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Research Article

New Method for Solving Linear Fractional Differential Equations

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We develop a new application of the Mittag-Leffler Function method that will extend the application of the method to linear differential equations with fractional order. A new solution is constructed in power series. The fractional derivatives are described in the Caputo sense. To illustrate the reliability of the method, some examples are provided. The results reveal that the technique introduced here is very effective and convenient for solving linear differential equations of fractional order.

1. Introduction

Fractional differential equations have excited, in recent years, a considerable interest both in mathematics and in applications. They were used in modeling of many physical and chemical processes and engineering (see, e.g., [1–6]). In its turn, mathematical aspects of fractional differential equations and methods of their solutions were discussed by many authors: the iteration method in [7], the series method in [8], the Fourier transform technique in [9, 10], special methods for fractional differential equations of rational order or for equations of special type in [11–16], the Laplace transform technique in [3–6, 16, 17], and the operational calculus method in [18–23]. Recently, several mathematical methods including the Adomian decomposition method [18–25], variational iteration method [23–26] and homotopy perturbation method [27, 28] have been developed to obtain the exact and approximate analytic solutions. Some of these methods use transformation in order to reduce equations into simpler equations or systems of equations, and some other methods give the solution in a series form which converges to the exact solution.

The reason of using fractional order differential (FOD) equations is that FOD equations are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. The results derived from the fractional system are of a more general nature. Respectively, solutions to the fractional diffusion equation spread at a faster rate than the classical diffusion equation and may exhibit asymmetry. However, the fundamental solutions of these equations still exhibit useful scaling properties that make them attractive for applications.

The concept of fractional or noninteger order derivation and integration can be traced back to the genesis of integer order calculus itself [29]. Almost all of the mathematical theory applicable to the study of noninteger order calculus was developed through the end of the 19th century. However, it is in the past hundred years that the most intriguing leaps in engineering and scientific application have been found. The calculation techniques in some cases meet the requirement of physical reality. The use of fractional differentiation for the mathematical modeling of real-world physical problems has been widespread in recent years, for example, the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, and measurement of viscoelastic material properties. Applications of fractional derivatives in other fields and related mathematical tools and techniques could be found in [30–41]. In fact, real-world processes generally or most likely are fractional order systems.

The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses.

2. Fractional Calculus

There are several approaches to the generalization of the notion of differentiation to fractional orders, for example, the Riemann-Liouville, Grünwald-Letnikov, Caputo, and generalized functions approach [42]. The Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real-world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations [42]. Unlike the Riemann-Liouville approach, which derives its definition from repeated integration, the Grünwald-Letnikov formulation approaches the problem from the derivative side. This approach is mostly used in numerical algorithms.

Here, we mention the basic definitions of the Caputo fractional-order integration and differentiation, which are used in the upcoming paper and play the most important role in the theory of differential and integral equation of fractional order.

The main advantages of Caputo approach are the initial conditions for fractional differential equations with the Caputo derivatives taking on the same form as for integer order differential equations.

Definition 2.1. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$\begin{aligned} D^{\alpha} f(x) &= I^{m-\alpha} D^{m} f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-t)^{m-\alpha+1} f^{(m)}(t) dt \end{aligned} \tag{2.1}$$

for $m-1 < \alpha \leq m$, $m \in N$, $x > 0$.

For the Caputo derivative we have $D^{\alpha}C = 0$, C is constant,

$$D^{\alpha}t^{n} = \left\{ \begin{array}{ll} 0, & (n \le \alpha - 1), \\ \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}, & (n > \alpha - 1). \end{array} \right\}. \tag{2.2}$$

Definition 2.2. For m to be the smallest integer that exceeds α, the Caputo fractional derivative of order $\alpha > 0$ is defined as

$$\begin{aligned} D^{\alpha}u(x,t) &= \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} \\ &= \left\{ \begin{array}{ll} \dfrac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha+1}\dfrac{\partial^{m}u(x,\tau)}{\partial\tau^{m}}d\tau, & \text{for } m-1 < \alpha < m \\ \dfrac{\partial^{m}u(x,t)}{\partial t^{m}}, & \text{for } \alpha = m \in N \end{array} \right\}. \end{aligned} \tag{2.3}$$

3. Analysis of the Method

The Mittag-Leffler (1902–1905) functions E_{α} and $E_{\alpha,\beta}$ [42], defined by the power series

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \quad \beta > 0, \tag{3.1}$$

have already proved their efficiency as solutions of fractional order differential and integral equations and thus have become important elements of the fractional calculus theory and applications.

In this paper, we will explain how to solve some of differential equations with fractional level through the imposition of the generalized Mittag-Leffler function $E_{\alpha}(z)$. The generalized Mittag-Leffler method suggests that the linear term $y(x)$ is decomposed by an infinite series of components:

$$y = E_{\alpha}(ax^{\alpha}) = \sum_{n=0}^{\infty}a^{n}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{3.2}$$

We will use the following definitions of fractional calculus:

$$D^{\alpha}y = \sum_{n=1}^{\infty}a^{n}\frac{x^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}, \tag{3.3}$$

$$D^{2\alpha}y = \sum_{n=2}^{\infty}a^{n}\frac{x^{(n-2)\alpha}}{\Gamma((n-2)\alpha+1)}. \tag{3.4}$$

This is based on the Caputo fractional is derivatives. The convergence of the Mittag Leffler function discussed in [42].

4. Applications and Results

In this section, we consider a few examples that demonstrate the performance and efficiency of the generalized Mittag-Leffler function method for solving linear differential equations with fractional derivatives.

Example 4.1. Consider the following fractional differential equation [43]:

$$\frac{d^{\alpha}y}{dx^{\alpha}} = Ay. \tag{4.1}$$

By using (3.3) into (4.1) we find

$$\sum_{n=1}^{\infty} a^{n} \frac{x^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} - A\sum_{n=0}^{\infty} a^{n} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0. \tag{4.2}$$

Combining the alike terms and replacing (n) by $(n+1)$ in the first sum, we assume the form

$$\begin{gathered}
\sum_{n=0}^{\infty} a^{n+1} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} - A\sum_{n=0}^{\infty} a^{n} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0, \\
\sum_{n=0}^{\infty} \left(a^{n+1} - Aa^{n}\right) \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0.
\end{gathered} \tag{4.3}$$

With the coefficient of $x^{n\alpha}$ equal to zero and identifying the coefficients, we obtain recursive

$$\begin{aligned}
&a^{n+1} - Aa^{n} = 0 \Longrightarrow a^{n+1} = Aa^{n}, \\
&\text{at } n = 0, \quad a^{1} = Aa^{0} = A, \\
&\text{at } n = 1, \quad a^{2} = Aa^{1} \Longrightarrow a^{2} = A^{2}, \\
&\text{at } n = 2, \quad a^{3} = Aa^{2} \Longrightarrow a^{3} = A^{3}.
\end{aligned} \tag{4.4}$$

Substituting into (3.2)

$$\begin{aligned}
y(x) &= a^{0} + a^{1}\frac{x^{\alpha}}{\Gamma(\alpha+1)} + a^{2}\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + a^{3}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots, \\
y(x) &= 1 + A\frac{x^{\alpha}}{\Gamma(\alpha+1)} + A^{2}\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + A^{3}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots.
\end{aligned} \tag{4.5}$$

The general solution is

$$y(x) = \sum_{n=0}^{\infty} \frac{A^{n}x^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{4.6}$$

We can write the general solution in the Mittag-Leffler function form as

$$y(x) = E_\alpha(A^n x^\alpha). \tag{4.7}$$

As $\alpha = 1$, we have the exact solution:

$$y(x) = \sum_{n=0}^{\infty} \frac{(Ax)^n}{\Gamma(n+1)} = e^{Ax}, \tag{4.8}$$

which is the exact solution of the standard form.

Example 4.2. Consider the fractional differential equation [44]

$$\frac{d^{2\alpha}y}{dx^{2\alpha}} - y = 0. \tag{4.9}$$

By using (3.2) and (3.4) into (4.9) we find

$$\sum_{n=2}^{\infty} a^n \frac{x^{(n-2)\alpha}}{\Gamma((n-2)\alpha+1)} - \sum_{n=0}^{\infty} a^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0. \tag{4.10}$$

Combining the alike terms and replacing (n) by $(n+2)$ in the first sum, we assume the form

$$\begin{gathered} \sum_{n=0}^{\infty} a^{n+2} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} - \sum_{n=0}^{\infty} a^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0, \\ \sum_{n=0}^{\infty} \left(a^{n+2} - a^n\right) \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = 0. \end{gathered} \tag{4.11}$$

With the Coefficient of $x^{n\alpha}$ equal to zero and identifying the coefficients, we obtain recursive

$$a^{n+2} = a^n. \tag{4.12}$$

Substituting into (3.2), we find that:

$$y(x) = 1 + a\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + a^2\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots. \tag{4.13}$$

If $a = 1$, we can write the general solution in the Mittag-Leffler function form as

$$y(x) = \sum_{n=0}^{\infty} \frac{x^\alpha}{\Gamma(n\alpha+1)} = E_\alpha(x^\alpha) \tag{4.14}$$

which is the exact solution of the linear fractional differential equation (4.9).

Example 4.3. Consider the fractional differential equation [43]

$$\frac{d^{2\alpha}y}{dx^{2\alpha}}+\frac{d^{\alpha}y}{dx^{\alpha}}-2y=0. \tag{4.15}$$

By using (3.2) and (3.4) into (4.15) we find

$$\sum_{n=2}^{\infty}a^{n}\frac{x^{(n-2)\alpha}}{\Gamma((n-2)\alpha+1)}+\sum_{n=1}^{\infty}a^{n}\frac{x^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}-2\sum_{n=0}^{\infty}a^{n}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=0. \tag{4.16}$$

Combining the alike terms and replacing (n) by $(n+2)$ in the first sum, we assume the form

$$\begin{gathered}\sum_{n=0}^{\infty}a^{n+2}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}+\sum_{n=0}^{\infty}a^{n+1}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}-2\sum_{n=0}^{\infty}a^{n}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=0,\\ \sum_{n=0}^{\infty}\left(a^{n+2}+a^{n+1}-2a^{n}\right)\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=0\end{gathered} \tag{4.17}$$

With the coefficient of $x^{n\alpha}$ equal to zero and identifying the coefficients, we obtain recursive

$$a^{n+2}=2a^{n}-a^{n+1}. \tag{4.18}$$

Substituting into (3.2), we find that:

$$y(x)=1+a\frac{x^{\alpha}}{\Gamma(\alpha+1)}+(2-a)\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+(a-2)\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots. \tag{4.19}$$

If $a=1$, we can write the general solution in the Mittag-Leffler function form as

$$y(x)=\sum_{n=0}^{\infty}\frac{x^{\alpha}}{\Gamma(n\alpha+1)}=E_{\alpha}(x^{\alpha}) \tag{4.20}$$

which is the solution of the linear fractional differential equation (4.15).

5. Conclusions

A new generalization of the Mittag-Leffler function method has been developed for linear differential equations with fractional derivatives. The new generalization is based on the Caputo fractional derivative. It may be concluded that this technique is very powerful and efficient in finding the analytical solutions for a large class of linear differential equations of fractional order.

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