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Research Article

Positive Solutions of a Second-Order Nonlinear Neutral Delay Difference Equation

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The purpose of this paper is to study solvability of the second-order nonlinear neutral delay difference equation $\Delta(a(n, y_{a_{1n}}, \ldots, y_{a_{rn}})\Delta(y_n + b_n y_{n-\tau})) + f(n, y_{f_{1n}}, \ldots, y_{f_{kn}}) = c_n$, $\forall n \geq n_0$. By making use of the Rothe fixed point theorem, Leray-Schauder nonlinear alternative theorem, Krasnoselskill fixed point theorem, and some new techniques, we obtain some sufficient conditions which ensure the existence of uncountably many bounded positive solutions for the above equation. Five nontrivial examples are given to illustrate that the results presented in this paper are more effective than the existing ones in the literature.

1. Introduction

It is well known that the oscillation, nonoscillation, asymptotic behavior, and existence of solutions for second-order difference equations with delays have been widely studied in many papers over the last 20 years, see, for example, [1–9] and the references cited therein.

Recently, Cheng [5] considered the second-order neutral delay linear difference equation with positive and negative coefficients

$$\Delta^2(y_n + p y_{n-m}) + p_n y_{n-k} - q_n y_{n-l} = 0, \quad \forall n \geq n_0 \tag{1.1}$$

and investigated the existence of a nonoscillatory solution of (1.1) under the condition $p \neq -1$ by using the Banach fixed point theorem. M. Migda and J. Migda [9] and Luo and Bainov [8] discussed the asymptotic behaviors of nonoscillatory solutions for the second-order neutral difference equation with maxima

$$\Delta^2(y_n + p_n y_{n-k}) + q_n \max\{y_s : n-l \le s \le n\} = 0, \quad \forall n \ge 1 \tag{1.2}$$

and the second-order neutral difference equation

$$\Delta^2(y_n + p y_{n-k}) + f(n, y_n) = 0, \quad \forall n \ge 1. \tag{1.3}$$

Cheng and Chu [2] got sufficient and necessary conditions of the oscillatory solutions for the second-order difference equation

$$\Delta(r_{n-1}\Delta y_{n-1}) + p_n y_n^{\gamma} = 0, \quad \forall n \ge 1. \tag{1.4}$$

Li and Yeh [6] established some oscillation criteria of the second-order delay difference equation

$$\Delta(a_{n-1}\Delta(y_{n-1} + p_{n-1}y_{n-1-\sigma})) + q_n f(y_{n-\tau}) = 0, \quad \forall n \ge 1. \tag{1.5}$$

Using the Leray-Schauder nonlinear alternative theorem, Agarwal et al. [1] studied the existence of nonoscillatory solutions for the discrete equation

$$\Delta(a_n\Delta(y_n + p y_{n-\tau})) + F(n+1, y_{n+1-\sigma}) = 0, \quad \forall n \ge 1 \tag{1.6}$$

under the condition $|p| \ne 1$. Very recently, Liu et al. [7] utilized the Banach contraction principle to establish the global existence and multiplicity of bounded nonoscillatory solutions for the second-order nonlinear neutral delay difference equation

$$\Delta(a_n\Delta(y_n + b y_{n-\tau})) + f(n, y_{n-d_{1n}}, y_{n-d_{2n}}, \ldots, y_{n-d_{kn}}) = c_n, \quad \forall n \ge n_0. \tag{1.7}$$

Motivated by the results in [1–9], in this paper, we discuss the solvability of the second-order nonlinear neutral delay difference equation

$$\Delta(a(n, y_{a_{1n}}, \ldots, y_{a_{rn}})\Delta(y_n + b_n y_{n-\tau})) + f(n, y_{f_{1n}}, \ldots, y_{f_{kn}}) = c_n, \quad \forall n \ge n_0, \tag{1.8}$$

where $\tau, r, k \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$, $\{b_n\}_{n\in\mathbb{N}_{n_0}} \cup \{c_n\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{R}$, $a \in C(\mathbb{N}_{n_0} \times \mathbb{R}^r, \mathbb{R}\setminus\{0\})$, $f \in C(\mathbb{N}_{n_0} \times \mathbb{R}^k, \mathbb{R})$, $\cup_{d=1}^{r}\{a_{dn}\}_{n\in\mathbb{N}_{n_0}} \subseteq \mathbb{Z}$, $\cup_{j=1}^{k}\{f_{jn}\}_{n\in\mathbb{N}_{n_0}} \subseteq \mathbb{Z}$ and

$$\lim_{n\to\infty} a_{dn} = \lim_{n\to\infty} f_{jn} = +\infty, \quad (d, j) \in J_r \times J_k. \tag{1.9}$$

It is clear that (1.1)–(1.7) are special cases of (1.8). By utilizing the Rothe fixed point theorem, Leray-Schauder nonlinear alternative theorem, Krasnoselskill fixed point theorem, and a few new techniques, we prove the existence of uncountably many bounded positive solutions of (1.8). Five examples are constructed to illuminate our results which extend essentially the corresponding results in [1, 7].

2. Preliminaries

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ stand for the sets of all integers, positive integers, and nonnegative integers, respectively,

$$\begin{gathered}\beta = \min\left\{n_0 - \tau, \inf\{a_{dn} : d \in J_r, n \in \mathbb{N}_{n_0}\}, \inf\{f_{jn} : j \in J_k, n \in \mathbb{N}_{n_0}\}\right\},\\ \mathbb{Z}_\beta = \{n : n \in \mathbb{Z} \text{ with } n \geq \beta\}, \qquad \mathbb{N}_{n_0} = \{n : n \in \mathbb{N}_0 \text{ with } n \geq n_0\},\\ J_l = \{1, \dots, l\} \quad \text{for } l \in \{r, k\}.\end{gathered} \tag{2.1}$$

l^∞_β denotes the Banach space of all bounded sequences $y = \{y_n\}_{n\in\mathbb{Z}_\beta}$ with the norm

$$\|y\| = \sup_{n\in\mathbb{Z}_\beta}|y_n| \quad \text{for } y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in l^\infty_\beta. \tag{2.2}$$

For any $M > N > 0$, put

$$\begin{aligned} V(N) &= \left\{y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in l^\infty_\beta : y_n \geq N, \forall n \in \mathbb{Z}_\beta\right\},\\ U(M) &= \{y \in V(N) : \|y\| < M\},\\ B(M,N) &= \left\{y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in l^\infty_\beta : \|y - M\| < N\right\},\\ A(N,M) &= \left\{y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in l^\infty_\beta : N \leq y_n \leq M, \forall n \in \mathbb{Z}_\beta\right\}.\end{aligned} \tag{2.3}$$

It is easy to see that $V(N)$ is a closed convex subset of l^∞_β, $U(M)$ is a bounded open subset of $V(N)$ and $B(M,N)$ is a bounded open convex subset of l^∞_β and $A(N,M)$ is a bounded closed and convex subset of l^∞_β.

By a solution of (1.8), we mean a sequence $\{y_n\}_{n\in\mathbb{Z}_\beta} \in l^\infty_\beta$ with a positive integer $T \geq n_0 + \tau + |\beta|$ such that (1.8) is satisfied for all $n \geq T$.

The following Lemmas play important roles in this paper.

Lemma 2.1 (Discrete Arzela-Ascoli's Theorem [3]). *A bounded, uniformly Cauchy subset Y of l^∞_β is relatively compact.*

Lemma 2.2 (Rothe Fixed Point Theorem [10]). *Let D be a bounded convex open subset of a Banach space E and $A : \overline{D} \to E$ be a continuous, condensing mapping, and $A(\partial D) \subseteq \overline{D}$. Then A has a fixed point in $\overline{D}$.*

Lemma 2.3 (Leray-Schauder Nonlinear Alternative Theorem [1]). *Let U be an open subset of a closed convex set K in a Banach space E with $p^* \in U$. Let $f : \overline{U} \to K$ be a continuous, condensing mapping with $f(\overline{U})$ bounded. Then either*

(a) *f has a fixed point in $\overline{U}$; or*

(b) *there exist an $x \in \partial U$ and a $\lambda \in (0,1)$ such that $x = (1-\lambda)p^* + \lambda f x$.*

Lemma 2.4 (Krasnoselskill Fixed Point Theorem [5]). *Let Y be a nonempty bounded closed convex subset of a Banach space X and f, g be mappings from Y into X such that $fx + gy \in Y$ for every pair $x, y \in Y$. If f is a contraction mapping and g is completely continuous, then the equation $fx + gx = x$ has at least one solution in Y.*

3. Main Results

Now we use the Rothe fixed point theorem to show the existence and multiplicity of bounded positive solutions of (1.8).

Theorem 3.1. *Assume that there exist two constants N and M with $M > N > 0$ and two positive sequences $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{p_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$\begin{aligned} |a(n,u_1,u_2,\ldots,u_r)| &\geq a_n, \quad \forall (n,u_d) \in \mathbb{N}_{n_0} \times [N,M],\ d \in J_r; \\ \left|f(n,u_1,u_2,\ldots,u_k)\right| &\leq p_n, \quad \forall (n,u_j) \in \mathbb{N}_{n_0} \times [N,M], \ \ j \in J_k; \end{aligned} \tag{3.1}$$

$$\sum_{s=n_0}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} \max\{p_i, |c_i|\} < +\infty; \tag{3.2}$$

$$b_n = 1 \text{ eventually}. \tag{3.3}$$

Then (1.8) *has uncountably many bounded positive solutions in $\overline{B(M,N)}$.*

Proof. Let $L \in (M-N, M+N)$. First of all, we show that there exists a mapping $S_L : \overline{B(M,N)} \to l_\beta^\infty$ with $S_L(\partial B(M,N)) \subseteq \overline{B(M,N)}$ such that S_L has a fixed point $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$, which is also a bounded positive solution of (1.8).

It follows from (3.2) and (3.3) that there exists $T \geq \max\{1, n_0 + \tau + |\beta|\}$ satisfying

$$b_n = 1, \quad \forall n \geq T; \tag{3.4}$$

$$\sum_{s=T}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) < \frac{1}{2} \min\{M + N - L, N - M + L\}. \tag{3.5}$$

Define a mapping $S_L : \overline{B(M,N)} \to l_\beta^\infty$ as follows:

$$(S_L y)_n = \begin{cases} L - \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \dfrac{1}{a(s, y_{a_{1s}}, \dots, y_{a_{rs}})} \sum_{i=s}^{\infty} [f(i, y_{f_{1i}}, \dots, y_{f_{ki}}) - c_i], & n \ge T \\ (S_L y)_T, & \beta \le n < T \end{cases} \tag{3.6}$$

for each $y = \{y_n\}_{n \in \mathbb{Z}_\beta} \in \overline{B(M,N)}$. On account of (3.1), (3.5), and (3.6), we conclude that for every $y = \{y_n\}_{n \in \mathbb{Z}_\beta} \in \partial B(M,N) \subseteq \overline{B(M,N)}$ and $n \ge T$

$$\begin{aligned} \left|(S_L y)_n - M\right| &= \left| L - M - \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \frac{1}{a(s, y_{a_{1s}}, \dots, y_{a_{rs}})} \sum_{i=s}^{\infty} [f(i, y_{f_{1i}}, \dots, y_{f_{ki}}) - c_i] \right| \\ &\le |L - M| + \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \frac{1}{|a(s, y_{a_{1s}}, \dots, y_{a_{rs}})|} \sum_{i=s}^{\infty} [|f(i, y_{f_{1i}}, \dots, y_{f_{ki}})| + |c_i|] \\ &\le |L - M| + \sum_{s=T+\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) \\ &< |L - M| + \frac{1}{2} \min\{M + N - L, N - M + L\} \\ &\le \frac{N}{2}, \end{aligned} \tag{3.7}$$

which means that

$$\|S_L y - M\| \le \frac{N}{2} < N, \tag{3.8}$$

that is, $S_L(\partial B(M,N)) \subseteq \overline{B(M,N)}$.

Now we assert that S_L is a continuous and condensing mapping in $\overline{B(M,N)}$. Let $y^\omega = \{y_n^\omega\}_{n \in \mathbb{Z}_\beta} \in \overline{B(M,N)}$ for each $\omega \in \mathbb{N}$ and $y = \{y_n\}_{n \in \mathbb{Z}_\beta} \in \overline{B(M,N)}$ with $\lim_{\omega \to \infty} y^\omega = y$. Let $\varepsilon > 0$. It follows from (3.2) and the continuity of a and f that there exist $T_1, T_2, T_3 \in \mathbb{N}$ with $T_1 > T$ and $T_2 > T_1 + \tau - 1$ satisfying

$$\max\left\{ \sum_{s=T_1+\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|), \sum_{s=T+\tau}^{T_1+\tau-1} \frac{1}{a_s} \sum_{i=T_2}^{\infty} (p_i + |c_i|) \right\} < \frac{\varepsilon}{16}; \tag{3.9}$$

$$\begin{aligned} \max&\left\{ \sum_{s=T+\tau}^{T_1+\tau-1} \frac{1}{a_s} \sum_{i=s}^{T_2-1} \left| f\left(i, y_{f_{1i}}^\omega, \dots, y_{f_{ki}}^\omega\right) - f(i, y_{f_{1i}}, \dots, y_{f_{ki}}) \right|, \right. \\ &\left. \sum_{s=T+\tau}^{T_1+\tau-1} \frac{\left|a(s, y_{a_{1s}}^\omega, \dots, y_{a_{rs}}^\omega) - a(s, y_{a_{1s}}, \dots, y_{a_{rs}})\right|}{a_s^2} \sum_{i=s}^{T_2-1} (p_i + |c_i|) \right\} < \frac{\varepsilon}{16}, \quad \forall \omega \ge T_3. \end{aligned} \tag{3.10}$$

In view of (3.1) and (3.6)–(3.10), we deduce that for any $\omega \geq T_3$

$$
\begin{aligned}
\left\|S_L y^{\omega} - S_L y\right\| = \sup_{n\geq T} & \left| \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \left[\frac{1}{a\left(s, y^{\omega}_{a_{1s}}, \dots, y^{\omega}_{a_{rs}}\right)} \sum_{i=s}^{\infty} \left[f\left(i, y^{\omega}_{f_{1i}}, \dots, y^{\omega}_{f_{ki}}\right) - c_i \right] \right.\right. \\
& \left.\left. - \frac{1}{a\left(s, y_{a_{1s}}, \dots, y_{a_{rs}}\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) - c_i \right] \right] \right| \\
= \sup_{n\geq T} & \left| \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \left[\frac{1}{a\left(s, y^{\omega}_{a_{1s}}, \dots, y^{\omega}_{a_{rs}}\right)} \sum_{i=s}^{\infty} \left[f\left(i, y^{\omega}_{f_{1i}}, \dots, y^{\omega}_{f_{ki}}\right) - c_i \right] \right.\right. \\
& - \frac{1}{a\left(s, y^{\omega}_{a_{1s}}, \dots, y^{\omega}_{a_{rs}}\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) - c_i \right] \\
& + \frac{1}{a\left(s, y^{\omega}_{a_{1s}}, \dots, y^{\omega}_{a_{rs}}\right)} \times \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) - c_i \right] \\
& \left.\left. - \frac{1}{a\left(s, y_{a_{1s}}, \dots, y_{a_{rs}}\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) - c_i \right] \right] \right| \\
= \sup_{n\geq T} & \left| \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \left[\frac{1}{a\left(s, y^{\omega}_{a_{1s}}, \dots, y^{\omega}_{a_{rs}}\right)} \right.\right. \\
& \times \sum_{i=s}^{\infty} \left[f\left(i, y^{\omega}_{f_{1i}}, \dots, y^{\omega}_{f_{ki}}\right) - f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) \right] \\
& + \left(\frac{1}{a\left(s, y^{\omega}_{a_{1s}}, \dots, y^{\omega}_{a_{rs}}\right)} - \frac{1}{a\left(s, y_{a_{1s}}, \dots, y_{a_{rs}}\right)} \right) \\
& \left.\left. \times \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) - c_i \right] \right] \right| \\
\leq \max & \left\{ \sup_{n\geq T_1} \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \frac{1}{\left|a\left(s, y^{\omega}_{a_{1s}}, \dots, y^{\omega}_{a_{rs}}\right)\right|} \right. \\
& \times \sum_{i=s}^{\infty} \left[\left| f\left(i, y^{\omega}_{f_{1i}}, \dots, y^{\omega}_{f_{ki}}\right) \right| + \left| f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) \right| \right] \\
& + \sup_{n\geq T_1} \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \left(\frac{1}{\left|a\left(s, y^{\omega}_{a_{1s}}, \dots, y^{\omega}_{a_{rs}}\right)\right|} + \frac{1}{\left|a\left(s, y_{a_{1s}}, \dots, y_{a_{rs}}\right)\right|} \right) \\
& \times \sum_{i=s}^{\infty} \left[\left| f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) \right| + |c_i| \right], \\
& \sup_{T\leq n\leq T_1-1} \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \frac{1}{\left|a\left(s, y^{\omega}_{a_{1s}}, \dots, y^{\omega}_{a_{rs}}\right)\right|} \\
& \times \sum_{i=s}^{\infty} \left| f\left(i, y^{\omega}_{f_{1i}}, \dots, y^{\omega}_{f_{ki}}\right) - f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) \right|
\end{aligned}
$$

$$
\begin{aligned}
&+\sup_{T\le n\le T_1-1}\sum_{l=1}^{\infty}\sum_{s=n+(2l-1)\tau}^{n+2l\tau-1}\left|\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\\
&\times\sum_{i=s}^{\infty}\left[\left|f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|+|c_i|\right]\Bigg\}\\
\le{}& \max\Bigg\{2\sum_{s=T_1+\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}p_i+2\sum_{s=T_1+\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|),\\
&\sum_{s=T+\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|\\
&+\sum_{s=T+\tau}^{\infty}\left|\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\sum_{i=s}^{\infty}(p_i+|c_i|)\Bigg\}\\
\le{}& \max\Bigg\{\frac{\varepsilon}{4},\sum_{s=T_1+\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|\\
&+\sum_{s=T+\tau}^{T_1+\tau-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|\\
&+\sum_{s=T+\tau}^{T_1+\tau-1}\frac{1}{a_s}\sum_{i=s}^{T_2-1}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|\\
&+\sum_{s=T_1+\tau}^{\infty}\left|\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\sum_{i=s}^{\infty}(p_i+|c_i|)\\
&+\sum_{s=T+\tau}^{T_1+\tau-1}\left|\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\sum_{i=T_2}^{\infty}(p_i+|c_i|)\\
&+\sum_{s=T+\tau}^{T_1+\tau-1}\left|\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\sum_{i=s}^{T_2-1}(p_i+|c_i|)\Bigg\}\\
\le{}& \max\Bigg\{\frac{\varepsilon}{4},2\sum_{s=T_1+\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}p_i+2\sum_{s=T+\tau}^{T_1+\tau-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}p_i+\frac{\varepsilon}{16}+2\sum_{s=T_1+\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|)\\
&+2\sum_{s=T+\tau}^{T_1+\tau-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}(p_i+|c_i|)+\frac{\varepsilon}{16}\Bigg\}\\
\le{}& \max\left\{\frac{\varepsilon}{4},\frac{5\varepsilon}{8}\right\}\\
<{}& \varepsilon,
\end{aligned}
\tag{3.11}
$$

which gives that $\lim_{\omega\to\infty}S_Ly^{\omega}=S_Ly$, that is, S_L is continuous in $\overline{B(M,N)}$.

In light of (3.1), (3.5), and (3.6), we get that for any $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$

$$\begin{aligned}
\|S_L y\| &= \sup_{n\geq T}\left| L - \sum_{l=1}^{\infty}\sum_{s=n+(2l-1)\tau}^{n+2l\tau-1}\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f(i,y_{f_{1i}},\dots,y_{f_{ki}})-c_i\right]\right| \\
&\leq L + \sum_{s=T+\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|) \\
&< L + \frac{1}{2}\min\{M+N-L, N-M+L\} \\
&\leq \frac{1}{2}(M+N+L),
\end{aligned} \tag{3.12}$$

which implies that $S_L(\overline{B(M,N)})$ is uniformly bounded.

Given $\varepsilon > 0$. Clearly (3.2) ensures that there exists $T^* > T$ satisfying

$$\sum_{s=T^*}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|) < \frac{\epsilon}{2}, \tag{3.13}$$

which together with (3.1) and (3.6) implies that for all $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$ and $t_2 > t_1 \geq T^*$

$$\begin{aligned}
\left|(S_L y)_{t_2} - (S_L y)_{t_1}\right| &= \left|\sum_{l=1}^{\infty}\sum_{s=t_2+(2l-1)\tau}^{t_2+2l\tau-1}\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f(i,y_{f_{1i}},\dots,y_{f_{ki}})-c_i\right]\right. \\
&\qquad\left. -\sum_{l=1}^{\infty}\sum_{s=t_1+(2l-1)\tau}^{t_1+2l\tau-1}\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f(i,y_{f_{1i}},\dots,y_{f_{ki}})-c_i\right]\right| \\
&\leq \sum_{l=1}^{\infty}\sum_{s=t_2+(2l-1)\tau}^{t_2+2l\tau-1}\frac{1}{\left|a(s,y_{a_{1s}},\dots,y_{a_{rs}})\right|}\sum_{i=s}^{\infty}\left[\left|f(i,y_{f_{1i}},\dots,y_{f_{ki}})\right|+|c_i|\right] \\
&\quad + \sum_{l=1}^{\infty}\sum_{s=t_1+(2l-1)\tau}^{t_1+2l\tau-1}\frac{1}{\left|a(s,y_{a_{1s}},\dots,y_{a_{rs}})\right|}\sum_{i=s}^{\infty}\left[\left|f(i,y_{f_{1i}},\dots,y_{f_{ki}})\right|+|c_i|\right] \\
&\leq 2\sum_{s=T^*+\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|) \\
&< \epsilon,
\end{aligned} \tag{3.14}$$

which yields that $S_L(\overline{B(M,N)})$ is uniformly Cauchy. Thus Lemma 2.1 means that $S_L(\overline{B(M,N)})$ is relatively compact. Consequently S_L is condensing in $\overline{B(M,N)}$.

It follows from Lemma 2.2 that S_L has a fixed point $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$, that is,

$$y_n = L - \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \frac{1}{a(s, y_{a_{1s}}, \dots, y_{a_{rs}})} \sum_{i=s}^{\infty} [f(i, y_{f_{1i}}, \dots, y_{f_{ki}}) - c_i], \quad \forall n \geq T, \tag{3.15}$$

which yields that

$$\begin{aligned} y_n + y_{n-\tau} &= 2L - \sum_{s=n}^{\infty} \frac{1}{a(s, y_{a_{1s}}, \dots, y_{a_{rs}})} \sum_{i=s}^{\infty} [f(i, y_{f_{1i}}, \dots, y_{f_{ki}}) - c_i], \quad \forall n \geq T + \tau, \\ a(n, y_{a_{1n}}, \dots, y_{a_{rn}}) \Delta(y_n + y_{n-\tau}) &= \sum_{i=n}^{\infty} [f(i, y_{f_{1i}}, \dots, y_{f_{ki}}) - c_i], \quad \forall n \geq T + \tau, \end{aligned} \tag{3.16}$$

which together with (3.3) implies that

$$\Delta(a(n, y_{a_{1n}}, \dots, y_{a_{rn}}) \Delta(y_n + b_n y_{n-\tau})) = -f(n, y_{f_{1n}}, \dots, y_{f_{kn}}) + c_n, \quad \forall n \geq T + \tau, \tag{3.17}$$

that is, (1.8) has a bounded positive solution $y \in \overline{B(M,N)}$.

Next we show that (1.8) has uncountably many bounded positive solutions in $\overline{B(M,N)}$. Let $L_1, L_2 \in (M-N, M+N)$ and $L_1 \neq L_2$. For every $\theta \in \{1,2\}$, we infer similarly that there exist a constant T_{L_θ} and a mapping S_{L_θ} satisfying (3.4)–(3.6), where L, T, and S_L are replaced by L_θ, T_{L_θ}, and S_{L_θ}, respectively, and the mapping S_{L_θ} has a fixed point $y^\theta = \{y_n^\theta\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$, which is a bounded positive solution of (1.8) in $\overline{B(M,N)}$, that is,

$$y_n^\theta = L_\theta - \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \frac{1}{a\left(s, y_{a_{1s}}^\theta, \dots, y_{a_{rs}}^\theta\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}^\theta, \dots, y_{f_{ki}}^\theta\right) - c_i\right], \quad \forall n \geq T_{L_\theta}. \tag{3.18}$$

Equation (3.2) ensures that there exists $T_* > \max\{T_{L_1}, T_{L_2}\}$ satisfying

$$\sum_{s=T_*}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) < \frac{|L_1 - L_2|}{4}. \tag{3.19}$$

In order to show that the set of bounded positive solutions of (1.8) is uncountable, it is sufficient to prove that $y^1 \neq y^2$. It follows from (3.1), (3.18), and (3.19) that for all $n \geq T_*$

$$\begin{aligned} \left|y_n^1 - y_n^2\right| = \Bigg| & L_1 - \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \frac{1}{a(s, y_{a_{1s}}^1, \dots, y_{a_{rs}}^1)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}^1, \dots, y_{f_{ki}}^1\right) - c_i\right] \\ & - L_2 + \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \frac{1}{a(s, y_{a_{1s}}^2, \dots, y_{a_{rs}}^2)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}^2, \dots, y_{f_{ki}}^2\right) - c_i\right] \Bigg| \end{aligned}$$

$$\begin{aligned}
&\geq |L_1 - L_2| - \sum_{l=1}^{\infty} \sum_{s=n+(2l-1)\tau}^{n+2l\tau-1} \frac{1}{a_s} \\
&\quad \times \sum_{i=s}^{\infty} \left[\left| f\left(i, y^1_{f_{1i}}, \ldots, y^1_{f_{ki}}\right) \right| + |c_i| + \left| f\left(i, y^2_{f_{1i}}, \ldots, y^2_{f_{ki}}\right) \right| + |c_i| \right] \\
&\geq |L_1 - L_2| - 2 \sum_{s=T_*+\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) \\
&> \frac{1}{2} |L_1 - L_2|,
\end{aligned} \tag{3.20}$$

that is, $y^1 \neq y^2$. This completes the proof. □

Theorem 3.2. *Assume that there exist two constants N and M with $M > N > 0$ and two positive sequences $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{p_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.1) *and*

$$\sum_{l=1}^{\infty} \sum_{s=n_0+l\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} \max\{p_i, |c_i|\} < +\infty; \tag{3.21}$$

$$b_n = -1 \quad \textit{eventually}. \tag{3.22}$$

Then (1.8) *has uncountably many bounded positive solutions.*

Proof. Let $L \in (M - N, M + N)$. Firstly, we show that there exists a mapping $S_L : \overline{B(M,N)} \to l^{\infty}_{\beta}$ with $S_L(\partial B(M,N)) \subseteq \overline{B(M,N)}$ such that S_L has a fixed point $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$, which is also a bounded positive solution of (1.8). In view of (3.21) and (3.22), we choose a sufficiently large integer $T \geq \max\{1, n_0 + \tau + |\beta|\}$ such that

$$b_n = -1, \quad \forall n \geq T; \tag{3.23}$$

$$\sum_{l=1}^{\infty} \sum_{s=T+l\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) < \frac{1}{2} \min\{M + N - L, N - M + L\}. \tag{3.24}$$

Define a mapping $S_L : \overline{B(M,N)} \to l^{\infty}_{\beta}$ as follows:

$$(S_L y)_n = \begin{cases} L + \displaystyle\sum_{l=1}^{\infty} \sum_{s=n+l\tau}^{\infty} \frac{1}{a(s, y_{a_{1s}}, \ldots, y_{a_{rs}})} \sum_{i=s}^{\infty} [f(i, y_{f_{1i}}, \ldots, y_{f_{ki}}) - c_i], & n \geq T \\ (S_L y)_T, & \beta \leq n < T \end{cases} \tag{3.25}$$

for each $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$. It follows from (3.1), (3.24), and (3.25) that for every $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \partial B(M,N) \subseteq \overline{B(M,N)}$ and $n \geq T$

$$\begin{aligned}
\left|(S_L y)_n - M\right| &= \left| L - M + \sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\frac{1}{a(s, y_{a_{1s}},\dots,y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f(i, y_{f_{1i}},\dots,y_{f_{ki}}) - c_i\right]\right| \\
&\leq |L-M| + \sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\frac{1}{\left|a(s, y_{a_{1s}},\dots,y_{a_{rs}})\right|}\sum_{i=s}^{\infty}\left[\left|f(i, y_{f_{1i}},\dots,y_{f_{ki}})\right| + |c_i|\right] \\
&\leq |L-M| + \sum_{l=1}^{\infty}\sum_{s=T+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i + |c_i|) \\
&< |L-M| + \frac{1}{2}\min\{M+N-L, N-M+L\} \\
&\leq \frac{N}{2},
\end{aligned} \tag{3.26}$$

which means that

$$\|S_L y - M\| \leq \frac{N}{2} < N, \tag{3.27}$$

that is, $S_L(\partial \overline{B(M,N)} \subseteq \overline{B(M,N)}$.

Now we prove that S_L is a continuous and condensing mapping in $\overline{B(M,N)}$. Put $y^\omega = \{y_n^\omega\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$ for each $\omega \in \mathbb{N}$ and $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{B(M,N)}$ with $\lim_{\omega\to\infty} y^\omega = y$. Let $\varepsilon > 0$. Using (3.21) and the continuity of a and f, we conclude that there exist four positive integers T_1, T_2, T_3, and T_4 with $T_3 > T$, $T_2 > T_3 + T_1\tau$ satisfying

$$\begin{aligned}
\max\Bigg\{&\sum_{l=1}^{\infty}\sum_{s=T_3+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|), \sum_{l=T_1}^{\infty}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|), \\
&\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}(p_i+|c_i|)\Bigg\} < \frac{\varepsilon}{16};
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
\max\Bigg\{&\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{1}{a_s}\sum_{i=s}^{T_2-1}\left|f\left(i, y^\omega_{f_{1i}},\dots,y^\omega_{f_{ki}}\right) - f(i, y_{f_{1i}},\dots,y_{f_{ki}})\right|, \\
&\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{\left|a(s, y^\omega_{a_{1s}},\dots,y^\omega_{a_{rs}}) - a(s, y_{a_{1s}},\dots,y_{a_{rs}})\right|}{a_s^2}\sum_{i=s}^{T_2-1}(p_i+|c_i|)\Bigg\} < \frac{\varepsilon}{16}, \quad \forall \omega \geq T_4.
\end{aligned} \tag{3.29}$$

By virtue of (3.1) and (3.25)–(3.29), we infer that for each $\omega \geq T_4$

$$
\begin{aligned}
\|S_L y^\omega - S_L y\| &= \sup_{n\geq T}\Bigg|\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\frac{1}{a\left(s,y^\omega_{a_{1s}},\dots,y^\omega_{a_{rs}}\right)}\sum_{i=s}^{\infty}\left[f\left(i,y^\omega_{f_{1i}},\dots,y^\omega_{f_{ki}}\right)-c_i\right]\\
&\qquad -\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\sum_{i=s}^{\infty}\left[f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)-c_i\right]\Bigg|\\
&= \sup_{n\geq T}\Bigg|\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\frac{1}{a\left(s,y^\omega_{a_{1s}},\dots,y^\omega_{a_{rs}}\right)}\sum_{i=s}^{\infty}\left[f\left(i,y^\omega_{f_{1i}},\dots,y^\omega_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right]\\
&\qquad +\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\left(\frac{1}{a\left(s,y^\omega_{a_{1s}},\dots,y^\omega_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right)\\
&\qquad \times\sum_{i=s}^{\infty}\left[f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)-c_i\right]\Bigg|\\
&\leq \max\Bigg\{\sup_{n\geq T_3}\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\frac{1}{\left|a\left(s,y^\omega_{a_{1s}},\dots,y^\omega_{a_{rs}}\right)\right|}\\
&\qquad \times\sum_{i=s}^{\infty}\left[\left|f\left(i,y^\omega_{f_{1i}},\dots,y^\omega_{f_{ki}}\right)\right|+\left|f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|\right]\\
&\qquad +\sup_{n\geq T_3}\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\left(\frac{1}{\left|a\left(s,y^\omega_{a_{1s}},\dots,y^\omega_{a_{rs}}\right)\right|}+\frac{1}{\left|a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)\right|}\right)\\
&\qquad \times\sum_{i=s}^{\infty}\left[\left|f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|+|c_i|\right],\\
&\qquad \sup_{T\leq n\leq T_3-1}\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\frac{1}{\left|a\left(s,y^\omega_{a_{1s}},\dots,y^\omega_{a_{rs}}\right)\right|}\\
&\qquad \times\sum_{i=s}^{\infty}\left|f\left(i,y^\omega_{f_{1i}},\dots,y^\omega_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|\\
&\qquad +\sup_{T\leq n\leq T_3-1}\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\left|\frac{1}{a\left(s,y^\omega_{a_{1s}},\dots,y^\omega_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\\
&\qquad \times\sum_{i=s}^{\infty}\left[\left|f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|+|c_i|\right]\Bigg\}\\
&\leq \max\Bigg\{2\sum_{l=1}^{\infty}\sum_{s=T_3+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}p_i+2\sum_{l=1}^{\infty}\sum_{s=T_3+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|),\\
&\qquad \sum_{l=1}^{\infty}\sum_{s=T+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}\left|f\left(i,y^\omega_{f_{1i}},\dots,y^\omega_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|\\
&\qquad +\sum_{l=1}^{\infty}\sum_{s=T+l\tau}^{\infty}\left|\frac{1}{a\left(s,y^\omega_{a_{1s}},\dots,y^\omega_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\sum_{i=s}^{\infty}(p_i+|c_i|)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\le \max\Bigg\{\frac{\varepsilon}{4}, \sum_{l=1}^{\infty}\sum_{s=T_3+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right| \\
&\quad+\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{1}{a_s}\sum_{i=s}^{T_2-1}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right| \\
&\quad+\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right| \\
&\quad+\sum_{l=T_1}^{\infty}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{1}{a_s}\sum_{i=s}^{\infty}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right| \\
&\quad+\sum_{l=1}^{\infty}\sum_{s=T_3+l\tau}^{\infty}\left|\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\sum_{i=s}^{\infty}\left(p_i+|c_i|\right) \\
&\quad+\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\left|\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\sum_{i=s}^{T_2-1}\left(p_i+|c_i|\right) \\
&\quad+\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\left|\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\sum_{i=T_2}^{\infty}\left(p_i+|c_i|\right) \\
&\quad+\sum_{l=T_1}^{\infty}\sum_{s=T+l\tau}^{T_3+l\tau-1}\left|\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right|\sum_{i=s}^{\infty}\left(p_i+|c_i|\right)\Bigg\} \\
&\le \max\Bigg\{\frac{\varepsilon}{4}, 2\sum_{l=1}^{\infty}\sum_{s=T_3+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}p_i+\frac{\varepsilon}{16}+2\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}p_i+2\sum_{l=T_1}^{\infty}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{1}{a_s}\sum_{i=s}^{\infty}p_i \\
&\quad+2\sum_{l=1}^{\infty}\sum_{s=T_3+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}\left(p_i+|c_i|\right)+\frac{\varepsilon}{16}+2\sum_{l=1}^{T_1-1}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}\left(p_i+|c_i|\right) \\
&\quad+2\sum_{l=T_1}^{\infty}\sum_{s=T+l\tau}^{T_3+l\tau-1}\frac{1}{a_s}\sum_{i=s}^{\infty}\left(p_i+|c_i|\right)\Bigg\} \\
&< \max\left\{\frac{\varepsilon}{4},\frac{7\varepsilon}{8}\right\} \\
&< \varepsilon,
\end{aligned}
\tag{3.30}
$$

which implies that $\lim_{\omega\to\infty}S_L y^{\omega}=S_L y$, that is, S_L is continuous in $\overline{B(M,N)}$.

From (3.1), (3.24), and (3.25), we infer that for any $y=\{y_n\}_{n\in\mathbb{Z}_\beta}\in\overline{B(M,N)}$

$$
\|S_L y\| = \sup_{n\ge T}\left|L+\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\sum_{i=s}^{\infty}\left[f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)-c_i\right]\right|
$$

$$\begin{aligned}
&\le L+\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|)\\
&< L+\frac{1}{2}\min\{M+N-L,N-M+L\}\\
&\le \frac{1}{2}(M+N+L),
\end{aligned}\tag{3.31}$$

which implies that $S_L(\overline{B(M,N)})$ is uniformly bounded.

Let $\varepsilon>0$. It follows from (3.21) that there exists $T^*>T$ satisfying

$$\sum_{l=1}^{\infty}\sum_{s=T^*+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|)<\frac{\varepsilon}{2},\tag{3.32}$$

which together with (3.1) and (3.25) yields that for all $y=\{y_n\}_{n\in\mathbb{Z}_\beta}\in\overline{B(M,N)}$ and $t_2>t_1\ge T^*$

$$\begin{aligned}
\left|(S_Ly)_{t_2}-(S_Ly)_{t_1}\right| &= \Bigg|\sum_{l=1}^{\infty}\sum_{s=t_2+l\tau}^{\infty}\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)-c_i\right]\\
&\quad -\sum_{l=1}^{\infty}\sum_{s=t_1+l\tau}^{\infty}\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)-c_i\right]\Bigg|\\
&\le \sum_{l=1}^{\infty}\sum_{s=t_2+l\tau}^{\infty}\frac{1}{\left|a(s,y_{a_{1s}},\dots,y_{a_{rs}})\right|}\sum_{i=s}^{\infty}\left[\left|f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|+|c_i|\right]\\
&\quad +\sum_{l=1}^{\infty}\sum_{s=t_1+l\tau}^{\infty}\frac{1}{\left|a(s,y_{a_{1s}},\dots,y_{a_{rs}})\right|}\sum_{i=s}^{\infty}\left[\left|f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|+|c_i|\right]\\
&\le 2\sum_{l=1}^{\infty}\sum_{s=T^*+l\tau}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|)\\
&<\varepsilon,
\end{aligned}\tag{3.33}$$

which gives that $S_L(\overline{B(M,N)})$ is uniformly Cauchy. Hence Lemma 2.1 implies that $S_L(\overline{B(M,N)})$ is relatively compact, that is, S_L is condensing in $\overline{B(M,N)}$.

It is clear that Lemma 2.2 means that S_L possesses a fixed point $y=\{y_n\}_{n\in\mathbb{Z}_\beta}\in\overline{B(M,N)}$, that is,

$$\begin{aligned}
y_n &= L+\sum_{l=1}^{\infty}\sum_{s=n+l\tau}^{\infty}\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)-c_i\right],\quad \forall n\ge T,\\
y_{n-\tau} &= L+\sum_{l=1}^{\infty}\sum_{s=n+(l-1)\tau}^{\infty}\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)-c_i\right],\quad \forall n\ge T+\tau
\end{aligned}\tag{3.34}$$

which lead to

$$y_n - y_{n-\tau} = -\sum_{s=n}^{\infty} \frac{1}{a(s, y_{a_{1s}}, \ldots, y_{a_{rs}})} \sum_{i=s}^{\infty} [f(i, y_{f_{1i}}, \ldots, y_{f_{ki}}) - c_i], \quad \forall n \geq T + \tau,$$
$$\Delta(y_n - y_{n-\tau}) = \frac{1}{a(n, y_{a_{1n}}, \ldots, y_{a_{rn}})} \sum_{i=n}^{\infty} [f(i, y_{f_{1i}}, \ldots, y_{f_{ki}}) - c_i], \quad \forall n \geq T + \tau, \tag{3.35}$$

which together with (3.23) yields that

$$\Delta(a(n, y_{a_{1n}}, \ldots, y_{a_{rn}}) \Delta(y_n + b_n y_{n-\tau})) = -f(n, y_{f_{1n}}, \ldots, y_{f_{kn}}) + c_n, \quad \forall n \geq T + \tau, \tag{3.36}$$

that is, (1.8) has a bounded positive solution in $\overline{B(M, N)}$.

Next we show that (1.8) has uncountably many bounded positive solutions in $\overline{B(M, N)}$. Let $L_1, L_2 \in (M - N, M + N)$ and $L_1 \neq L_2$. Similarly we infer that for each $\theta \in \{1, 2\}$, there exist a constant T_{L_θ} and a mapping S_{L_θ} satisfying (3.23)–(3.25), where L, T and S_L are replaced by L_θ, T_{L_θ}, and S_{L_θ}, respectively, and the mapping S_{L_θ} has a fixed point $y^\theta = \{y_n^\theta\}_{n \in \mathbb{Z}_\beta} \in \overline{B(M, N)}$, which is a bounded positive solution of (1.8) in $\overline{B(M, N)}$, that is,

$$y_n^\theta = L_\theta + \sum_{l=1}^{\infty} \sum_{s=n+l\tau}^{\infty} \frac{1}{a\left(s, y_{a_{1s}}^\theta, \ldots, y_{a_{rs}}^\theta\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}^\theta, \ldots, y_{f_{ki}}^\theta\right) - c_i\right], \quad \forall n \geq T_{L_\theta}. \tag{3.37}$$

It follows from (3.21) that there exists $T_* > \max\{T_{L_1}, T_{L_2}\}$ such that

$$\sum_{l=1}^{\infty} \sum_{s=T_*+l\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) < \frac{|L_1 - L_2|}{4}. \tag{3.38}$$

In order to show that the set of bounded positive solutions of (1.8) is uncountable, it is sufficient to prove that $y^1 \neq y^2$. By means of (3.1), (3.37) and (3.38), we infer that for each $n \geq T_*$

$$\begin{aligned}
\left|y_n^1 - y_n^2\right| &= \Bigg| L_1 + \sum_{l=1}^{\infty} \sum_{s=n+l\tau}^{\infty} \frac{1}{a\left(s, y_{a_{1s}}^1, \ldots, y_{a_{rs}}^1\right)} \sum_{i=s}^{\infty}\left[f\left(i, y_{f_{1i}}^1, \ldots, y_{f_{ki}}^1\right) - c_i\right] \\
&\qquad - L_2 - \sum_{l=1}^{\infty} \sum_{s=n+l\tau}^{\infty} \frac{1}{a\left(s, y_{a_{1s}}^2, \ldots, y_{a_{rs}}^2\right)} \sum_{i=s}^{\infty}\left[f\left(i, y_{f_{1i}}^2, \ldots, y_{f_{ki}}^2\right) - c_i\right] \Bigg| \\
&\geq |L_1 - L_2| - \sum_{l=1}^{\infty} \sum_{s=n+l\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty}\left[\left|f\left(i, y_{f_{1i}}^1, \ldots, y_{f_{ki}}^1\right)\right| + |c_i| + \left|f\left(i, y_{f_{1i}}^2, \ldots, y_{f_{ki}}^2\right)\right| + |c_i|\right] \\
&\geq |L_1 - L_2| - 2\sum_{l=1}^{\infty} \sum_{s=T_*+l\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty}(p_i + |c_i|) \\
&> \frac{1}{2}|L_1 - L_2|,
\end{aligned} \tag{3.39}$$

that is, $y^1 \neq y^2$. This completes the proof. □

Next we use the Leray-Schauder nonlinear alternative theorem to show the existence and multiplicity of bounded positive solutions of (1.8).

Theorem 3.3. *Assume that there exist four constants N, M, b_*, and b^* and two positive sequences $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{p_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.1), (3.2) *and*

$$0 < N < (1 - b_* - b^*)M, \quad b_* \geq 0,\ b^* \geq 0,\ b_* + b^* < 1,\ -b_* \leq b_n \leq b^* \ \text{eventually}. \tag{3.40}$$

Then (1.8) *has uncountably many bounded positive solutions in $\overline{U(M)}$.*

Proof. Let $L \in (b^*M + N, (1 - b_*)M)$. Now we prove that there exists a mapping $S_L : \overline{U(M)} \to V(N)$ such that it has a fixed point $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$, which is also a bounded positive solution of (1.8). It follows from (3.2), (3.40) and that there exists a sufficiently large number $T \geq \max\{1, n_0 + \tau + |\beta|\}$ satisfying

$$-b_* \leq b_n \leq b^*, \quad \forall n \geq T; \tag{3.41}$$

$$\sum_{s=T}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty}(p_i + |c_i|) < \min\{L - b^*M - N, (1 - b_*)M - L\}. \tag{3.42}$$

Put $p^* = M - \varepsilon^*$, where $\varepsilon^* \in (0, \min\{L - b^*M - N, (1 - b_*)M - L, (M - N)/2\})$ is enough small and

$$\sum_{s=T}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty}(p_i + |c_i|) < \min\{L - b^*M - N, (1 - b_*)M - L\} - \varepsilon^*. \tag{3.43}$$

Obviously, $p^* \in U(M)$. Define a mapping $S_L : \overline{U(M)} \longrightarrow l_\beta^\infty$ by

$$\left(S_L y\right)_n = \left(S_{1L} y\right)_n + \left(S_{2L} y\right)_n, \quad n \geq \beta \tag{3.44}$$

for each $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$, where the mappings $S_{1L}, S_{2L} : \overline{U(M)} \longrightarrow l_\beta^\infty$ are defined by

$$\left(S_{1L} y\right)_n = \begin{cases} L - b_n y_{n-\tau}, & n \geq T \\ \left(S_{1L} y\right)_T, & \beta \leq n < T, \end{cases} \tag{3.45}$$

$$\left(S_{2L} y\right)_n = \begin{cases} -\displaystyle\sum_{s=n}^{\infty} \frac{1}{a\left(s, y_{a_{1s}}, \ldots, y_{a_{rs}}\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}, \ldots, y_{f_{ki}}\right) - c_i\right], & n \geq T \\ \left(S_{2L} y\right)_T, & \beta \leq n < T. \end{cases} \tag{3.46}$$

It follows from (3.1), (3.41), and (3.43)–(3.46) that for any $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$ and $n \geq T$

$$\begin{aligned} \left(S_L y\right)_n &= \left(S_{1L} y\right)_n + \left(S_{2L} y\right)_n \\ &= L - b_n y_{n-\tau} - \sum_{s=n}^{\infty} \frac{1}{a\left(s, y_{a_{1s}}, \ldots, y_{a_{rs}}\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}, \ldots, y_{f_{ki}}\right) - c_i\right] \\ &\geq L - b^* M - \sum_{s=n}^{\infty} \frac{1}{\left|a\left(s, y_{a_{1s}}, \ldots, y_{a_{rs}}\right)\right|} \sum_{i=s}^{\infty} \left(p_i + |c_i|\right) \\ &\geq L - b^* M - \sum_{s=T}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} \left(p_i + |c_i|\right) \\ &> L - b^* M - \min\{L - b^* M - N, (1 - b_*)M - L\} + \varepsilon^* \\ &\geq N + \varepsilon^* \\ &> N, \end{aligned} \tag{3.47}$$

which yields that $S_L(\overline{U(M)}) \subseteq V(N)$.

Next we show that $S_{2L} : \overline{U(M)} \longrightarrow l_\beta^\infty$ is a continuous and relatively compact mapping. Let $y^\omega = \{y_n^\omega\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$ and $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$ with $\lim_{\omega\to\infty} y^\omega = y$. By virtue of (3.2) and the continuity of a and f, we infer that there exist $T_1, T_2, T_3 \in \mathbb{N}$ with $T_2 > T_1 - 1 > T$ satisfying

$$\max\left\{\sum_{s=T_1}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|),\sum_{s=T}^{T_1-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}(p_i+|c_i|)\right\}<\frac{\varepsilon}{16}; \tag{3.48}$$

$$\begin{aligned}\max\Bigg\{&\sum_{s=T}^{T_1-1}\frac{1}{|a_s|}\sum_{i=s}^{T_2-1}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|,\\ &\sum_{s=T}^{T_1-1}\frac{\left|a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)-a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)\right|}{a_s^2}\sum_{i=s}^{T_2-1}(p_i+|c_i|)\Bigg\}<\frac{\varepsilon}{16},\quad\forall\omega\ge T_3.\end{aligned} \tag{3.49}$$

It follows from (3.1) and (3.46)–(3.49) that for each $\omega\ge T_3$

$$\begin{aligned}\left\|S_{2L}y^{\omega}-S_{2L}y\right\|&=\sup_{n\ge T}\Bigg|\sum_{s=n}^{\infty}\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}\sum_{i=s}^{\infty}\left[f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-c_i\right]\\ &\qquad-\sum_{s=n}^{\infty}\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\sum_{i=s}^{\infty}\left[f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)-c_i\right]\Bigg|\\ &=\sup_{n\ge T}\Bigg|\sum_{s=n}^{\infty}\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}\sum_{i=s}^{\infty}\left[f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right]\\ &\qquad+\sum_{s=n}^{\infty}\left(\frac{1}{a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)}-\frac{1}{a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)}\right)\\ &\qquad\times\sum_{i=s}^{\infty}\left[f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)-c_i\right]\Bigg|\\ &\le\max\Bigg\{\sup_{n\ge T_1}\sum_{s=n}^{\infty}\frac{1}{\left|a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)\right|}\\ &\qquad\times\sum_{i=s}^{\infty}\left[\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)\right|+\left|f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|\right]\\ &\qquad+\sup_{n\ge T_1}\sum_{s=n}^{\infty}\left(\frac{1}{\left|a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)\right|}+\frac{1}{\left|a\left(s,y_{a_{1s}},\dots,y_{a_{rs}}\right)\right|}\right)\\ &\qquad\times\sum_{i=s}^{\infty}\left[\left|f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|+|c_i|\right],\\ &\qquad\sup_{T\le n\le T_1-1}\sum_{s=n}^{\infty}\frac{1}{\left|a\left(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}}\right)\right|}\\ &\qquad\times\sum_{i=s}^{\infty}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f\left(i,y_{f_{1i}},\dots,y_{f_{ki}}\right)\right|\end{aligned}$$

$$
\begin{aligned}
&\quad + \sup_{T\le n\le T_1-1}\sum_{s=n}^{\infty}\left|\frac{1}{a(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}})}-\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\right| \\
&\quad \times\sum_{i=s}^{\infty}\left[\left|f(i,y_{f_{1i}},\dots,y_{f_{ki}})\right|+|c_i|\right]\Bigg\} \\
&\le \max\Bigg\{2\sum_{s=T_1}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}p_i+2\sum_{s=T_1}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|), \\
&\quad \sum_{s=T}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}\left|f\left(i,y^{\omega}_{f_{1i}},y^{\omega}_{f_{2i}},\dots,y^{\omega}_{f_{ki}}\right)-f(i,y_{f_{1i}},y_{f_{2i}},\dots,y_{f_{ki}})\right| \\
&\quad +\sum_{s=T}^{\infty}\left|\frac{1}{a(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}})}-\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\right|\left|\sum_{i=s}^{\infty}(p_i+|c_i|)\right\} \\
&\le \max\Bigg\{\frac{\varepsilon}{4},\sum_{s=T}^{T_1-1}\frac{1}{a_s}\sum_{i=s}^{T_2-1}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f(i,y_{f_{1i}},\dots,y_{f_{ki}})\right| \\
&\quad +\sum_{s=T}^{T_1-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f(i,y_{f_{1i}},\dots,y_{f_{ki}})\right| \\
&\quad +\sum_{s=T_1}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}\left|f\left(i,y^{\omega}_{f_{1i}},\dots,y^{\omega}_{f_{ki}}\right)-f(i,y_{f_{1i}},\dots,y_{f_{ki}})\right| \\
&\quad +\sum_{s=T}^{T_1-1}\left|\frac{1}{a(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}})}-\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\right|\left|\sum_{i=s}^{T_2-1}(p_i+|c_i|)\right. \\
&\quad +\sum_{s=T}^{T_1-1}\left|\frac{1}{a(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}})}-\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\right|\left|\sum_{i=T_2}^{\infty}(p_i+|c_i|)\right. \\
&\quad +\sum_{s=T_1}^{\infty}\left|\frac{1}{a(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}})}-\frac{1}{a(s,y_{a_{1s}},\dots,y_{a_{rs}})}\right|\left|\sum_{i=s}^{\infty}(p_i+|c_i|)\right\} \\
&\le \max\Bigg\{\frac{\varepsilon}{4},\frac{\varepsilon}{16}+2\sum_{s=T}^{T_1-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}p_i+2\sum_{s=T_1}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}p_i \\
&\quad +\sum_{s=T}^{T_1-1}\frac{\left|a(s,y^{\omega}_{a_{1s}},\dots,y^{\omega}_{a_{rs}})-a(s,y_{a_{1s}},\dots,y_{a_{rs}})\right|}{a_s^2}\sum_{i=s}^{T_2-1}(p_i+|c_i|) \\
&\quad +2\sum_{s=T}^{T_1-1}\frac{1}{a_s}\sum_{i=T_2}^{\infty}(p_i+|c_i|)+2\sum_{s=T_1}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i+|c_i|)\Bigg\} \\
&\le \max\left\{\frac{\varepsilon}{4},\frac{5\varepsilon}{8}\right\} \\
&<\varepsilon,
\end{aligned}
\tag{3.50}
$$

which yields that $\lim_{\omega\to\infty}\|S_{2L}y^{\omega}-S_{2L}y\|=0$, that is, S_{2L} is continuous in $\overline{U(M)}$.

In light of (3.1) and (3.43)–(3.46), we deduce that for all $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$

$$\begin{aligned}
\|S_L y\| &= \sup_{n\ge T}\left|(S_{1L}y)_n + (S_{2L}y)_n\right| \\
&\le \sup_{n\ge T}\left|(S_{1L}y)_n\right| + \sup_{n\ge T}\left|(S_{2L}y)_n\right| \\
&\le \sup_{n\ge T}\left|L - b_n y_{n-\tau}\right| + \sup_{n\ge T}\left|\sum_{s=n}^{\infty}\frac{1}{a(s, y_{a_{1s}}, \dots, y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) - c_i\right]\right| \\
&\le \sup_{n\ge T}\left(L + |b_n|\left|y_{n-\tau}\right|\right) + \sup_{n\ge T}\sum_{s=n}^{\infty}\frac{1}{\left|a(s, y_{a_{1s}}, \dots, y_{a_{rs}})\right|}\sum_{i=s}^{\infty}\left[\left|f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right)\right| + |c_i|\right] \\
&\le L + (b_* + b^*)M + \sum_{s=T}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i + |c_i|) \\
&\le L + M + \min\{L - b^*M - N, (1-b_*)M - L\} - \varepsilon^* \\
&< 2L + M,
\end{aligned} \tag{3.51}$$

which means that $S_L(\overline{U(M)})$ and $S_{2L}(\overline{U(M)})$ are bounded.

Let $\varepsilon > 0$. Notice that (3.2) ensures that there exists $T^* > T$ satisfying

$$\sum_{s=T^*}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i + |c_i|) < \frac{\epsilon}{2}, \tag{3.52}$$

which together with (3.1) and (3.46) implies that for all $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$ and $t_2 > t_1 \ge T^*$

$$\begin{aligned}
\left|(S_{2L}y)_{t_2} - (S_{2L}y)_{t_1}\right| &= \left|\sum_{s=t_2}^{\infty}\frac{1}{a(s, y_{a_{1s}}, \dots, y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) - c_i\right]\right. \\
&\qquad \left. -\sum_{s=t_1}^{\infty}\frac{1}{a(s, y_{a_{1s}}, \dots, y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) - c_i\right]\right| \\
&\le \sum_{s=t_2}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i + |c_i|) + \sum_{s=t_1}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i + |c_i|) \\
&\le 2\sum_{s=T^*}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i + |c_i|) \\
&< \epsilon,
\end{aligned} \tag{3.53}$$

which means that $S_{2L}(\overline{U(M)})$ is uniformly Cauchy. Thus $S_{2L}(\overline{U(M)})$ is relatively compact.

By virtue of (3.41) and (3.45), we infer that for all $x = \{x_n\}_{n\in\mathbb{Z}_\beta}, y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$ and $n \geq T$

$$\left|(S_{1L}x)_n - (S_{1L}y)_n\right| = |b_n|\left|x_{n-\tau} - y_{n-\tau}\right| \leq (b_* + b^*)\|x - y\|, \tag{3.54}$$

which yields that

$$\|S_{1L}x - S_{1L}y\| \leq (b_* + b^*)\|x - y\|, \tag{3.55}$$

that is, S_{1L} is a contraction mapping in $\overline{U(M)}$. It follows that S_L is a continuous and condensing mapping.

Put

$$P = \left\{y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in l_\beta^\infty : N \leq y_n \leq M, n \geq \beta,\ \|y\| = M\right\}, \tag{3.56}$$

$$Q = \left\{y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in l_\beta^\infty : N \leq y_n \leq M, n \geq \beta \text{ and there exists } n^* \geq \beta \text{ satisfying } y_{n^*} = N\right\}. \tag{3.57}$$

It is easy to see that $\partial U(M) = P \cup Q$. Suppose that there exist $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \partial U(M)$ and $\lambda \in (0,1)$ with

$$y = (1-\lambda)p^* + \lambda S_L y. \tag{3.58}$$

Now we consider two possible cases as follows.

Case 1. Let $y \in P$. Obviously (3.41), (3.43)–(3.46), (3.56), and (3.58) guarantee that

$$\begin{aligned}
y_n &= (1-\lambda)p^* + \lambda S_L y_n \\
&= (1-\lambda)p^* + \lambda\left[L - b_n y_{n-\tau} - \sum_{s=n}^{\infty}\frac{1}{a(s, y_{a_{1s}}, \ldots, y_{a_{rs}})}\sum_{i=s}^{\infty}\left[f(i, y_{f_{1i}}, \ldots, y_{f_{ki}}) - c_i\right]\right] \\
&\leq (1-\lambda)(M-\varepsilon^*) + \lambda\left[L + b_* M + \sum_{s=T}^{\infty}\frac{1}{a_s}\sum_{i=s}^{\infty}(p_i + |c_i|)\right] \\
&< (1-\lambda)(M-\varepsilon^*) + \lambda[L + b_* M + \min\{L - b^* M - N, (1-b_*)M - L\} - \varepsilon^*] \\
&\leq M - \varepsilon^*, \quad \forall n \geq T,
\end{aligned} \tag{3.59}$$

which implies that

$$M = \|y\| = \sup_{n\geq\beta}|y_n| \leq M - \varepsilon^* < M, \tag{3.60}$$

which is a contradiction.

Case 2. Let $y \in Q$. It follows from (3.41), (3.43)–(3.46), (3.57), and (3.58) that

$$\begin{aligned}
N &= y_{n^*} = (1-\lambda)p^* + \lambda S_L y_{n^*} \\
&= (1-\lambda)p^* + \lambda\left[L - b_{n^*} y_{n^*-\tau} - \sum_{s=\max\{n^*,T\}}^{\infty} \frac{1}{a\left(s, y_{a_{1s}}, \ldots, y_{a_{rs}}\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}, \ldots, y_{f_{ki}}\right) - c_i\right]\right] \\
&\geq (1-\lambda)(M-\varepsilon^*) + \lambda\left[L - b^* M - \sum_{s=\max\{n^*,T\}}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|)\right] \\
&> (1-\lambda)(M-\varepsilon^*) + \lambda\left[L - b^* M - \min\{L - b^* M - N, (1-b_*)M - L\} + \varepsilon^*\right] \\
&\geq (1-\lambda)(M-\varepsilon^*) + \lambda(N+\varepsilon^*) \\
&\geq \min\{M-\varepsilon^*, N+\varepsilon^*\} \\
&= N + \varepsilon^*,
\end{aligned} \tag{3.61}$$

which is absurd. Thus Lemma 2.3 ensures that there exists $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$ satisfying $S_L y = S_{1L} y + S_{2L} y = y$, that is,

$$y_n = L - b_n y_{n-\tau} - \sum_{s=n}^{\infty} \frac{1}{a\left(s, y_{a_{1s}}, \ldots, y_{a_{rs}}\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}, \ldots, x_{h_{ki}}\right) - c_i\right], \quad \forall n \geq T, \tag{3.62}$$

which means that

$$a\left(n, y_{a_{1n}}, \ldots, y_{a_{rn}}\right) \Delta\left(y_n + b_n y_{n-\tau}\right) = \sum_{i=n}^{\infty} \left[f\left(i, y_{f_{1i}}, \ldots, y_{f_{ki}}\right) - c_i\right], \quad \forall n \geq T, \tag{3.63}$$

which yields that

$$\Delta\left(a\left(n, y_{a_{1n}}, \ldots, y_{a_{rn}}\right) \Delta\left(y_n + b_n y_{n-\tau}\right)\right) = -f\left(n, y_{f_{1n}}, \ldots, y_{f_{kn}}\right) + c_n, \quad \forall n \geq T, \tag{3.64}$$

that is, $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$ is a bounded positive solution of (1.8).

Finally we show that (1.8) has uncountably many bounded positive solutions in $\overline{U(M)}$. Let $L_1, L_2 \in (b^*M+N, (1-b_*)M)$ and $L_1 \neq L_2$. Similarly we infer that for each $\theta \in \{1,2\}$, there exists a mapping $S_{L_\theta} : \overline{U(M)} \to V(N)$ satisfying (3.41)–(3.46), where L, β, T, S_{1L}, S_{2L}, and S_L are replaced by L_θ, β_θ, T_{L_θ}, S_{1L_θ}, S_{2L_θ} and S_{L_θ}, respectively, and the mapping S_{L_θ} has a fixed point $y^\theta = \{y_n^\theta\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$, which is a bounded positive solution of (1.8) in $\overline{U(M)}$, that is,

$$y_n^\theta = L_\theta - b_n y_{n-\tau}^\theta - \sum_{s=n}^{\infty} \frac{1}{a\left(s, y_{a_{1s}}^\theta, \ldots, y_{a_{rs}}^\theta\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}^\theta, \ldots, y_{f_{ki}}^\theta\right) - c_i\right], \quad \forall n \geq T_{L_\theta}. \tag{3.65}$$

It follows from (3.2) that there exists $T_* > \max\{T_{L_1}, T_{L_2}\}$ satisfying

$$\sum_{s=T_*}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) < \frac{|L_1 - L_2|}{4}. \tag{3.66}$$

In order to prove that the set of bounded positive solutions of (1.8) is uncountable, it is sufficient to verify that $y^1 \neq y^2$. In terms of (3.1), (3.65), and (3.66), we deduce that for $n \geq T_*$

$$\begin{aligned}
\left|y_n^1 - y_n^2\right| &= \Bigg| L_1 - L_2 - b_n y_{n-\tau}^1 + b_n y_{n-\tau}^2 - \sum_{s=n}^{\infty} \frac{1}{a\left(s, y_{a_{1s}}^1, \ldots, y_{a_{rs}}^1\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}^1, \ldots, y_{f_{ki}}^1\right) - c_i \right] \\
&\quad + \sum_{s=n}^{\infty} \frac{1}{a\left(s, y_{a_{1s}}^2, \ldots, y_{a_{rs}}^2\right)} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}^2, \ldots, y_{f_{ki}}^2\right) - c_i \right] \Bigg| \\
&\geq |L_1 - L_2| - |b_n| \left| y_{n-\tau}^1 - y_{n-\tau}^2 \right| \\
&\quad - \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} \left[\left| f\left(i, y_{f_{1i}}^1, \ldots, y_{f_{ki}}^1\right) \right| + |c_i| + \left| f\left(i, y_{f_{1i}}^2, y_{f_{2i}}^2, \ldots, y_{f_{ki}}^2\right) \right| + |c_i| \right] \\
&\geq |L_1 - L_2| - (b_* + b^*) \left\| y^1 - y^2 \right\| - 2 \sum_{s=T_*}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) \\
&> \frac{1}{2} |L_1 - L_2| - (b_* + b^*) \left\| y^1 - y^2 \right\|,
\end{aligned} \tag{3.67}$$

which means that

$$\left\| y^1 - y^2 \right\| > \frac{|L_1 - L_2|}{2(1 + b_* + b^*)} > 0, \tag{3.68}$$

that is, $y^1 \neq y^2$. This completes the proof. □

Theorem 3.4. *Assume that there exist four constants N, M, b_*, and b^* and two positive sequences $\{a_n\}_{n \in \mathbb{N}_{n_0}}$ and $\{p_n\}_{n \in \mathbb{N}_{n_0}}$ satisfying* (3.1), (3.2) *and*

$$(1 + b^*)M < (1 + b_*)N < 0, \quad b_* \leq b_n \leq b^* < -1 \text{ eventually}. \tag{3.69}$$

Then (1.8) *has uncountably many bounded positive solutions in $\overline{U(M)}$.*

Proof. Let $L \in ((1+b^*)M, (1+b_*)N)$. Now we show that there exists a mapping $S_L : \overline{U(M)} \to V(N)$ such that it has a fixed point $y = \{y_n\}_{n \in \mathbb{Z}_\beta} \in \overline{U(M)}$, which is also a bounded positive solution of (1.8). It follows from (3.2) and (3.69) that there exists $T \geq \max\{1, n_0 + \tau + |\beta|\}$ satisfying

$$b_* \leq b_n \leq b^* < -1, \quad \forall n \geq T; \tag{3.70}$$

$$\sum_{s=T}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) < \min\left\{L - (1 + b^*)M, \frac{b^*}{b_*}[N(1 + b_*) - L]\right\}. \tag{3.71}$$

Let $p^* = M - \varepsilon^*$, where $\varepsilon^* \in (0, \min\{L - (1+b^*)M, (b^*/b_*)[N(1+b_*) - L, b^*(M-N)/(b^*-1)]\})$ is enough small and

$$\sum_{s=T}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) < \min\left\{L - (1 + b^*)M, \frac{b^*}{b_*}[N(1 + b_*) - L]\right\} - \varepsilon^*. \tag{3.72}$$

Obviously, $p^* \in U(M)$. Define a mapping $S_L : \overline{U(M)} \to l_\beta^\infty$ by (3.44), where the mappings $S_{1L}, S_{2L} : \overline{U(M)} \to l_\beta^\infty$ are defined by

$$(S_{1L}y)_n = \begin{cases} \dfrac{L}{b_{n+\tau}} - \dfrac{y_{n+\tau}}{b_{n+\tau}}, & n \geq T \\ (S_{1L}y)_T, & \beta \leq n < T, \end{cases} \tag{3.73}$$

$$(S_{2L}y)_n = \begin{cases} -\dfrac{1}{b_{n+\tau}} \displaystyle\sum_{s=n+\tau}^{\infty} \frac{1}{a(s, y_{a_{1s}}, \ldots, y_{a_{rs}})} \sum_{i=s}^{\infty} [f(i, y_{f_{1i}}, \ldots, y_{f_{ki}}) - c_i], & n \geq T \\ (S_{2L}y)_T, & \beta \leq n < T \end{cases} \tag{3.74}$$

for each $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$. By virtue of (3.1), (3.70), and (3.72)–(3.74), we get that for any $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \overline{U(M)}$ and $n \geq T$

$$\begin{aligned} (S_L y)_n &= (S_{1L}y)_n + (S_{2L}y)_n \\ &= \frac{L}{b_{n+\tau}} - \frac{y_{n+\tau}}{b_{n+\tau}} - \frac{1}{b_{n+\tau}} \sum_{s=n+\tau}^{\infty} \frac{1}{a(s, y_{a_{1s}}, \ldots, y_{a_{rs}})} \sum_{i=s}^{\infty} [f(i, y_{f_{1i}}, \ldots, y_{f_{ki}}) - c_i] \\ &\geq \frac{L}{b_*} - \frac{N}{b_*} + \frac{1}{b^*} \sum_{s=T+\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) \\ &\geq \frac{L}{b_*} - \frac{N}{b_*} + \frac{1}{b^*} \min\left\{L - (1 + b^*)M, \frac{b^*}{b_*}[N(1 + b_*) - L]\right\} - \frac{1}{b^*}\varepsilon^* \\ &\geq N - \frac{1}{b^*}\varepsilon^* \\ &> N, \end{aligned} \tag{3.75}$$

which gives that $S_L(\overline{U(M)}) \subseteq V(N)$. The rest of the proof is similar to that of Theorem 3.3 and is omitted. This completes the proof. □

Now we employ the Krasnoselskii fixed point theorem to prove the existence and multiplicity of bounded positive solutions of (1.8).

Theorem 3.5. *Assume that there exist four constants N, M, b_* and b^* and two positive sequences $\{a_n\}_{n\in\mathbb{N}_{n_0}}$, $\{p_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.1), (3.2) *and*

$$0 < Nb^*b_* < M\left(b_*^2 - b^*\right), \quad 1 < b_* \le b_n \le b^* < b_*^2 \text{ eventually}. \tag{3.76}$$

Then (1.8) *has uncountably many bounded positive solutions in $A(N,M)$.*

Proof. Let $L \in ((b^*/b_*)M + b^*N, b_*M)$. Now we show that there exist two mappings $S_{1L}, S_{2L} : A(N,M) \to l_\beta^\infty$ such that the equation $S_{1L}y + S_{2L}y = y$ has a solution $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$, which is also a bounded positive solution of (1.8). It follows from (3.2) and (3.76) that there exists $T \ge \max\{1, n_0 + \tau + |\beta|\}$ satisfying

$$1 < b_* \le b_n \le b^* < b_*^2, \quad \forall n \ge T; \tag{3.77}$$

$$\sum_{s=T}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) < \min\left\{b_*M - L, \frac{b_*L}{b^*} - M - b_*N\right\}. \tag{3.78}$$

Define two mappings S_{1L} and $S_{2L} : A(N,M) \to l_\beta^\infty$ by (3.73) and (3.74), respectively. It follows from (3.1), (3.73), (3.74), (3.77), and (3.78) that for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$, $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$ and $n \ge T$

$$\begin{aligned}
\left|(S_{1L}x)_n - (S_{1L}y)_n\right| &= \frac{1}{b_{n+\tau}}\left|x_{n+\tau} - y_{n+\tau}\right| \le \frac{1}{b_*}\|x - y\|, \\
(S_{1L}x)_n + (S_{2L}y)_n &= \frac{L}{b_{n+\tau}} - \frac{x_{n+\tau}}{b_{n+\tau}} - \frac{1}{b_{n+\tau}} \sum_{s=n+\tau}^{\infty} \frac{1}{a(s, y_{a_{1s}}, \dots, y_{a_{rs}})} \sum_{i=s}^{\infty} \left[f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right) - c_i\right] \\
&\ge \frac{L}{b^*} - \frac{M}{b_*} - \frac{1}{b_*} \sum_{s=n+\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} \left[\left|f\left(i, y_{f_{1i}}, \dots, y_{f_{ki}}\right)\right| + |c_i|\right] \\
&\ge \frac{L}{b^*} - \frac{M}{b_*} - \frac{1}{b_*} \sum_{s=T+\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) \\
&> \frac{L}{b^*} - \frac{M}{b_*} - \frac{1}{b_*} \min\left\{b_*M - L, \frac{b_*L}{b^*} - M - b_*N\right\} \\
&\ge N,
\end{aligned}$$

$$
\begin{aligned}
(S_{1L}x)_n + (S_{2L}y)_n &= \frac{L}{b_{n+\tau}} - \frac{x_{n+\tau}}{b_{n+\tau}} - \frac{1}{b_{n+\tau}} \sum_{s=n+\tau}^{\infty} \frac{1}{a(s, y_{a_{1s}}, \dots, y_{a_{rs}})} \sum_{i=s}^{\infty} [f(i, y_{f_{1i}}, \dots, y_{f_{ki}}) - c_i] \\
&\le \frac{L}{b_*} + \frac{1}{b_*} \sum_{s=n+\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} [|f(i, y_{f_{1i}}, \dots, y_{f_{ki}})| + |c_i|] \\
&\le \frac{L}{b_*} + \frac{1}{b_*} \sum_{s=T+\tau}^{\infty} \frac{1}{a_s} \sum_{i=s}^{\infty} (p_i + |c_i|) \\
&< \frac{L}{b_*} + \frac{1}{b_*} \min\left\{ b_* M - L, \frac{b_* L}{b^*} - M - b_* N \right\} \\
&\le M,
\end{aligned}
\tag{3.79}
$$

which yield that

$$
\|S_{1L}x - S_{1L}y\| \le \frac{1}{b_*} \|x - y\|, \qquad S_{1L}x + S_{2L}y \in A(N, M), \quad \forall x, y \in A(N, M). \tag{3.80}
$$

As in the proof of Theorem 3.3, we infer similarly that S_{2L} is continuous in $A(N, M)$ and $S_{2L}(A(N, M))$ is relatively compact. Thus S_{2L} is completely continuous, which together with (3.77), (3.80), and Lemma 2.4, ensures that the equation $S_{1L}y + S_{2L}y = y$ has a solution $y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in A(N, M)$, which is also a bounded positive solution of (1.8) in $A(N, M)$. The rest of the proof is similar to that of Theorem 3.3 and is omitted. This completes the proof. □

Remark 3.6. Theorems 3.1–3.5 extend and improve Theorem 2.1 in [1] and Theorems 2.1–2.7 in [7], respectively. The examples in Section 4 show that our results are indeed generalizations of the corresponding results in [1, 7].

4. Examples

Now we construct five examples to show the applications of the results presented in Section 3. Note that none of the known results can be applied to the five examples.

Example 4.1. Consider the second-order nonlinear neutral delay difference equation

$$
\begin{aligned}
&\Delta\Big((-1)^n \Big(n^6 - n^5 + 1\Big)\Big(y_n^2 + 1\Big) \Delta (y_n + y_{n-\tau})\Big) + \frac{-3n^3 y_{n+1}^4 + \sqrt{n} y_{n+1} + (y_{n+1} - 1)^{4/5}}{n^5 \ln^2(n+2) + |y_{n^2} - n^2|^3 + 1} \\
&\quad = \frac{(-1)^n n^2 - 5n + 1}{n^4 + n^2 + \sin\sqrt{n^3 + 1}}, \quad \forall n \ge 1,
\end{aligned}
\tag{4.1}
$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 1$, $r = 1$, $k = 2$, $N = 2$, $M = 3$, $\beta = 1 - \tau$ and

$$a_{1n} = n, \qquad f_{1n} = n + 1, \qquad f_{2n} = n^2, \qquad b_n = 1, \qquad c_n = \frac{(-1)^n n^2 - 5n + 1}{n^4 + n^2 + \sin\sqrt{n^3 + 1}},$$
$$a(n,u) = (-1)^n\left(n^6 - n^5 + 1\right)\left(u^2 + 1\right), \qquad a_n = 3\left(n^6 - n^5 + 1\right),$$
$$f(n,u,v) = \frac{-3n^3u^4 + \sqrt{n}u + (u-1)^{4/5}}{n^5\ln^2(n+2) + |v - n^2|^3 + 1}, \qquad p_n = \frac{243n^3 + 3\sqrt{n} + 2}{n^5}, \quad \forall (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2. \tag{4.2}$$

It is easy to verify that (3.1)–(3.3) hold. Thus Theorem 3.1 guarantees that (4.1) has uncountably many bounded positive solutions in $\overline{B(M,N)}$. But the results in [1, 7] are not applicable for (4.1).

Example 4.2. Consider the second-order nonlinear neutral delay difference equation

$$\Delta\left(\left(n^5\sqrt{n+2} + y_{n^2-60}^2 y_{2n+(-1)^n}^4\right)\Delta(y_n - y_{n-\tau})\right) + \frac{n^3(y_{n+4} - 2)^9 - (n^3 - 2n + 1)y_{2n-1}^3}{n^5 + \left(y_{n^2-3}^4 - 3n\right)^2 + \sin(ny_{n+4}) + 2}$$
$$= \frac{n^5 - 6n^3\sqrt{2n-4} - 5}{n^{11} + 4}, \quad \forall n \geq 5, \tag{4.3}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 5$, $r = 2$, $k = 3$, $N = 1$, $M = 3$, $\beta = \min\{5 - \tau, -35\}$ and

$$a_{1n} = n^2 - 60, \qquad a_{2n} = 2n + (-1)^n, \qquad f_{1n} = n + 4, \qquad f_{2n} = 2n - 1,$$
$$f_{3n} = n^2 - 3, \qquad b_n = -1, \qquad c_n = \frac{n^5 - 6n^3\sqrt{2n-4} - 5}{n^{11} + 4}, \qquad a(n,u,v) = n^5\sqrt{n+2} + u^2v^4,$$
$$a_n = n^5\sqrt{n+2} + 1, \qquad f(n,u,v,w) = \frac{n^3(u-2)^9 - (n^3 - 2n + 1)v^3}{n^5 + (w^4 - 3n)^2 + \sin(nu) + 2},$$
$$p_n = \frac{28n^3 + 54n + 27}{n^5}, \quad \forall (n,u,v,w) \in \mathbb{N}_{n_0} \times \mathbb{R}^3. \tag{4.4}$$

It is clear that (3.1), (3.21), and (3.22) hold. Hence Theorem 3.2 ensures that (4.3) has uncountably many bounded positive solutions in $\overline{B(M,N)}$. But the results in [1, 7] are not valid for (4.3).

Example 4.3. Consider the second-order nonlinear neutral delay difference equation

$$\begin{aligned}&\Delta\left((-1)^{n(n-1)/2}\left(\left(n^3+2\right)\left|y_{3n}^3\right|+n^2y_{n-8}^2+1\right)\Delta\left(x_n+\frac{(-1)^n\left(n^7-1\right)}{3n^7+1}x_{n-\tau}\right)\right)\\&\quad+\frac{\left(x_{n-5}-1\right)^{1/3}+\sqrt{\left|x_{n(n+1)/2}^2-2\right|}}{n\ln^2n+1}\\&=\frac{n^6-30n^4+8\cos^5\left(4n^8-1\right)}{n^9+n^5+2\ln^2n},\quad\forall n\geq 3,\end{aligned}\tag{4.5}$$

where $\tau\in\mathbb{N}$ is fixed. Let $n_0=3,\ r=k=2,\ b_*=b^*=1/3,\ N=2,\ M=7,\ \beta=\min\{3-\tau,-5\}$ and

$$\begin{gathered}a_{1n}=3n,\qquad a_{2n}=n-8,\qquad f_{1n}=n-5,\qquad f_{2n}=\frac{n(n+1)}{2},\qquad b_n=\frac{(-1)^n\left(n^7-1\right)}{3n^7+1},\\ c_n=\frac{n^6-30n^4+8\cos^5\left(4n^8-1\right)}{n^9+n^5+2\ln^2n},\qquad a(n,u,v)=(-1)^{n(n-1)/2}\left(\left(n^3+2\right)\left|u^3\right|+n^2v^2+1\right),\\ a_n=8n^3+4n^2+1,\qquad f(n,u,v)=\frac{(u-1)^{1/3}+\sqrt{\left|v^2-2\right|}}{n\ln^2n+1},\\ p_n=\frac{6^{1/3}+\sqrt{47}}{n\ln^2n},\quad\forall(n,u,v)\in\mathbb{N}_{n_0}\times\mathbb{R}^2.\end{gathered}\tag{4.6}$$

It is clear that (3.1), (3.2) and (3.40) are satisfied. Hence Theorem 3.3 implies that (4.5) has uncountably many bounded positive solutions in $\overline{U(M)}$. But the results in [1, 7] are unapplicable for (4.5).

Example 4.4. Consider the second-order nonlinear neutral delay difference equation

$$\begin{aligned}&\Delta\left((-1)^{n-1}n^2\left(y_{n^3-1}^4+1\right)\Delta\left(x_n-\frac{3n^2-3n-3\cos(n/2-1)+1}{n^2-n-\cos(n/2-1)}x_{n-\tau}\right)\right)\\&\quad+\frac{\left(3-x_{n^2-9}\right)^{1/3}-\sqrt{n-1}x_{2n-3}^3}{n^4+nx_{3n-7}^4+1}\\&=\frac{n^3+(-1)^nn^2+\ln^3\left(1+n^2\right)}{n^6+4n^5-3n^4+\sin^3\left(4n^2-5\right)+1},\quad n\geq 2,\end{aligned}\tag{4.7}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 2$, $r = 1$, $k = 3$, $b_* = -4$, $b^* = -3$, $N = 1$, $M = 3$, $\beta = \min\{2-\tau, -5\}$ and

$$\begin{gathered} a_{1n} = n^3 - 1, \qquad f_{1n} = n^2 - 9, \qquad f_{2n} = 3n - 7, \qquad f_{3n} = 2n - 3, \\ b_n = -\frac{3n^2 - 3n - 3\cos(n/2 - 1) + 1}{n^2 - n - \cos(n/2 - 1)}, \qquad c_n = \frac{n^3 + (-1)^n n^2 + \ln^3(1+n^2)}{n^6 + 4n^5 - 3n^4 + \sin^3(4n^2 - 5) + 1}, \\ a(n,u) = (-1)^{n-1} n^2 \left(u^4 + 1\right), \qquad a_n = n^2 \ln^3 n, \qquad f(n,u,v,w) = \frac{(3-u)^{1/3} - \sqrt{2n-3}\,v^3}{n^4 + nw^4 + 1}, \\ p_n = \frac{2 + 27\sqrt{2n-3}}{n^4}, \quad \forall (n,u,v,w) \in \mathbb{N}_{n_0} \times \mathbb{R}^3. \end{gathered} \tag{4.8}$$

It is easy to verify that (3.1), (3.2), and (3.69) hold. Hence Theorem 3.4 implies that (4.7) has uncountably many bounded positive solutions in $\overline{U(M)}$. But the results in [1, 7] are not valid for (4.7).

Example 4.5. Consider the second-order nonlinear neutral delay difference equation

$$\begin{aligned} &\Delta\Bigg((-1)^{n(n-1)/2} \left((n+1)^5 + y_{2n-3}^8 + 2n^3 y_{n^2-1}^2 + \ln\left(1 + n\left|y_{n(n-1)}\right|\right)\right) \\ &\quad \times \Delta\left(x_n + \frac{11\ln(1+n^2) + 11\sin n + 10}{\ln(1+n^2) + \sin n + 1} x_{n-\tau}\right)\Bigg) + \frac{|x_{n-5} - 2|^{3/4} + \left(x_{n(n-2)} - 2\right)^2}{(n^2+3)^3 + \left|x_{n-5} - n^2 x_{n(n-2)}\right| + 1} \\ &\quad = \frac{n^7 - 5n^4 - 9}{n^{11} + 7n^8 - 6n^7 + 5n^3 + 1}, \quad n \ge 0, \end{aligned} \tag{4.9}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 0$, $r = 3$, $k = 2$, $b_* = 10$, $b^* = 11$, $N = 2$, $M = 3$, $\beta = \min\{-\tau, -5\}$ and

$$\begin{gathered} a_{1n} = 2n - 3, \qquad a_{2n} = n^2 - 1, \qquad a_{3n} = n(n-1), \qquad f_{1n} = n - 5, \qquad f_{2n} = n(n-2), \\ b_n = \frac{11\ln(1+n^2) + 11\sin n + 10}{\ln(1+n^2) + \sin n + 1}, \qquad c_n = \frac{n^7 - 5n^4 - 9}{n^{11} + 7n^8 - 6n^7 + 5n^3 + 1}, \\ a(n,u,v,w) = (-1)^{n(n-1)/2}\left((n+1)^5 + u^8 + 2n^3 v^2 + \ln(1 + n|w|)\right), \qquad a_n = n^5, \\ f(n,u,v) = \frac{|u-2|^{3/4} + (v-2)^2}{(n^2+3)^3 + |u - n^2 v| + 1}, \qquad p_n = \frac{2}{n^6+1}, \quad \forall (n,u,v,w) \in \mathbb{N}_{n_0} \times \mathbb{R}^3. \end{gathered} \tag{4.10}$$

It is easy to see that (3.1), (3.2), and (3.76) hold. Hence Theorem 3.5 guarantees that (4.9) possesses uncountably many bounded positive solutions in $A(N, M)$. But the results in [1, 7] are inapplicable for (4.9).

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