Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2011, Article ID 356356, 17 pages
doi:10.1155/2011/356356

Research Article

L^∞-Solutions for Some Nonlinear Degenerate Elliptic Equations

Albo Carlos Cavalheiro

Department of Mathematics, State University of Londrina, 86051-990 Londrina, PR, Brazil

Correspondence should be addressed to Albo Carlos Cavalheiro, accava@gmail.com

Received 17 May 2011; Revised 6 October 2011; Accepted 6 October 2011

Academic Editor: Toka Diagana

We are interested in the existence of solutions for Dirichlet problem associated to the degenerate quasilinear elliptic equations $-\sum_{j=1}^{n} D_j[\omega_2(x)\mathcal{A}_j(x,u,\nabla u)] + \omega_1(x)g(x,u(x)) + H(x,u,\nabla u)\ \omega_2(x) = f(x),$ on Ω in the setting of the weighted Sobolev spaces $W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

1. Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev spaces $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ for the homogeneous Dirichlet problem:

$$\begin{aligned} Lu(x) &= f(x), \quad \text{on } \Omega, \\ u(x) &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{P}$$

where L is the partial differential operator:

$$Lu(x) = -\operatorname{div}[\omega_2(x)\mathcal{A}(x,u,\nabla u)] + g(x,u)\omega_1(x) + H(x,u,\nabla u)\omega_2(x), \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$, ω_1 and ω_2 are two weight functions, and the functions $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$, and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions.

By a *weight*, we will mean a locally integrable function ω on $\mathbb{R}^N$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^N$. Every weight ω_i $(i = 1, 2)$ gives rise to a measure on the measurable subsets on $\mathbb{R}^N$ through integration. This measure will be denoted by μ_i. Thus, $\mu_i(E) = \int_E \omega_i(x)dx$ $(i = 1, 2)$ for measurable sets $E \subset \mathbb{R}^N$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, that is, equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1–4]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by Muckenhoupt (see [5]). These classes have found many useful applications in harmonic analysis (see [6, 7]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^N$ often belong to A_p (see [8]). There are, in fact, many interesting examples of weights (see [4] for p-admissible weights).

Equations like (1.1) have been studied by many authors in the nondegenerate case (i.e., with $\omega(x) \equiv 1$) (see, e.g., [9] and the references therein). The degenerate case with different conditions has been studied by many authors. In [2] Drábek et al. proved that under certain condition, the Dirichlet problem associated with the equation $-\operatorname{div}(a(x, u, \nabla u)) = h$, $h \in [W_0^{1,p}(\Omega, \omega)]^*$ has at least one solution $u \in W_0^{1,p}(\Omega, \omega)$, and in [1] the author proved the existence of solution when the nonlinear term $H(x, \eta, \xi)$ is equal to zero.

Firstly, we prove an L^∞ estimate for the bounded solutions of (P): we assume that $f/\omega_1 \in L^q(\Omega, \omega_1)$, with $r_2/(r_2 - 1) < q < \infty$ (where $r_2 > 1$ as in Theorem 2.5), and we prove that any $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2) \cap L^\infty(\Omega)$ that solves (P) satisfies $\|u\|_{L^\infty(\Omega)} \leq C$, where C depends only on the data, that is, Ω, $N, p, q, \alpha_1, \alpha_2, C_0, C_1$ and $\|f/\omega_1\|_{L^q(\Omega,\omega_1)}$. After that, we prove the existence of solution for problem (P) if $f/\omega_1 \in L^q(\Omega, \omega_1)$, with $p' r_2/(r_2 - 1) < q < \infty$.

Note that, in the proof of our main result, many ideas have been adapted from [9–11].

The following theorem will be proved in Section 3.

Theorem 1.1. *Let ω_1 and ω_2 be A_p-weights, $1 < p < \infty$, with $\omega_1 \leq \omega_2$. Suppose the following.*

(H1) *$x \mapsto \mathcal{A}(x, \eta, \xi)$ is measurable in Ω for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$: $(\eta, \xi) \mapsto \mathcal{A}(x, \eta, \xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for almost all $x \in \Omega$.*

(H2) *$[\mathcal{A}(x, \eta, \xi) - \mathcal{A}(x, \eta', \xi')] \cdot (\xi - \xi') > 0$, whenever $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$.*

(H3) *$\mathcal{A}(x, \eta, \xi) \cdot \xi \geq \alpha_1 |\xi|^p$, with $1 < p < \infty$, where $\alpha_1 > 0$.*

(H4) *$|\mathcal{A}(x, \eta, \xi)| \leq K_2(x) + h_1(x)|\eta|^{p/p'} + h_2(x)|\xi|^{p/p'}$, where K_2, h_1, and h_2 are positive functions, with h_1 and $h_2 \in L^\infty(\Omega)$, and $K_2 \in L^{p'}(\Omega, \omega_2)$ $(1/p + 1/p' = 1)$.*

(H5) *$x \mapsto g(x, \eta)$ is measurable in Ω for all $\eta \in \mathbb{R}$: $\eta \mapsto g(x, \eta)$ is continuous in $\mathbb{R}$ for almost all $x \in \Omega$.*

(H6) *$|g(x, \eta)| \leq K_1(x) + h_3(x)|\eta|^{p/p'}$, where K_1 and h_3 are positive functions, with $h_3 \in L^\infty(\Omega)$ and $K_1 \in L^{p'}(\Omega, \omega_1)$.*

(H7) *$g(x, \eta)\ \eta \geq \alpha_0 |\eta|^p$, for all $\eta \in \mathbb{R}$, where $\alpha_0 > 0$.*

(H8) *$x \mapsto H(x, \eta, \xi)$ is measurable in Ω for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$: $(\eta, \xi) \mapsto H(x, \eta, \xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for almost all $x \in \Omega$.*

(H9) *$|H(x, \eta, \xi)| \leq C_0 + C_1 |\xi|^p$, where C_0 and C_1 are positive constants.*

(H10) *$f/\omega_1 \in L^q(\Omega, \omega_1)$, with $r_2/(r_2 - 1) < q < \infty$ (where $r_2 > 1$ as in Theorem 2.5) and $\omega_2/\omega_1 \in L^q(\Omega, \omega_1)$.*

Let $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2) \cap L^\infty(\Omega)$ be a solution of problem (P). Then there exists a constant $C > 0$, which depends only on Ω, $n, p, \alpha_1, \alpha_0, C_0, C_1$ and $\|f/\omega_1\|_{L^q(\Omega,\omega_1)}$, such that $\|u\|_{L^\infty(\Omega)} \leq C$.

The main result of this article is given in the next theorem, which is proved in Section 4.

Theorem 1.2. *Assume that (H1)–(H9) hold true and suppose that*

(H11) $f/\omega_1 \in L^q(\Omega, \omega_1)$, *with* $p' r_2/(r_2 - 1) < q < \infty$;

(H12) $H(x, \eta, \xi)\ \eta \geq 0$, *for all* $\eta \in \mathbb{R}$.

Then there exists at least one solution $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2) \cap L^\infty(\Omega)$ *of the problem* (P).

Theorem 1.2 will be proved by approximating problem (P) with the following problems:

$$
\begin{gathered}
-\operatorname{div}[\omega_2\, \mathcal{A}(x, u, \nabla u)] + g(x, u)\omega_1 + H_m(x, u, \nabla u)\omega_2 = f(x), \quad \text{on } \Omega, \\
u(x) = 0, \quad \text{on } \partial\Omega,
\end{gathered}
\tag{P_m}
$$

where $H_m(x, \eta, \xi) = H(x, \eta, \xi)/1 + (1/m)|H(x, \eta, \varepsilon)|$, for $m \in \mathbb{N}$. Note that $|H_m| \leq |H|$ and that $|H_m| \leq m$.

2. Definitions and Basic Results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^N$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$
\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/1-p}(x)dx\right)^{p-1} \leq C_{p,\omega} \tag{2.1}
$$

for all balls $B \subset \mathbb{R}^N$, where $|\cdot|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [4, 7, 12] or [13] for more information about A_p-weights). The weight ω satisfies the doubling condition if $\mu(2B) \leq C\mu(B)$, for all balls $B \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)dx$ and $2B$ denotes the ball with the same center as B which is twice as large. If $\omega \in A_p$, then ω is doubling (see Corollary 15.7 in [4]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p-1)$ (see Corollary 4.4, Chapter IX in [7]). If $\varphi \in \mathrm{BMO}(\mathbb{R}^N)$, then $\omega(x) = e^{\alpha\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [6]).

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^N$ be open. For $0 < p < \infty$, we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$
\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{1/p} < \infty. \tag{2.2}
$$

Remark 2.2. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega, \omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [13]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.3. Let $\Omega \subset \mathbb{R}^N$ be open, $1 < p < \infty$, and let ω_1 and ω_2 be A_p-weights, $1 < p < \infty$. We define the weighted Sobolev space $W^{1,p}(\Omega, \omega_1, \omega_2)$ as the set of functions $u \in L^p(\Omega, \omega_1)$ with weak derivatives $D_j u \in L^p(\Omega, \omega_2)$, for $j = 1, \dots, N$. The norm of u in $W^{1,p}(\Omega, \omega_1, \omega_2)$ is given by

$$\|u\|_{W^{1,p}(\Omega,\omega_1,\omega_2)} = \left(\int_\Omega |u(x)|^p \omega_1(x)\, dx + \sum_{j=1}^{N} \int_\Omega \left|D_j u(x)\right|^p \omega_2(x) dx \right)^{1/p}. \tag{2.3}$$

The space $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p}(\Omega,\omega_1,\omega_2)}$. The dual space of $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is the space

$$\begin{aligned}\left[W_0^{1,p}(\Omega, \omega_1, \omega_2)\right]^* &= W^{-1,p'}(\Omega, \omega_1, \omega_2) \\ &= \left\{ T = f - \operatorname{div} g \,:\, g = (g_1, \dots, g_N), \frac{f}{\omega_1} \in L^{p'}(\Omega, \omega_1), \frac{g_j}{\omega_2} \in L^{p'}(\Omega, \omega_2) \right\}.\end{aligned} \tag{2.4}$$

Remark 2.4. (a) If $\omega \in A_p$, $1 < p < \infty$, then $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega, \omega) = W^{1,p}(\Omega, \omega, \omega)$ (see Corollary 2.16 in [13]).

(b) If $\omega_1 \le \omega_2$, then

$$W_0^{1,p}(\Omega, \omega_2) \subset W_0^{1,p}(\Omega, \omega_1, \omega_2) \subset W_0^{1,p}(\Omega, \omega_1). \tag{2.5}$$

In this paper we use the following four results.

Theorem 2.5 (The Weighted Sobolev Inequality). *Let Ω be an open bounded set in $\mathbb{R}^N (N \ge 2)$ and $\omega_2 \in A_p (1 < p < \infty)$. There exist constants C_Ω and δ positive such that for all $u \in C_0^\infty(\Omega)$ and all r_2 satisfying $1 \le r_2 \le N/(N-1) + \delta$,*

$$\|u\|_{L^{p^*}(\Omega,\omega_2)} \le C_\Omega \, \|\nabla u\|_{L^p(\Omega,\omega_2)}, \tag{2.6}$$

where $p^* = p\, r_2$.

Proof. See Theorem 1.3 in [3]. □

The following lemma is due to Stampacchia (see [14], Lemma 4.1).

Lemma 2.6. *Let α, β, C, and k_0 be real positive numbers, where $\beta > 1$.*

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function such that

$$\varphi(l) \le \frac{C}{(l-k)^\alpha} \left[\varphi(k)\right]^\beta \tag{2.7}$$

for all $l > k \ge k_0$. Then $\varphi(k_0 + d) = 0$, where $d^\alpha = C\, [\varphi(k_0)]^{\beta-1} 2^{\alpha\, \beta/(\beta-1)}$.

Lemma 2.7. *If $\omega \in A_p$, then $(|E|/|B|)^p \leq C_{p,\omega}(\mu(E)/\mu(B))$, whenever B is a ball in $\mathbb{R}^N$ and E is a measurable subset of B.*

Proof. See Theorem 15.5 *Strong doubling of A_p-weights* in [4]. □

By Lemma 2.7, if $\mu(E) = 0$, then $|E| = 0$.

Lemma 2.8. *Let ω_1 and ω_2 be A_p-weights, $1 < p < \infty$, $\omega_1 \leq \omega_2$, and a sequence $\{u_n\}$, $u_n \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ satisfies the following:*

(i) $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ *and* μ_2*-a.e. in* Ω;

(ii) $\int_\Omega \langle \mathcal{A}(x, u_n, \nabla u_n) - \mathcal{A}(x, u_n, \nabla u), \nabla(u_n - u)\rangle\, \omega_2\, dx \to 0$ *with* $n \to \infty$.

Then $u_n \to u$ *in* $W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

Proof. The proof of this lemma follows the line of Lemma 5 in [10]. □

Definition 2.9. We say that $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2) \cap L^\infty(\Omega)$ is a (weak) solution of problem (P) if

$$\int_\Omega \omega_2 \mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi\, dx + \int_\Omega g(x, u)\varphi \omega_1 dx + \int_\Omega H(x, u, \nabla u)\, \varphi \omega_2 dx = \int_\Omega f\varphi\, dx, \quad (2.8)$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2) \cap L^\infty(\Omega)$.

3. Proof of Theorem 1.1

Set $\lambda = (C_1/\alpha_1) + 1$ and define for $k > 0$ the functions $\phi \in C^1(\mathbb{R})$ and $G_k \in W^{1,\infty}(\mathbb{R})$ by

$$\phi(s) = \begin{cases} e^{\lambda s} - 1, & \text{if } s \geq 0, \\ -e^{-\lambda s} + 1, & \text{if } s \leq 0, \end{cases}$$
$$G_k(s) = \begin{cases} s - k, & \text{if } s \geq k, \\ 0, & \text{if } -k \leq s \leq k, \\ s + k, & \text{if } s \leq -k. \end{cases} \quad (3.1)$$

If $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2) \cap L^\infty(\Omega)$ is a solution of problem (P), define the set $A(k) = \{x \in \Omega : |u(x)| > k\}$. We will use the test functions $v(x) = \phi(G_k(u(x)))$. Since $u \in W_0^{1,2}(\Omega, \omega_1, \omega_2) \cap L^\infty(\Omega)$ and Ω is a bounded set in $\mathbb{R}^N$, then $v(x) = \phi(G_k(u(x))) \in W_0^{1,p}(\Omega, \omega_1, \omega_2) \cap L^\infty(\Omega)$ and

$$v(x) = \phi\left((|u| - k)^+\right)\chi_{A(k)}\, \text{sign}(u),$$
$$\nabla v = \phi'\left((|u| - k)^6\right)\chi_{A(k)} \nabla u, \quad (3.2)$$

where $\chi_{A(k)}$ is the characteristic function of the set $A(k)$ (by $\operatorname{sign}(u(x))$ we mean the function equal to $+1$ for $u(x)>0$ and to -1 for $u(x)<0$).

Since $u\in W_0^{1,p}(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega)$, we have that $v\in W_0^{1,p}(\Omega,\omega_1,\omega_2)\cap L^\infty(\Omega)$.

Using the function v in (2.8) we obtain

$$\int_\Omega \omega_2\,\mathcal{A}(x,u,\nabla u)\cdot\nabla v\,dx+\int_\Omega g(x,u)v\;\omega_1 dx+\int_\Omega H(x,u,\nabla u)\;v\;\omega_2 dx\;=\int_\Omega f\,v\,dx. \tag{3.3}$$

We have the following estimates.

(i) By (H3) we obtain

$$\begin{aligned}\int_\Omega \omega_2\;\mathcal{A}(x,u,\nabla u)\cdot\nabla v dx&=\int_{A(k)}\phi'\big((|u|-k)^+\big)\mathcal{A}(x,u,\nabla u)\cdot\nabla u\,\omega_2 dx\\&\geq\alpha_1\int_{A(k)}|\nabla u|^p\phi'\big((|u|-k)^+\big)\omega_2 dx.\end{aligned} \tag{3.4}$$

(ii) By (H7) we obtain

$$\begin{aligned}\int_\Omega g(x,u)v\;\omega_1 dx&=\int_{A(k)}g(x,u)\;\phi\big((|u|-k)^+\big)\omega_1 dx\\&\geq\alpha_0\int_{A(k)}|u|^{p-1}\phi\big((|u|-k)^+\big)\omega_1 dx.\end{aligned} \tag{3.5}$$

(iii) Using (H9) we obtain

$$\begin{aligned}\left|\int_\Omega H(x,u,\nabla u)v\;\omega_2 dx\right|&\leq\int_\Omega |H(x,u,\nabla u)||v|\omega_2 dx\\&\leq\int_{A(k)}\big(C_0+C_1|\nabla u|^p\big)\phi\big((|u|-k)^+\big)\omega_2 dx.\end{aligned} \tag{3.6}$$

And we also have

$$\left|\int_\Omega f\;v\;dx\right|\;\leq\;\int_{A(k)}|f|\phi\big((|u|-k)^+\big)dx. \tag{3.7}$$

Hence in (3.3) we obtain

$$\begin{aligned}&\alpha_1\int_{A(k)}|\nabla u|^p\phi'\big((|u|-k)^+\big)\omega_2 dx+\alpha_0\int_{A(k)}|u|^{p-1}\phi\big((|u|-k)^+\big)\omega_1 dx\\&\leq\int_{A(k)}\big(C_0+C_1|\nabla u|^p\big)\phi\big((|u|-k)^+\big)\omega_2\;dx+\int_{A(k)}|f|\phi\big((|u|-k)^+\big)dx.\end{aligned} \tag{3.8}$$

Since $\lambda = (C_1/\alpha_1) + 1$, we have for $s \geq 0$

$$\begin{aligned}\alpha_1\phi'(s) - C_1\phi(s) &= \alpha_1\lambda e^{\lambda s} - C_1\left(e^{\lambda s} - 1\right)\\ &= (\alpha_1\lambda - C_1)e^{\lambda s} + C_1 = \alpha_1 e^{\lambda s} + C_1\\ &\geq \alpha_1 e^{\lambda s} = \frac{\alpha_1}{\lambda^p}\left[\lambda e^{\lambda s/p}\right]^p = \frac{\alpha_1}{\lambda^p}\left[\phi'\left(\frac{s}{p}\right)\right]^p.\end{aligned} \tag{3.9}$$

Hence in (3.8) we obtain

$$\begin{aligned}&\int_{A(k)}\left[\alpha_1|\nabla u|^p\ \phi'((|u|-k)^+) - C_1|\nabla u|^p\phi((|u|-k)^+)\right]\omega_2 dx\\ &\quad+\ \alpha_0\int_{A(k)}|u|^{p-1}\phi((|u|-k)^+)\omega_1 dx \leq \int_{A(k)}\left(\left|f\right| + C_0\ \omega_2\right)\phi((|u|-k)^+)dx.\end{aligned} \tag{3.10}$$

Using (3.9) and $k < |u(x)|$ if $x \in A(k)$, we obtain

$$\begin{aligned}&\frac{\alpha_1}{\lambda^p}\int_{A(k)}\left|\phi'\left(\frac{(|u|-k)^+}{p}\right)\nabla u\right|^p\omega_2 dx + \alpha_0 k^{p-1}\int_{A(k)}\phi((|u|-k)^+)\omega_1 dx\\ &\quad\leq \int_{A(k)}\left(\left|f\right| + C_0\ \omega_2\right)\ \phi((|u|-k)^+)dx.\end{aligned} \tag{3.11}$$

Let us define the function ψ_k by $\psi_k(x) = \phi((|u(x)|-k)^+/p)$. We have that $\psi_k \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ and

$$\nabla\psi_k = \frac{1}{p}\ \phi'\left(\frac{(|u|-k)^+}{p}\right)\ \chi_{A(k)}\ \mathrm{sign}(u)\ \nabla u. \tag{3.12}$$

We have the following.

(a) For all $s \geq 0$, $e^{\lambda s} - 1 \geq (e^{\lambda s/p} - 1)^p$.

(b) There exists, a constant $C_2 > 0$ ($C_2 = C_2(\lambda, p)$) such that for all $s \geq 1$

$$e^{\lambda s} - 1 \leq C_2\left(e^{\lambda s/p} - 1\right)^p, \qquad \lambda e^{\lambda s} \leq C_2\ \lambda\left(e^{\lambda s/p} - 1\right)^p. \tag{3.13}$$

This implies the following.

(I1) $\phi((|u|-k)^+) = e^{\lambda(|u|-k)^+} - 1 \geq (e^{\lambda(|u|-k)^+/p} - 1)^p = |\psi_k|^p$ a.e. on Ω.

(I2) If $x \in \mathrm{A}(k+1)$, then

$$\begin{aligned}\phi((|u|-k)^+) &= e^{\lambda(|u|-k)^+} - 1 \leq C_2\left(e^{\lambda(|u|-k)^+/p} - 1\right) = C_2\left|\psi_k\right|^p,\\ \phi'((|u|-k)^+) &= (|u|-k)^+e^{\lambda(|u|-k)^+} \leq C_2\ \lambda\ \left(e^{\lambda(|u|-k)^+/p} - 1\right)^p = C_2\ \lambda\left|\psi_k\right|^p.\end{aligned} \tag{3.14}$$

Combining (I1) and (I2) with (3.11) and (3.12) we obtain

$$
\begin{aligned}
\frac{\alpha_1 p^p}{\lambda^p}\int_\Omega |\nabla\psi_k|^p\omega_2 dx + \alpha_0 k^{p-1}\int_\Omega |\psi_k|^p\ \omega_1 dx
&\le \int_{A(k)} (|f| + C_0\ \omega_2)\phi((|u|-k)^+) dx \\
&\le \int_{A(k+1)} (|f| + C_0\ \omega_2)\ C_2\ |\psi_k|^p\ dx \\
&\quad + \int_{A(k)-A(k+1)} (|f| + C_0\ \omega_2)\phi((|u|-k)^+)\ dx.
\end{aligned} \tag{3.15}
$$

Define the function $h = |f| + C_0\,\omega_2$. Since $f/\omega_1 \in L^q(\Omega,\omega_1)$ and $\omega_2/\omega_1 \in L^q(\Omega,\omega_1)$, we have that $h/\omega_1 \in L^q(\Omega,\omega_1)$. Hence

$$
\begin{aligned}
\int_\Omega h|\psi_k|^p dx &= \int_\Omega \frac{h}{\omega_1}\ |\psi_k|^p\ \omega_1^{1/q}\ \omega_1^{1/q'}\ dx \\
&\le \left\|\frac{h}{\omega_1}\right\|_{L^q(\Omega,\omega_1)} \|\psi_k\|^p_{L^{p\,q'}(\Omega,\omega_1)}.
\end{aligned} \tag{3.16}
$$

If $x \in A(k) - A(k+1)$, we have $k < |u| < k+1$. Hence

$$
\phi((|u|-k)^+) = \mathrm{e}^{\lambda(|u|-k)^+} - 1\ \le\ \mathrm{e}^{\lambda} - 1, \tag{3.17}
$$

and we obtain

$$
\begin{aligned}
\int_{A(k)-A(k+1)} (|f| + C_0\ \omega_2)\ \phi((|u|-k)^+)\ dx &\le \int_{A(k)-A(k+1)} \left(\mathrm{e}^{\lambda} - 1\right)\ h\ dx \\
&\le\ \mathrm{e}^{\lambda}\int_{A(k)} h\, dx.
\end{aligned} \tag{3.18}
$$

By Theorem 2.5, (3.15), and (3.18) we have

$$
\begin{aligned}
&\frac{\alpha_1\ p^p}{\lambda^p}\ \frac{1}{C_\Omega^p}\left(\int_\Omega |\psi_k|^{p^*}\omega_2 dx\right)^{p/p^*} +\ \alpha_0\ k_0^{p-1}\int_\Omega |\psi_k|^p\ \omega_1 dx \\
&\quad\le\ C_2\int_\Omega h\ |\psi_k|^p\ dx + \mathrm{e}^{\lambda}\int_{A(k)} h\ dx.
\end{aligned} \tag{3.19}
$$

Therefore, there exist positive constants C_3 and C_4 (depending only on Ω, α_1, p, λ, C_2, and C_Ω) such that

$$C_3\left(\int_\Omega |\varphi_k|^{p^*}\omega_2 dx\right)^{p/p^*} + C_4\ \alpha_0\ k_0^{p-1}\int_\Omega |\varphi_k|^p\ \omega_1 dx \\ \leq \int_\Omega h\ |\varphi_k|^p\ dx + \int_{A(k)} h\ dx. \tag{3.20}$$

Since $r_2/(r_2-1) < q$, then $q' < r_2$ and $p < p\,q' < p^*$. For $0 < \theta < 1$ such that $1/p\,q' = (\theta/p) + (1-\theta)/p^*$, using an interpolation inequality, Young's inequality (with $0 < \gamma < \infty$), and Hölder's inequality with exponents q and q' we thus obtain

$$\begin{aligned}\int_\Omega h|\varphi_k|^p dx &\leq \left\|\frac{h}{\omega_1}\right\|_{L^q(\Omega,\omega_1)}\|\varphi_k\|^p_{L^{p\,q'}(\Omega,\omega_1)} \\ &\leq \left\|\frac{h}{\omega_1}\right\|_{L^q(\Omega,\omega_1)}\|\varphi_k\|^{\theta p}_{L^p(\Omega,\omega_1)}\|\varphi_k\|^{(1-\theta)p}_{L^{p^*}(\Omega,\omega_1)} \\ &\leq (1-\theta)\gamma^{1/(1-\theta)}\|\varphi_k\|^p_{L^{p^*}(\Omega,\omega_1)} + \theta\ \gamma^{-1/\theta}\left\|\frac{h}{\omega_1}\right\|^{1/\theta}_{L^q(\Omega,\omega_1)}\|\varphi_k\|^p_{L^p(\Omega,\omega_1)}.\end{aligned} \tag{3.21}$$

Hence in (3.20) we obtain

$$\begin{aligned}C_3\ &\left(\int_\Omega |\varphi_k|^{p^*}\omega_2 dx\right)^{p/p^*} + C_4\ \alpha_0\ k_0^{p-1}\int_\Omega |\varphi_k|^p\ \omega_1 dx \\ &\leq (1-\theta)\gamma^{1/(1-\theta)}\|\varphi_k\|^p_{L^{p^*}(\Omega,\omega_1)} \\ &\quad + \theta\ \gamma^{-1/\theta}\left\|\frac{h}{\omega_1}\right\|^{1/\theta}_{L^q(\Omega,\omega_1)}\|\varphi_k\|^p_{L^p(\Omega,\omega_1)} \\ &\quad + \int_{A(k)} h\ dx \leq\ (1-\theta)\gamma^{1/(1-\theta)}\|\varphi_k\|^p_{L^{p^*}(\Omega,\omega_2)} \\ &\quad + \theta\ \gamma^{-1/\theta}\left\|\frac{h}{\omega_1}\right\|^{1/\theta}_{L^q(\Omega,\omega_1)}\|\varphi_k\|^p_{L^p(\Omega,\omega_1)} + \int_{A(k)} h\,dx.\end{aligned} \tag{3.22}$$

Now, we can choose γ in order to have $(1-\theta)\ \gamma^{1/(1-\theta)} = C_3/2$ and k_0 such that

$$C_4\alpha_0 k_0^{p-1} = \theta\ \gamma^{-1/\theta}\left\|\frac{h}{\omega_1}\right\|^{1/\theta}_{L^q(\Omega,\omega_1)}. \tag{3.23}$$

We obtain, from (3.22), that for every $k\ \geq\ k_0$ it results

$$\frac{C_3}{2}\left(\int_\Omega |\varphi_k|^{p^*}\ \omega_2 dx\right)^{p/p^*} \leq \int_{A(k)} h\,dx \leq \left\|\frac{h}{\omega_1}\right\|_{L^q(\Omega,\omega_1)}\left[\mu_1(A_k)\right]^{1/q'}, \tag{3.24}$$

where $\mu_1(A_k) = \int_{A_k} \omega_1 \, dx$. Hence for all $k \geq k_0$ we have

$$\begin{aligned}
\int_\Omega |\varphi_k|^{p^*} \omega_2 dx &\leq \left(\frac{2}{C_3} \left\| \frac{h}{\omega_1} \right\|_{L^q(\Omega,\omega_1)} [\mu_1(A(k))]^{1/q'} \right)^{p^*/p} \\
&= \left(\frac{2}{C_3} \right)^{p^*/p} \left\| \frac{h}{\omega_1} \right\|_{L^q(\Omega,\omega_1)}^{p^*/p} [\mu_1(A(k))]^{p^*/pq'} \\
&= C_5 [\mu_1(A(k))]^{p^*/pq'}.
\end{aligned} \tag{3.25}$$

Let us now take $l > k \geq k_0$. Then we have

$$\begin{aligned}
\mu_1(A(l)) \left[\lambda \left(\frac{l-k}{p} \right) \right]^{p^*} &\leq \mu_1(A(k)) \left| \phi\left(\frac{l-k}{p} \right) \right|^{p^*} \leq \int_{A(k)} |\varphi_k|^{p^*} \omega_1 dx \\
&\leq \int_\Omega |\varphi_k|^{p^*} \omega_2 dx.
\end{aligned} \tag{3.26}$$

Therefore for all $l > k \geq k_0$ we obtain (by (3.25) and (3.26))

$$(l-k)^{p^*} \mu_1(A(l)) \leq \frac{p^{p^*}}{\lambda^{p^*}} C_5 [\mu_1(A(k))]^{p^*/pq'} = C_6 [\mu_1(A(k))]^{p^*/pq'}, \tag{3.27}$$

that is, $\mu_1(A(l)) \leq (C_6/(l-k)^{p^*})[\mu_1(A(k))]^{p^*/pq'}$.

Let $\varphi(k) = \mu_1(A(k))$. Since $\beta = p^*/p\ q' > 1$, by Lemma 2.6 there exists a constant $C_7 > 0$ such that

$$\mu_1(A(k)) = 0, \quad \forall k \geq C_7. \tag{3.28}$$

Using Lemma 2.7 we have $|A(k)| = 0$ for all $k \geq C_7$. Therefore any solution u of problem (P) satisfies the estimate $\|u\|_{L^\infty(\Omega)} \leq C_7$.

4. Proof of Theorem 1.2

Step 1. Let us define for $m \in \mathbb{N}$ the approximation

$$H_m(x,\eta,\xi) = \frac{H(x,\eta,\xi)}{1 + (1/m)|H(x,\eta,\xi)|}. \tag{4.1}$$

We have that $|H_m(x,\eta,\xi)| \leq |H(x,\eta,\xi)|$, $|H_m(x,\eta,\xi)| < m$, and $H_m(x,\eta,\xi)$ satisfies the conditions (H9) and (H12). We consider the approximate problem

$$\begin{aligned}
-\operatorname{div}[\omega_2\, \mathcal{A}(x,u,\nabla u)] + g(x,u)\, \omega_1 + H_m(x,u,\nabla u)\, \omega_2, \quad &\text{on } \Omega, \\
u(x) = 0, \quad &\text{on } \partial\Omega.
\end{aligned} \tag{P_m}$$

We say that $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is (weak) solution of problem (P_m) if

$$\begin{aligned}\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\omega_2 dx + \int_\Omega g(x,u)\ \varphi\omega_1 dx \\ + \int_\Omega H_m(x,u,\nabla u)\ \varphi\omega_2 dx = \int_\Omega f\ \varphi\ dx,\end{aligned} \tag{4.2}$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. We will prove that there exists at least one solution u_m of the problem (P_m). For $u, v, \varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ we define

$$\begin{aligned}B(u,v,\varphi) &= \int_\Omega \omega_2\ \mathcal{A}(x,u,\nabla v)\cdot\nabla\varphi\, dx, \\ B_m(u,\varphi) &= \int_\Omega g(x,u)\ \varphi\omega_1 dx + \int_\Omega H_m(x,u,\nabla u)\ \varphi\omega_2 dx, \\ T(\varphi) &= \int_\Omega f\varphi\, dx.\end{aligned} \tag{4.3}$$

Then $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a (weak) solution of problem (P_m) if

$$B(u,u,\varphi) + B_m(u,\varphi) = T(\varphi), \quad \text{for all } \varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2). \tag{4.4}$$

Let $a(u,v,\varphi) = B(u,v,\varphi) + B_m(u,\varphi)$.

(i) Using (H4) we obtain

$$\begin{aligned}\left|B(u,v,\varphi)\right| \le \Big(&\|K_2\|_{L^{p'}(\Omega,\omega_2)} + \|h_1\|_{L^\infty(\Omega)}\ \|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} + \|h_2\|_{L^\infty(\Omega)}\|v\|^{p/p'}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\Big) \\ &\times \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.\end{aligned} \tag{4.5}$$

(ii) Using (H6) and $|H_m(x,\eta,\xi)| \le m$, we obtain

$$\begin{aligned}\left|B_m(u,\varphi)\right| \le \Big(&\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_3\|_{L^\infty(\Omega)}\|u\|^{p'/p}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} +\ mC_\Omega\left[\mu_2(\Omega)\right]^{1'/p}\Big) \\ &\times \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.\end{aligned} \tag{4.6}$$

Hence,

$$\begin{aligned}\left|\ a(u,v,\varphi)\right| \le \Big(&\|K_2\|_{L^{p'}(\Omega,\omega_2)} + \left(\|h_1\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)}\right)\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} + \|K_1\|_{L^{p'}(\Omega,\omega_1)} \\ &+\|h_2\|_{L^\infty(\Omega)}\|v\|^{p/p'}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} + m\ C_\Omega\ \left[\mu_2(\Omega)\right]^{1/p'}\Big)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.\end{aligned} \tag{4.7}$$

For each $(u,v) \in W_0^{1,p}(\Omega,\omega_1,\omega_2) \times W_0^{1,p}(\Omega,\omega_1,\omega_2)$ we have that $a(u,v,.)$ is linear and continuous. Hence, there exists a linear and continuous operator

$$A(u,v) : W_0^{1,p}(\Omega,\omega_1,\omega_2) \longrightarrow \left[W_0^{1,p}(\Omega,\omega_1,\omega_2)\right]^* \tag{4.8}$$

such that $\langle A(u,v),\varphi\rangle = a(u,v,\varphi)$. We set

$$\widetilde{A}(u) = A(u,u), \quad \forall u \in W_0^{1,p}(\Omega,\omega_1,\omega_2). \tag{4.9}$$

The operator $\widetilde{A} : W_0^{1,p}(\Omega,\omega_1,\omega_2) \to [W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$ is semimonotone; that is, by similar arguments as in the proof of Theorem 2 in [11] we have the following.

(i) $\langle A(u,u) - A(u,v), u-v\rangle \geq 0$ for all $u,v \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

(ii) For each $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, the operator $v \mapsto A(u,v)$ is hemicontinuous and bounded from $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ to $[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$ and,

for each $v \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ the operator $u \mapsto A(u,v)$ is hemicontinuous and bounded from $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ to $[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$.

(iii) If $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ and $\langle A(u_n,u_n)-A(u_n,u),u_n-u\rangle \to 0$, then $A(u_n,u) \rightharpoonup A(u,v)$ in $[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$ as $n \to \infty$ for all $v \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

(iv) If $v \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$, and $A(u_n,v) \rightharpoonup \tilde{v}$ in $[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$, then $\langle A(u_n,v),u_n\rangle \to \langle \tilde{v},u\rangle$ as $n \to \infty$.

(v) The operator $\widetilde{A} : W_0^{1,p}(\Omega,\omega_1,\omega_2) \to [W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$ is bounded.

Hence the operator $\widetilde{A} : W_0^{1,p}(\Omega,\omega_1,\omega_2) \to [W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$ is pseudomonotone (see [15]).

(vi) By (H3), (H7), and (H12) we have

$$\left\langle \widetilde{A}(u),u\right\rangle \geq \alpha_1 \int_\Omega |\nabla u|^p \omega_2 dx + \alpha_0 \int_\Omega |u|^p \omega_1 dx \geq \alpha \|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}, \tag{4.10}$$

where $\alpha = \min\{\alpha_0,\alpha_1\}$. Since $p > 1$, we have

$$\frac{\left\langle \widetilde{A}(u),u\right\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}} \longrightarrow \infty \quad \text{on } \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \longrightarrow \infty; \tag{4.11}$$

that is, the operator $\widetilde{A}$ is coercive. Then, by Theorem 27.B in [15], for each $T \in [W_0^{1,p}(\Omega,\omega_1,\omega_2)]^*$, the equation

$$\widetilde{A}u = T, \quad u \in W_0^{1,p}(\Omega,\omega_1,\omega_2) \tag{4.12}$$

has a solution. Therefore, the problem (P_m) , has a solution $u_m \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Step 2. We will show that $u_m \in L^\infty(\Omega)$ and $\|u_m\|_{L^\infty(\Omega)} \le C$, where C is independent of m. If $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is a solution of problem (P_m), we define

$$u_n(x) = T_n(u(x)) = \begin{cases} u(x), & \text{if } |u(x)| \le n, \\ n, & \text{if } u(x) > n, \\ -n, & \text{if } u(x) < -n. \end{cases} \tag{4.13}$$

We have $D_i u_n = D_i u$ if $|u(x)| \le n$. For $k > 0$, let us define the function $\psi_n(x) = \operatorname{sign}(u_n(x)) \max\{|u_n(x)| - k, 0\}$. We have $\psi_n \in W_0^{1,p}(\Omega,\omega_1,\omega_2) \cap L^\infty(\Omega)$.

Now consider the function

$$\Phi(t) = \begin{cases} t+k, & \text{if } t \le -k, \\ 0, & \text{if } |t| \le k, \\ t-k, & \text{if } t \ge k. \end{cases} \tag{4.14}$$

Since Φ is a Lipschitz function and $\Phi(0) = 0$, then $\Phi(\psi_n) \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$. Moreover, $D_i\Phi(\psi_n) = \Phi'(\psi_n) D_i\psi_n$ and

$$\Phi'(u_n)\nabla u_n \longrightarrow \Phi'(u)\nabla u, \quad \mu_2 - a.e. \text{ in } \Omega. \tag{4.15}$$

We also have, for all measurable subset $E \subset \Omega$,

$$\int_E \left|\Phi'(u_n)\nabla u_n\right|^p \omega_2 dx \le \int_E |\nabla u_n|^p \omega_2 dx. \tag{4.16}$$

By applying Vitali's Convergence Theorem, with $\psi = \Phi(u)$, we obtain

$$\nabla\psi_n \longrightarrow \nabla\psi \quad \text{in } L^p(\Omega,\omega_2). \tag{4.17}$$

Since $\omega_1 \le \omega_2$, we obtain

$$\left\|\psi_n - \psi\right\|_{L^p(\Omega,\omega_1)} \le \left\|\psi_n - \psi\right\|_{L^p(\Omega,\omega_2)} \le C_\Omega \left\|\nabla\psi_n - \nabla\psi\right\|_{L^p(\Omega,\omega_2)}. \tag{4.18}$$

Hence $\psi_n \to \psi$ in $L^p(\Omega,\omega_1)$. Since $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is a solution of problem (P_m) and $\psi_n \in W_0^{1,p}(\Omega,\omega_1,\omega_2) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\psi_n\omega_2 dx + \int_\Omega g(x,u)\psi_n\omega_1 dx + \int_\Omega H_m(x,u,\nabla u)\psi_n\omega_2 dx = \int_\Omega f\psi_n dx. \tag{4.19}$$

Using (H4), (H6), $|H_m(x,\eta,\xi)| \leq m$, (4.17), and (4.18), we obtain in (4.19) as $n \to \infty$

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\psi\omega_2 dx + \int_\Omega g(x,u)\psi\omega_1 dx + \int_\Omega H_m(x,u,\nabla u)\psi\omega_2 dx = \int_\Omega f\psi\ dx. \quad (4.20)$$

Using $\varphi = \psi\chi_{A(k)}$ in (4.2) (where $A(k) = \{x \in \Omega : |u(x)| > k\}$) we obtain

$$\begin{aligned}\int_{A(k)} \mathcal{A}(x,u,\nabla u)\cdot\nabla\psi\omega_2 dx &+ \int_{A(k)} g(x,u)\psi\omega_1 dx + \int_{A(k)} H_m(x,u,\nabla u)\psi\omega_2 dx \\ &= \int_{A(k)} f\ \psi\, dx.\end{aligned} \quad (4.21)$$

Since

$$\psi = \Phi(u) = \begin{cases} u+k, & \text{if } u \leq -k, \\ 0, & \text{if } |u| \leq k, \\ u-k, & \text{if } u \geq k, \end{cases} \quad (4.22)$$

we obtain the following.

(i) By (H7) we have $g(x,\eta)\ \eta \geq 0$ for all $\eta \in \mathbb{R}$, and

$$\begin{aligned}\int_{A(k)} g(x,u)\psi\omega_1 dx = &\int_{\{u\leq -k\}} g(x,u)(u+k)\omega_1 dx \\ &+ \int_{\{u\geq k\}} g(x,u)(u-k)\omega_1 dx \geq 0.\end{aligned} \quad (4.23)$$

(ii) Using (H12) we have $H_m(x,\eta,\xi)\ \eta \geq 0$ for all $\eta \in \mathbb{R}$, and

$$\begin{aligned}\int_{A(k)} H_m(x,u,\nabla u)\psi\ \omega_2 dx = &\int_{\{u\leq -k\}} H_m(x,u,\nabla u)(u+k)\omega_2 dx \\ &+ \int_{\{u\geq k\}} H_m(x,u,\nabla u)(u-k)\omega_2 dx \geq 0.\end{aligned} \quad (4.24)$$

We have $\nabla\psi = \nabla u$ in $A(k)$. Using (H3), (i), and (ii) we obtain in (4.21)

$$\alpha_1 \int_{A(k)} |\nabla u|^p\ \omega_2 dx \leq \int_{A(k)} f\psi\, dx. \quad (4.25)$$

By Theorem 2.1.14 in [13] there is a positive constant C such that

$$\int_{\omega} |\varphi|^p \omega_1 dx \leq \int_{\Omega} |\varphi|^p \omega_2 dx \leq C \int_{\Omega} |\nabla\varphi|^p \omega_2 dx. \tag{4.26}$$

Then we obtain

$$\begin{aligned}
\int_{A(k)} f\ \varphi\ dx &\leq \left(\int_{A(k)} \left|\frac{f}{\omega_1}\right|^{p'} \omega_1\right)^{1/p'} \left(\int_{A(k)} |\varphi|^p \omega_1 dx\right)^{1/p} \\
&\leq C \left(\int_{A(k)} \left|\frac{f}{\omega_1}\right|^{p'} \omega_1\right)^{1/p'} \left(\int_{A(k)} |\nabla\varphi|^p \omega_2 dx\right)^{1/p} \\
&= C \left(\int_{A(k)} \left|\frac{f}{\omega_1}\right|^{p'} \omega_1\right)^{1/p'} \left(\int_{A(k)} |\nabla u|^p \omega_2 dx\right)^{1/p}.
\end{aligned} \tag{4.27}$$

Using (4.27) and Young's inequality we obtain in (4.25) (for all $\varepsilon > 0$)

$$\begin{aligned}
\alpha_1 \int_{A(k)} |\nabla u|^p \omega_2 dx &\leq C \left(\int_{A(k)} \left|\frac{f}{\omega_1}\right|^{p'} \omega_1\right)^{1/p'} \left(\int_{A(k)} |\nabla u|^p \omega_2 dx\right)^{1/p} \\
&\leq C\left[\varepsilon \int_{A(k)} |\nabla u|^p \omega_2 dx + C(\varepsilon) \int_{A(k)} \left|\frac{f}{\omega_1}\right|^{p'} \omega_1 dx\right],
\end{aligned} \tag{4.28}$$

where $C(\varepsilon) = (\varepsilon p)^{-p'/p}/p'$. We can choose $\varepsilon > 0$ so that $C\varepsilon = \alpha_1/2$, and there exists a constant C_8 such that

$$\int_{A(k)} |\nabla u|^p \omega_2 dx \leq C_8 \int_{A(k)} \left|\frac{f}{\omega_1}\right|^{p'} \omega_1 dx. \tag{4.29}$$

Using Sobolev's inequality (Theorem 2.5) and Hölder's inequality with exponents q and q' we obtain (since $q > p'\,(r/(r-1)) > p'$)

$$\begin{aligned}
\left(\int_{A(k)} (|u|-k)^{p^*} \omega_1 dx\right)^{p/p^*} &= \left(\int_{A(k)} |\varphi|^{p^*} \omega_1 dx\right)^{p/p^*} \leq \left(\int_{A(k)} |\varphi|^{p^*} \omega_2 dx\right)^{p/p^*} \\
&\leq C \int_{A(k)} |\nabla\varphi|^p \omega_2 dx = C \int_{A(k)} |\nabla u|^p \omega_2 dx \\
&\leq C\, C_8 \int_{A(k)} \left|\frac{f}{\omega_1}\right|^{p'} \omega_2 dx \\
&\leq C_9 \left(\int_{\Omega} \left|\frac{f}{\omega_1}\right|^{q} \omega_1 dx\right)^{p'/q} [\mu_1(\Omega)]^{1-(p'/q)}.
\end{aligned} \tag{4.30}$$

Let us now take $l > k > 0$ and observe that $A(l) \subset A(k)$. Then, from the previous inequality, it follows that

$$\begin{aligned}
\mu_1(A(l))(l-k)^{p^*} &= \int_{A(l)} (l-k)^{p^*}\omega_1 dx \\
&\le \int_{A(l)} (|u|-k)^{p^*}\omega_1 dx \le \int_{A(k)} (|u|-k)^{p^*}\omega_1 dx \\
&\le C_9 \left(\int_\Omega \left|\frac{f}{\omega_1}\right|^q \omega_1 dx\right)^{p'p^*/qp} \left[\mu_1(A(k))\right]^{(1-(p'/q))(p^*/p)}.
\end{aligned} \tag{4.31}$$

Hence we obtain

$$\mu_1(A(l)) \le \frac{C_9\left(\int_\Omega |f/\omega_1|^q \omega_1 dx\right)^{(p'p^*)/q\,p}}{(l-k)^{p^*}} \left[\mu_1(A(k))\right]^{(1-(p'/q))\,(p^*/p)}. \tag{4.32}$$

Since $(1-(p'/q))(p^*/p) > 1$, by Lemma 2.6 there exists a constant $C_{10} > 0$ such that $\mu_1(A(k)) = 0$ for all $k \ge C_{10}$, and using Lemma 2.7 we obtain $|A(k)| = 0$. Therefore if u_m is a solution of problem (P_m), we have $\|u_m\|_{L^\infty(\Omega)} \le C_{10}$ and C_{10} is independent of m.

Step 3. Since $u_m \in W_0^{1,p}(\Omega,\omega_1,\omega_2) \cap L^\infty(\Omega)$ and $\|u_m\|_{L^\infty(\Omega)} \le C_{10}$, then the sequence $\{u_m\}$ is relative compact in the strong topology of $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ (by apply the analogous results of [10] and Lemma 2.8). Then, by extracting a subsequence $\{u_{m_k}\}$ which strongly converges in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ (i.e., there exists $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ such that $u_{m_k} \to u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$), we have for all $\varphi \in W_0^{1,p}(\Omega,\omega_1,\omega_2) \cap L^\infty(\Omega)$

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x,u_{m_k},\nabla u_{m_k}) \cdot \nabla\varphi\omega_2 + \int_\Omega g(x,u_{m_k})\varphi\omega_1 dx \\
&\quad + \int_\Omega H_{m_k}(x,u_{m_k},\nabla u_{m_k})\varphi\omega_2 dx \longrightarrow \int_\Omega \mathcal{A}(x,u,\nabla u) \cdot \nabla\varphi\omega_2 \\
&\quad + \int_\Omega g(x,u)\varphi\omega_1 dx + \int_\Omega H(x,u,\nabla u)\varphi\omega_2 dx.
\end{aligned} \tag{4.33}$$

Therefore $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2) \cap L^\infty(\Omega)$ is the solution of problem (P).

Example 4.1. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and consider the weights $\omega_1(x,y) = (x^2 + y^2)^{1/2}$ and $\omega_2 = (x^2+y^2)^{1/4}$ (ω_1 and $\omega_2 \in A_2$), and the functions $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$, and $H : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned}
\mathcal{A}((x,y),\eta,\xi) &= h_2(x,y)\ \xi, \\
g((x,y),\eta) &= \eta\left(\cos^2(xy) + 1\right), \\
H((x,y),\eta,\xi) &= |\xi|^2 \sin(xy) + \arctan(\eta),
\end{aligned} \tag{4.34}$$

where $h_2(x,y) = 2e^{x^2+y^2}$. Let us consider the partial differential operator

$$\begin{aligned} Lu(x,y) = &-\operatorname{div}\left[\omega_2(x,y)\mathcal{A}((x,y),u,\nabla u)\right] \\ &+\omega_1(x,y)g((x,y),u) \\ &+\ \omega_2(x,y)H((x,y),\eta,\xi), \end{aligned} \tag{4.35}$$

and $f(x,y) = (x^2+y^2)^{3/2q}\cos(1/(x^2+y^2))$, with $2r_2/(r_2-1) < q < 6$. Therefore, by Theorem 1.2, the problem

$$Lu(x,y) = f(x,y), \quad \text{on } \Omega, \qquad u(x,y) = 0, \quad \text{on } \partial\Omega \tag{P}$$

has a solution $u \in W_0^{1,2}(\Omega,\omega_1,\omega_2) \cap L^\infty(\Omega)$.

References

[1] A. C. Cavalheiro, "Existence results for degenerate quasilinear elliptic equations in weighted Sobolev spaces," *Bulletin of the Belgian Mathematical Society. Simon Stevin*, vol. 17, no. 1, pp. 141–153, 2010.

[2] P. Drábek, A. Kufner, and V. Mustonen, "Pseudo-monotonicity and degenerated or singular elliptic operators," *Bulletin of the Australian Mathematical Society*, vol. 58, no. 2, pp. 213–221, 1998.

[3] E. Fabes, C. Kenig, and R. Serapioni, "The local regularity of solutions of degenerate elliptic equations," *Communications in Partial Differential Equations*, vol. 7, no. 1, pp. 77–116, 1982.

[4] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, The Clarendon Press, New York, NY, USA, 1993.

[5] B. Muckenhoupt, "Weighted norm inequalities for the Hardy maximal function," *Transactions of the American Mathematical Society*, vol. 165, pp. 207–226, 1972.

[6] E. Stein, *Harmonic Analysis*, vol. 43, Princeton University Press, Princeton, NJ, USA, 1993.

[7] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, vol. 123, Academic Press, San Diego, Calif, USA, 1986.

[8] A. Kufner, *Weighted Sobolev Spaces*, John Wiley & Sons, New York, NY, USA, 1985.

[9] L. Boccardo, F. Murat, and J.-P. Puel, "L^∞-estimate for some nonlinear elliptic partial differential equations and application to an existence result," *SIAM Journal on Mathematical Analysis*, vol. 23, no. 2, pp. 326–333, 1992.

[10] L. Boccardo, F. Murat, and J.-P. Puel, "Existence of bounded solutions for nonlinear elliptic unilateral problems," *Annali di Matematica Pura ed Applicata*, vol. 152, pp. 183–196, 1988.

[11] J. Leray and J.-L. Lions, "Quelques résulatats de Višik sur les problèmes elliptiques nonlinéaires par les mèthodes de Minty-Browder," *Bulletin de la Socièté Mathèmatique de France*, vol. 93, pp. 97–107, 1965.

[12] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, vol. 116, North-Holland Mathematics Studies, Amsterdam, The Netherlands, 1985.

[13] B. O. Turesson, *Nonlinear Potential Theory and Weighted Sobolev Spaces*, vol. 1736 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.

[14] G. Stampacchia, *Èquations Elliptiques du Second ordre à Coefficients Discontinus*, Séminaire de Mathématiques Supérieures, Les Presses de l'Université de Montréal, Montreal, Quebec, 1966.

[15] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, vol. II/B, Springer, New York, NY, USA, 1990.