|)A} f(s)ds\right\|_{E_\alpha} + \operatorname{sgn}(t)M(M+1)\int_{-t}^{t}\|Au(s)\|_{E_\alpha}ds. \end{aligned} \tag{3.34}$$

By [17, Chapter 1, Theorem 4.1], we obtain

$$\left\|\int_{-|t|}^{|t|} Ae^{-(|t|-|s|)A} f(s)ds\right\|_{E_\alpha} \leq \frac{M}{\alpha(1-\alpha)}\max_{-1\leq t\leq 1}\|f(t)\|_{E_\alpha}. \tag{3.35}$$

So,

$$\|Au(t)\|_{E_\alpha} \leq \frac{M}{\alpha(1-\alpha)} \max_{-1\leq t\leq 1} \|f(t)\|_{E_\alpha} + \operatorname{sgn}(t) M(M+1) \int_{-t}^{t} \|Au(s)\|_{E_\alpha} ds \tag{3.36}$$

for $-1 \leq t \leq 1$. Then, using the Theorem 2.4, we have

$$\|Au(t)\|_{E_\alpha} \leq \frac{M}{\alpha(1-\alpha)} \max_{-1\leq t\leq 1} \|f(t)\|_{E_\alpha} e^{2M(M+1)|t|}, \quad -1 \leq t \leq 1. \tag{3.37}$$

So,

$$\max_{-1\leq t\leq 1} \|Au(t)\|_{E_\alpha} \leq \frac{M}{\alpha(1-\alpha)} \max_{-1\leq t\leq 1} \|f(t)\|_{E_\alpha} e^{2M(M+1)}. \tag{3.38}$$

Then, using the triangle inequality in (1.1) yields

$$\max_{-1\leq t\leq 1} \left\| \frac{du(t)}{dt} \right\|_{E_\alpha} \leq \frac{M(M+1)}{\alpha(1-\alpha)} \max_{-1\leq t\leq 1} \|f(t)\|_{E_\alpha} e^{2M(M+1)}. \tag{3.39}$$

Combining last two inequalities, we obtain (3.30) with $M^*(\alpha) = (M(M+2)/\alpha(1-\alpha))e^{2M(M+1)}$. The proof of Theorem 3.3 is complete. □

Now, we consider the Rothe difference scheme for approximate solutions of (1.1).

$$\begin{gathered}
\frac{u_k - u_{k-1}}{\tau} + Au_k = \sum_{i=-k+1}^{k-1} B_i u_i \tau + \varphi_k, \quad k = 1, \ldots, N, \\
\frac{u_k - u_{k-1}}{\tau} - Au_{k-1} = -\sum_{i=k}^{-k} B_i u_i \tau + \varphi_k, \quad k = -N+1, \ldots, 0, \\
B_k = B(t_k), \quad t_k = k\tau, \ k = -N, \ldots, N, \\
u_0 = 0.
\end{gathered} \tag{3.40}$$

Theorem 3.4. *Suppose that the requirements of the Theorem 3.2 are satisfied. Then for the solution of difference scheme* (3.40)*, the following stability inequalities*

$$\max_{k=-N+1,\ldots,N} \left\| \frac{u_k - u_{k-1}}{\tau} \right\|_E + \max_{k=-N,\ldots,N} \|Au_k\|_E \leq M^* \left[\|\varphi_0\|_E + \sum_{k=-N+1}^{N} \|\varphi_k - \varphi_{k-1}\|_E \right] \tag{3.41}$$

hold, where M^* does not depend on φ_k, $k = -N, \ldots, N$.

Proof. By induction we can prove that the initial value problem

$$\frac{u_k - u_{k-1}}{\tau} + Au_k = \varphi_k, \quad k = 1, \ldots, N, \ u_0 = 0 \tag{3.42}$$

for difference equations in an arbitrary Banach space E with positive operator A has a unique solution

$$u_k = \sum_{i=1}^{k} R^{k-i+1}\varphi_i\tau, \quad k = 1,\dots,N, \tag{3.43}$$

where $R = (I+\tau A)^{-1}$. By putting $\varphi_k = \sum_{i=-k+1}^{k-1} B_i u_i \tau + \varphi_k$, we obtain

$$Au_k = A\sum_{i=1}^{k} R^{k-i+1}\varphi_i\tau + A\sum_{i=1}^{k} R^{k-i+1}\sum_{j=-i+1}^{i-1} B_j u_j \tau^2, \quad k = 1,\dots,N. \tag{3.44}$$

Using

$$\begin{aligned} &\tau\sum_{i=\pm j+1}^{k} R^{k-i+1} \\ &\quad = \tau\left(R + R^2 + \cdots + R^{k\mp j}\right) = \tau R(I-R)^{-1}\left(I - R^{k\mp j}\right) = A^{-1}\left(I - R^{k\mp j}\right), \quad k = 1,\dots,N, \end{aligned} \tag{3.45}$$

we have

$$\begin{aligned} A\sum_{i=1}^{k} R^{k-i+1}\sum_{j=-i+1}^{i-1} B_j u_j \tau^2 &= A\sum_{j=-k+1}^{0}\tau\sum_{i=-j+1}^{k} R^{k-i+1} B_j u_j \tau + A\sum_{j=1}^{k-1}\tau\sum_{i=j+1}^{k} R^{k-i+1} B_j u_j \tau \\ &= A\sum_{j=-k+1}^{0} A^{-1}\left(I - R^{k+j}\right) B_j u_j \tau + A\sum_{j=1}^{k-1} A^{-1}\left(I - R^{k-j}\right) B_j u_j \tau \\ &= \sum_{i=-k+1}^{k-1}\left[I - R^{k-|i|}\right] B_i u_i \tau. \end{aligned} \tag{3.46}$$

Furthermore,

$$\begin{aligned} A\sum_{i=1}^{k} R^{k-i+1}\varphi_i\tau &= \sum_{i=1}^{k} (I-R)R^{k-i}\varphi_i \\ &= \sum_{i=1}^{k} R^{k-i}\varphi_i - \sum_{i=1}^{k} R^{k-i+1}\varphi_i \\ &= \sum_{i=2}^{k+1} R^{k-i+1}\varphi_{i-1} - \sum_{i=1}^{k} R^{k-i+1}\varphi_i \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{k} R^{k-i+1}\varphi_{i-1} - R^k\varphi_0 + \varphi_k - \sum_{i=1}^{k} R^{k-i+1}\varphi_i \\
&= \varphi_k - R^k\varphi_0 - \sum_{i=1}^{k} R^{k-i+1}\left(\varphi_i - \varphi_{i-1}\right).
\end{aligned} \tag{3.47}$$

Putting (3.46)-(3.47) in (3.44), we get

$$Au_k = \varphi_k - R^k\varphi_0 - \sum_{i=1}^{k} R^{k-i+1}\left(\varphi_i - \varphi_{i-1}\right) + \sum_{\substack{i=-k+1 \\ i\neq 0}}^{k-1} \left[I - R^{k-|i|}\right] B_i u_i \tau \tag{3.48}$$

for $k = 1, \ldots, N$.

Since

$$R^k = (I + \tau A)^{-k} = \frac{1}{(k-1)!}\int_0^\infty t^{k-1}e^{-t}e^{-\tau t A}dt, \quad k = 1, \ldots, N, \tag{3.49}$$

applying estimates (1.4) gives

$$\left\|R^k\right\|_{E\to E} \leq \frac{M}{(k-1)!}\int_0^\infty t^{k-1}e^{-t}dt = M, \quad k = 1, \ldots, N. \tag{3.50}$$

Then, applying the triangle inequality and the estimate (1.2) in (3.48), we obtain

$$\begin{aligned}
\|Au_k\|_E &\leq \|\varphi_k\|_E + \left\|R^k\right\|_{E\to E}\|\varphi_0\|_E + \sum_{i=1}^{k}\left\|R^{k-i+1}\right\|_{E\to E}\|\varphi_i - \varphi_{i-1}\|_E \\
&\quad + \sum_{\substack{i=-k+1 \\ i\neq 0}}^{k-1}\left(1 + \left\|R^{k-|i|}\right\|_{E\to E}\right)\left\|B_iA^{-1}\right\|_{E\to E}\|Au_i\|_E\tau \\
&\leq \left\|\sum_{i=1}^{k}\left(\varphi_i - \varphi_{i-1}\right) + \varphi_0\right\|_E + M\|\varphi_0\|_E + M\sum_{i=1}^{k}\|\varphi_i - \varphi_{i-1}\|_E + M(M+1)\sum_{\substack{i=-k+1 \\ i\neq 0}}^{k-1}\|Au_i\|_E\tau \\
&\leq (M+1)\sum_{i=-N+1}^{N}\|\varphi_i - \varphi_{i-1}\|_E + (M+1)\|\varphi_0\|_E \\
&\quad + M(M+1)\sum_{\substack{i=-k+1 \\ i\neq 0}}^{k-1}\|Au_i\|_E\tau, \quad k = 1, \ldots, N.
\end{aligned} \tag{3.51}$$

So, for $k = 1, \ldots, N$,

$$\|Au_k\|_E \leq \widetilde{M}\left(\|\varphi_0\|_E + \sum_{i=-N+1}^{N} \|\varphi_i - \varphi_{i-1}\|_E \right) + M(M+1) \sum_{\substack{i=-k+1 \\ i \neq 0}}^{k-1} \|Au_i\|_E \tau \tag{3.52}$$

holds, where $\widetilde{M} = \max\{M(1+\tau(M+1)), M+1\}$.

In a similar way, we can prove that the initial value problem

$$\frac{u_k - u_{k-1}}{\tau} - Au_{k-1} = -\sum_{i=k}^{-k} B_i u_i \tau + \varphi_k, \quad k = -N+1, \ldots, 0, \ u_0 = 0 \tag{3.53}$$

for difference equations in an arbitrary Banach space E with positive operator A has a unique solution and inequalities

$$\|Au_k\|_E \leq \widetilde{M}\left(\|\varphi_0\|_E + \sum_{i=-N+1}^{N} \|\varphi_i - \varphi_{i-1}\|_E \right) + M(M+1) \sum_{\substack{i=k+1 \\ i \neq 0}}^{-k-1} \|Au_i\|_E \tau \tag{3.54}$$

hold for $k = -N \ldots, 0$.

Now, the proof of this theorem is based on the Theorem 2.2 and the inequalities (3.52) and (3.54). The proof of Theorem 3.4 is complete. □

Note that it does not hold, generally speaking

$$\max_{k=-N+1,\ldots,N} \left\| \frac{u_k - u_{k-1}}{\tau} \right\|_E + \max_{k=-N,\ldots,N} \|Au_k\|_E \leq M^* \max_{k=-N,\ldots,N} \|\varphi_k\|_E \tag{3.55}$$

in the arbitrary Banach space E for the general strong positive operator A, see [17, 18].

This approach and theory of difference schemes of [17] permit us to obtain the following two theorems on stability estimates for the solution of difference scheme (3.40).

Theorem 3.5. *Suppose that the requirements of the Theorem 3.2 are satisfied. Then for the solution of difference scheme* (3.40) *the following stability inequalities*

$$\begin{aligned} &\max_{k=-N+1,\ldots,N} \left\| \frac{u_k - u_{k-1}}{\tau} \right\|_E + \max_{k=-N,\ldots,N} \|Au_k\|_E \\ &\quad \leq M^* \min\left\{ \ln \frac{1}{\tau}, 1 + |\ln \|A\|_{E \to E}| \right\} \max_{k=-N,\ldots,N} \|\varphi_k\|_E \end{aligned} \tag{3.56}$$

hold, where M^* does not depend on φ_k, $k = -N, \ldots, N$.

Theorem 3.6. *Suppose that the requirements of the Theorem 3.3 are satisfied. Then for the solution of difference scheme* (3.40)*, the following stability inequalities*

$$\max_{k=-N+1,\dots,N}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_{E'_\alpha}+\max_{k=-N,\dots,N}\|Au_k\|_{E'_\alpha}\le M^*(\alpha)\max_{k=-N,\dots,N}\left\|\varphi_k\right\|_{E'_\alpha} \tag{3.57}$$

hold, where $M^*(\alpha)$ does not depend on φ_k, $k=-N,\dots,N$. Here the fractional spaces $E'_\alpha=E'_\alpha(E,A)$ $(0<\alpha<1)$, consisting of all $v\in E$ for which the following norms are finite:

$$\|v\|_{E'_\alpha}=\sup_{0<z} z^\alpha\left\|A(z+A)^{-1}v\right\|_E. \tag{3.58}$$

Stability estimates could be also proved for the more general Pade difference schemes of the high order of accuracy, see [17, 19].

4. Conclusion

In this paper, the integral-differential equation of the parabolic type with two dependent limits in a Banach space is studied. The unique solvability of this equation is established. The stability estimates for the solution of this equation are obtained. The Rothe difference scheme approximately solving this equation is presented. The stability estimates for the solution of this difference scheme are obtained.

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