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Research Article

On the Nevanlinna's Theory for Vector-Valued Mappings

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The purpose of this paper is to establish the first and second fundamental theorems for an E-valued meromorphic mapping from a generic domain $D \subset \mathbb{C}$ to an infinite dimensional complex Banach space E with a Schauder basis. It is a continuation of the work of C. Hu and Q. Hu. For $f(z)$ defined in the disk, we will prove Chuang's inequality, which is to compare the relationship between $T(r, f)$ and $T(r, f')$. Consequently, we obtain that the order and the lower order of $f(z)$ and its derivative $f'(z)$ are the same.

1. Introduction

In 1980s, Ziegler [1] established Nevanlinna's theory for the vector-valued meromorphic functions in finite dimensional spaces. After Ziegler some works in finite dimensional spaces were done in 1990s [2–4]. In 2006, C. Hu and Q. Hu [5] considered the case of infinite dimensional spaces and they investigated the E-valued meromorphic mappings defined in the disk $C_r = \{z : |z| < r\}$. In this article, by using Green function technique, we will consider this theory defined in generic domain $D \subseteq \mathbb{C}$ (see Section 2). In Section 3, motivated by the work of [6–8], we will prove Chuang's inequality, which is to compare the relationship between $T(r, f)$ and $T(r, f')$. Consequently, we obtain that the order and the lower order of $f(z)$ and its derivative $f'(z)$ are the same. This is an extension of an important result for meromorphic functions.

2. First and Second Fundamental Theorem in Generic Domains

Let $(E, \|\cdot\|)$ be a complex Banach space with a Schauder basis $\{\mathbf{e}_i\}$ and the norm $\|\cdot\|$. Thus an E-valued meromorphic mapping $f(z)$ defined in a domain $D \subseteq \mathbb{C}$ can be written as $f(z) =$

$(f_1(z), f_2(z), \ldots, f_k(z), \ldots)$. The elements of E are called vectors and are usually denoted by letters from the alphabet: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \ldots$. The symbol $\mathbf{0}$ denotes the zero vector of E. We denote vector infinity, complex number infinity, and the norm infinity by $\widehat{\infty}$, ∞, and $+\infty$, respectively. A vector-valued mapping is called holomorphic(meromorphic) if all $f_j(z)$ are holomorphic (meromorphic). The jth derivative ($j = 1, 2, \ldots$) and the integration of $f(z)$ are defined by

$$f^{(j)}(z) = \left(f_1^{(j)}(z), f_2^{(j)}(z), \ldots, f_k^{(j)}(z), \ldots\right),$$
$$\int^z f(\zeta)d\zeta = \left(\int^z f_1(\zeta)d\zeta, \int^z f_2(\zeta)d\zeta, \ldots, \int^z f_k(\zeta)d\zeta, \ldots\right), \tag{2.1}$$

respectively. We assume that $f^{(0)}(z) = f(z)$. A point $z_0 \in D$ is called a "pole" or "$\widehat{\infty}$-point" of $f(z) = (f_1(z), \ldots, f_k(z), \ldots)$ if z_0 is a pole of at least one of the component functions $f_k(z)$ $(k = 1, 2, \ldots)$. We define $\|f(z_0)\| = +\infty$ when z_0 is a pole. A point $z_0 \in D$ is called "zero" of $f(z)$ if all the component functions $f_k(z)$ $(k = 1, 2, \ldots)$ have zeros at z_0.

Remark 2.1. The integrals are well defined because the set of singularities making $\widehat{\infty} - \widehat{\infty}$ meaningless is zero measurable.

In order to make our statement clear, we first recall some knowledge of Green functions.

Definition 2.2. Let D be a domain surrounded by finitely many piecewise analytic curves. Then for any $a \in D$, there exists a Green function, denoted by $G_D(z, a)$, for D with singularity at $a \in D$ which is uniquely determined by the following:

(1) $G_D(z, a)$ is harmonic in $D \setminus \{a\}$;

(2) in a neighborhood of a, $G_D(z, a) = -\log|z - a| + w(z, a)$ for some function $w(z, a)$ harmonic in D;

(3) $G_D(z, a) \equiv 0$, on the boundary of D.

By ∂D we denote the boundary of D and $\vec{n}$ the inner normal of ∂D with respect to D. Using Green function we can establish the following general Poisson formula for the E-valued meromorphic mapping, which is similar with [5, Lemma 2.2] (see [9, Theorem 2.1], or [10, Theorem 2.1.1]). We do not give the details here.

Theorem 2.3. *Let $f : \overline{D}(\subset \mathbb{C}) \to E$ be an E-valued meromorphic mapping, which does not reduce to the constant zero element $\mathbf{0} \in E$. Then*

$$\log\|f(z)\| = \frac{1}{2\pi}\int_{\partial D} \log\|f(\zeta)\| \frac{\partial G_D(\zeta, z)}{\partial \vec{n}} ds - \sum_{a_m \in D} G_D(a_m, z)$$
$$+ \sum_{b_n \in D} G_D(b_n, z) - \frac{1}{2\pi}\int_D G_D(\zeta, z)\Delta \log\|f(\zeta)\| dx \wedge dy, \tag{2.2}$$

where $\zeta = x + iy$, $\{a_m\}$ are the zeros of $f(z)$ and $\{b_n\}$ are the poles of $f(z)$ according to their multiplicities.

Remark 2.4. A simple inspection to the $\mathbb{C}^2$-valued case shows that $\log\|f(z)\|$ is not harmonic for a holomorphic (or meromorphic) E-valued function. Therefore we have an additional term in formula (2.2).

Following Theorem 2.3, we introduce some notations.

$$\begin{aligned} N(D,a,f) &= \sum_{b_n\in D} G_D(b_n,a), \\ m(D,a,f) &= \frac{1}{2\pi}\int_{\partial D}\log^+\|f(\zeta)\|\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds, \\ V(D,a,f) &= \frac{1}{2\pi}\int_D G(\zeta,a)\Delta\log\|f(\zeta)\|dx\wedge dy, \end{aligned} \tag{2.3}$$

where a is a point in D and $\{b_n\}$ are the poles of $f(z)$ in D appearing according to their multiplicities, $\log^+ x = \log\max\{x,1\}$. Define

$$T(D,a,f) = m(D,a,f) + N(D,a,f). \tag{2.4}$$

$T(D,a,f)$ is called the Nevanlinna characteristic function of $f(z)$ with the center $a\in D$.

Next, we give the first (FFT) and the second (SFT) fundamental theorems for $f(z)$.

Theorem 2.5 (FFT). *Let $f(z)$ be an E-valued meromorphic mapping on $\overline{D}$. Then for a fixed vector $\mathbf{b}\in E$ and for any $a\in D$ such that $f(a)\neq\mathbf{b}$, one has*

$$T\left(D,a,\frac{1}{f-\mathbf{b}}\right) = T(D,a,f) - V(D,a,f-\mathbf{b}) - \log\|f(a)-\mathbf{b}\| + \varepsilon(\mathbf{b},D), \tag{2.5}$$

where

$$|\varepsilon(\mathbf{b},D)| \le \begin{cases} \log^+\|\mathbf{b}\| + \log 2, & \mathbf{b}\neq\mathbf{0}, \\ 0, & \mathbf{b}=\mathbf{0}. \end{cases} \tag{2.6}$$

Proof. We can rewrite Theorem 2.3 as follows:

$$T(D,a,f) = T\left(D,a,\frac{1}{f}\right) + V(D,a,f) + \log\|f(a)\|. \tag{2.7}$$

Applying this formula to the function $f(z)-\mathbf{b}$, we can prove the theorem. □

Theorem 2.6 (SFT). *Let $f(z)$ be an E-valued meromorphic mapping on $\overline{D}$, let $\mathbf{a}^{[j]}(j=1,2,\ldots,q) \in E\bigcup\{\widehat{\infty}\}$ be q distinct vectors, and let $f(a) \neq \mathbf{a}^{[j]}$. Then,*

$$\begin{aligned}(q-2)T(D,a,f) \leq \sum_{j=1}^{q}\Big[N\Big(D,a,f=\mathbf{a}^{[j]}\Big)+V\Big(D,a,f-\mathbf{a}^{[j]}\Big)\Big]&\\ -V(D,a,f')-N_1(D,a,f)+S(D,a,f),&\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}S(D,a,f) &= \frac{1}{2\pi}\int_{\partial D}\log^{+}\left[\sum_{j=1}^{q}\frac{\|f'(\zeta)\|}{\|f(\zeta)-\mathbf{a}^{[j]}\|}\right]\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds\\ &\quad -\log\|f'(a)\|+q\log^{+}\frac{2q}{\delta}+\sum_{i=1}^{q}\log\|f(a)-\mathbf{a}^{[i]}\|,\\ N_1(D,a,f) &= 2N(D,a,f)-N(D,a,f')+N\left(D,a,\frac{1}{f'}\right),\\ \delta &= \min_{i\neq j}\|\mathbf{a}^{[i]}-\mathbf{a}^{[j]}\|>0.\end{aligned} \tag{2.9}$$

Furthermore, one has the following form:

$$\begin{aligned}(q-2)T(D,a,f) \leq \sum_{j=1}^{q}\left[\overline{N}\left(D,a,\frac{1}{f-\mathbf{a}^{[j]}}\right)+V\Big(D,a,f-\mathbf{a}^{[j]}\Big)\right]&\\ -V(D,a,f')+S(D,a,f).&\end{aligned} \tag{2.10}$$

Proof. Set

$$F(\zeta)=\sum_{j=1}^{q}\frac{1}{\|f(\zeta)-\mathbf{a}^{[j]}\|}. \tag{2.11}$$

According to the property of the logarithm function, we get

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial D}\log^{+}F(\zeta)\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds\\ &\quad\leq m\left(D,a,\frac{1}{f'}\right)+\frac{1}{2\pi}\int_{\partial D}\log^{+}\left[F(\zeta)\|f'(\zeta)\|\right]\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds.\end{aligned} \tag{2.12}$$

Denote $\delta=\min_{i\neq j}\|a^{[i]}-a^{[j]}\|$, and fix $\mu\in\{1,2,\ldots,q\}$. Then we obtain

$$\|f(z)-\mathbf{a}^{[v]}\| \geq \|\mathbf{a}^{[\mu]}-\mathbf{a}^{[v]}\|-\|\mathbf{a}^{[\mu]}-f(z)\|>\frac{3\delta}{4} \tag{2.13}$$

for $\mu \neq v$ by

$$\|f(z) - \mathbf{a}^{[\mu]}\| < \frac{\delta}{2q} \leq \frac{\delta}{4}. \tag{2.14}$$

So either the set of points on ∂D which is determined by (2.14) is empty or any two of some sets for different μ have intersection. In any case, on ∂D we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial D}\log^{+}F(\zeta)\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds &\geq \frac{1}{2\pi}\sum_{\mu=1}^{q}\int_{\|f-\mathbf{a}^{[\mu]}\|<\delta/2q}\log^{+}F(\zeta)\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds \\ &\geq \frac{1}{2\pi}\sum_{\mu=1}^{q}\int_{\|f-\mathbf{a}^{[\mu]}\|<\delta/2q}\log^{+}\frac{1}{\|f(\zeta)-a^{[\mu]}\|}\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds.\end{aligned} \tag{2.15}$$

Since

$$\begin{aligned}&\frac{1}{2\pi}\int_{\|f-\mathbf{a}^{[\mu]}\|<\delta/2q}\log^{+}\frac{1}{\|f(\zeta)-\mathbf{a}^{[\mu]}\|}\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds \\ &\quad= m\left(D,a,\mathbf{a}^{[\mu]}\right) - \frac{1}{2\pi}\int_{\|f-\mathbf{a}^{[\mu]}\|>\delta/2q}\log^{+}\frac{1}{\|f(\zeta)-\mathbf{a}^{[\mu]}\|}\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds \\ &\quad\geq m\left(D,a,\mathbf{a}^{[\mu]}\right) - \log^{+}\frac{2q}{\delta},\end{aligned} \tag{2.16}$$

it follows that

$$\frac{1}{2\pi}\int_{\partial D}\log^{+}F(\zeta)\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds \geq \sum_{\mu=1}^{q}m\left(D,a,\mathbf{a}^{[\mu]}\right) - q\log^{+}\frac{2q}{\delta}. \tag{2.17}$$

From (2.12), we get

$$m\left(D,a,\frac{1}{f'}\right) \geq \sum_{\mu=1}^{q}m\left(D,a,\mathbf{a}^{[\mu]}\right) - q\log^{+}\frac{2q}{\delta} - \frac{1}{2\pi}\int_{\partial D}\log^{+}\left[F(\zeta)\|f'(\zeta)\|\right]\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds. \tag{2.18}$$

Since $f(z)$ is nonconstant vector, $f'(z)$ does not reduce to the constant zero element $\mathbf{0}$. Applying FFT to $f'(z)$, we can obtain

$$T(D,a,f') = N\left(D,a,\frac{1}{f'}\right) + m\left(D,a,\frac{1}{f'}\right) + V(D,a,f') + \log\|f'(a)\|. \tag{2.19}$$

Using this formula, we have

$$\begin{aligned} T(D,a,f') \geq{}& \sum_{\mu=1}^{q} m\left(D,a,\mathbf{a}^{[\mu]}\right) - q\log^{+}\frac{2q}{\delta} \\ &- \frac{1}{2\pi}\int_{\partial D}\log^{+}\left[F(\zeta)\|f'(\zeta)\|\right]\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds \\ &+ N\left(D,a,\frac{1}{f'}\right) + V(D,a,f') + \log\|f'(a)\|. \end{aligned} \tag{2.20}$$

On the other hand, we have

$$\begin{aligned} T(D,a,f') &= m(D,a,f') + N(D,a,f') \\ &\leq m(D,a,f) + N(D,a,f') + \frac{1}{2\pi}\int_{\partial D}\log\frac{\|f'(\zeta)\|}{\|f(\zeta)\|}\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds. \end{aligned} \tag{2.21}$$

The two inequalities above give

$$\begin{aligned} &\sum_{\mu=1}^{q} m\left(D,a,\mathbf{a}^{[\mu]}\right) + V(D,a,f') \\ &\quad\leq m(D,a,f) + N(D,a,f') - N\left(D,a,\frac{1}{f'}\right) \\ &\qquad+ \frac{1}{2\pi}\int_{\partial D}\log^{+}\left[F(\zeta)\|f'(\zeta)\|\right]\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds + \frac{1}{2\pi}\int_{\partial D}\log\frac{\|f'(\zeta)\|}{\|f(\zeta)\|}\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds \\ &\qquad- \log\|f'(a)\| + q\log^{+}\frac{2q}{\delta}. \end{aligned} \tag{2.22}$$

That is to say,

$$\begin{aligned} &\sum_{\mu=1}^{q} m\left(D,a,\mathbf{a}^{[\mu]}\right) + V(D,a,f') \\ &\quad\leq m(D,a,f) + N(D,a,f') - N\left(D,a,\frac{1}{f'}\right) + S(D,a,f). \end{aligned} \tag{2.23}$$

Adding $\sum_{\mu=1}^{q} N(D,a,f=\mathbf{a}^{[\mu]})$ to the above inequality and applying FFT, we can formulate

$$\begin{aligned} (q-1)T(D,a,f) <{}& N(D,a,f) + \sum_{j=1}^{q}\left[N\left(D,a,f=\mathbf{a}^{[j]}\right) + V\left(D,a,f-\mathbf{a}^{[j]}\right)\right] \\ &- N_1(D,a,f) + S(D,a,f), \end{aligned} \tag{2.24}$$

where

$$S(D,a,f) = \frac{1}{2\pi}\int_{\partial D}\log^{+}\left[\sum_{j=0}^{q}\frac{\|f'(\zeta)\|}{\|f(\zeta)-\mathbf{a}^{[j]}\|}\right]\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds$$
$$-\log\|f'(a)\| + q\log^{+}\frac{2q}{\delta} + \sum_{i=0}^{q}\log\|f(a)-\mathbf{a}^{[i]}\|, \quad \mathbf{a}^{[0]} = \mathbf{0}. \tag{2.25}$$

Since $N(D,a,f) \le T(D,a,f)$, (2.24) can be written as

$$(q-2)T(D,a,f) < \sum_{j=1}^{q}\left[N\left(D,a,f=\mathbf{a}^{[j]}\right) + V\left(D,a,f-\mathbf{a}^{[j]}\right)\right]$$
$$-N_1(D,a,f) + S(D,a,f). \tag{2.26}$$

If $\{\mathbf{a}^{[j]}\}$ contains $\widehat{\infty}$, (2.26) also holds. Let $\mathbf{a}^{[q+1]} = \widehat{\infty}$, and substitute q with $q+1$; then we have (2.26), where $\mathbf{a}^{[q]} = \widehat{\infty}$, and

$$S(D,a,f) = \frac{1}{2\pi}\int_{\partial D}\log^{+}\left[\sum_{j=0}^{q-1}\frac{\|f'(\zeta)\|}{\|f(\zeta)-\mathbf{a}^{[j]}\|}\right]\frac{\partial G_D(\zeta,a)}{\partial\vec{n}}ds$$
$$-\log\|f'(a)\| + q\log^{+}\frac{2q}{\delta} + \sum_{i=0}^{q-1}\log\|f(a)-\mathbf{a}^{[i]}\|. \tag{2.27}$$

□

Next we establish Hiong King-Lai's inequality for $f(z)$.

Theorem 2.7. *Let $f(z)$ be an E-valued meromorphic mapping on $\overline{D}$, $l \in D$, let $\mathbf{a},\mathbf{b},\mathbf{c} \in E$ be three finite vectors, and let $\mathbf{b} \ne \mathbf{0}$, $\mathbf{c} \ne \mathbf{0}$, $\mathbf{b} \ne \mathbf{c}$, $f^{(k)}(l) \ne \mathbf{0},\mathbf{b},\mathbf{c}$. Then one has*

$$T(D,l,f) < N(D,l,f=\mathbf{a}) + N\left(D,l,f^{(k)}=\mathbf{b}\right) + N\left(D,l,f^{(k)}=\mathbf{c}\right)$$
$$+V\left(D,l,f^{(k)}\right) + V\left(D,l,f^{(k)}-\mathbf{b}\right) + V\left(D,l,f^{(k)}-\mathbf{c}\right) - N\left(D,l,\frac{1}{f^{(k+1)}}\right)$$
$$+S\left(D,l,f^{(k)}\right). \tag{2.28}$$

Proof. First, we have

$$\frac{1}{2\pi}\int_{\partial D}\log^{+}\left\|\frac{1}{f(\zeta)-\mathbf{a}}\right\|\frac{\partial G_D(\zeta,l)}{\partial\vec{n}}ds$$
$$\le \frac{1}{2\pi}\int_{\partial D}\log^{+}\left\|\frac{1}{f^{(k)}(\zeta)}\right\|\frac{\partial G_D(\zeta,l)}{\partial\vec{n}}ds + \frac{1}{2\pi}\int_{\partial D}\log^{+}\left\|\frac{f^{(k)}(\zeta)}{f(\zeta)-\mathbf{a}}\right\|\frac{\partial G_D(\zeta,l)}{\partial\vec{n}}ds. \tag{2.29}$$

Applying FFT to $f(z)$ and $f^{(k)}(z)$, respectively, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial D}\log^{+}\left\|\frac{1}{f(\zeta)-\mathbf{a}}\right\|\frac{\partial G_D(\zeta,l)}{\partial\vec{n}}ds\\
&\quad= T(D,l,f)-N(D,l,f=\mathbf{a})-V(D,l,f-\mathbf{a})-\log\|f(l)-\mathbf{a}\|+\varepsilon(\mathbf{a},D),\\
&\frac{1}{2\pi}\int_{\partial D}\log^{+}\left\|\frac{1}{f^{(k)}(\zeta)}\right\|\frac{\partial G_D(\zeta,l)}{\partial\vec{n}}ds\\
&\quad= T\left(D,l,f^{(k)}\right)-N\left(D,l,\frac{1}{f^{(k)}}\right)-V\left(D,l,f^{(k)}\right)-\log\|f^{(k)}(l)\|.
\end{aligned}\tag{2.30}$$

Thus we have

$$\begin{aligned}
T(D,l,f)\le{}& T\left(D,l,f^{(k)}\right)+N(D,l,f=\mathbf{a})+V(D,l,f-\mathbf{a})\\
&-N\left(D,l,\frac{1}{f^{(k)}}\right)-V\left(D,l,f^{(k)}\right)+\log\frac{\|f(l)-a\|}{\|f^{(k)}(l)\|}-\varepsilon(\mathbf{a},D).
\end{aligned}\tag{2.31}$$

Applying SFT to $f^{(k)}$ with $\mathbf{0},\mathbf{b},\mathbf{c}$, we have

$$\begin{aligned}
T\left(D,l,f^{(k)}\right)\le{}& \overline{N}\left(D,l,\frac{1}{f^{(k)}}\right)+\overline{N}\left(D,l,f^{(k)}=\mathbf{b}\right)+\overline{N}\left(D,l,f^{(k)}=\mathbf{c}\right)-N\left(D,l,f^{(k+1)}\right)\\
&+V\left(D,l,f^{(k)}\right)+V\left(D,l,f^{(k)}-\mathbf{b}\right)+V\left(D,l,f^{(k)}-\mathbf{c}\right)-V\left(D,l,f^{(k+1)}\right)\\
&+S\left(D,l,f^{(k)}\right).
\end{aligned}\tag{2.32}$$

Combining (2.31) with (2.32), we have

$$\begin{aligned}
T(D,l,f)\le{}& N(D,l,f=\mathbf{a})+\overline{N}\left(D,l,f^{(k)}=\mathbf{b}\right)+\overline{N}\left(D,l,f^{(k)}=\mathbf{c}\right)-N\left(D,l,f^{(k+1)}\right)\\
&+V(D,l,f-\mathbf{a})+V\left(D,l,f^{(k)}-\mathbf{b}\right)+V\left(D,l,f^{(k)}-\mathbf{c}\right)-V\left(D,l,f^{(k+1)}\right)\\
&+S\left(D,l,f^{(k)}\right).
\end{aligned}\tag{2.33}$$

□

3. The Vector-Valued Mapping and Its Derivative

In this section, we will discuss the value distribution theory of $f(z)$ defined in the disk $C_r = \{z : |z| < r\}$. We will prove Chuang's inequality. According to (2.3), we have the following terms:

$$\begin{aligned}
m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\|f\left(re^{i\theta}\right)\|d\theta,\\
N(r,f) &= \int_0^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)\log r,\\
V(r,f) &= \frac{1}{2\pi}\int_{C_r} \log\left|\frac{r}{\zeta}\right|\Delta \log\|f(\zeta)\|dx\wedge dy, \quad \zeta = x+iy,\\
T(r,f) &= m(r,f) + N(r,f),
\end{aligned} \tag{3.1}$$

where $n(r,f)$ denotes the number of poles of $f(z)$ in $\{z : |z| < r\}$. The order and the lower order of an E-valued meromorphic mapping $f(z)$ are defined by

$$\begin{aligned}
\lambda(f) &= \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},\\
\mu(f) &= \liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.
\end{aligned} \tag{3.2}$$

The following lemma is well known.

Lemma 3.1 (see [11, Boutroux-Cartan Theorem]). *Let $\{a_j\}_{j=1}^n$ be n complex numbers. Then the set of the point z satisfying*

$$\prod_{j=1}^{n}\left|z-a_j\right| < h^n \tag{3.3}$$

can be contained in several disks, denoted by (γ); the total sum of its radius does not exceed $2eh$.

The next lemma is a special case of Theorem 2.3.

Lemma 3.2 (see [5]). *Let $f : C_r \to E$ be an E-valued meromorphic mapping, which does not reduce to the constant zero element $\mathbf{0} \in E$. Then, for a $z \in C_r$, one has*

$$\begin{aligned}\log\|f(z)\| = {} & \frac{1}{2\pi}\int_0^{2\pi} \log\left\|f\left(re^{i\theta}\right)\right\| \frac{r^2 - t^2}{r^2 - 2rt\cos(\theta - \phi) + t^2}\, d\phi \\ & - \sum_{z_j(\mathbf{0})\in C_r} \log\left|\frac{r^2 - \overline{z_j(\mathbf{0})}z}{r(z - z_j(\mathbf{0}))}\right| + \sum_{z_j(\widehat{\infty})\in C_r} \log\left|\frac{r^2 - \overline{z_j(\widehat{\infty})}z}{r(z - z_j(\widehat{\infty}))}\right| \\ & - \frac{1}{2\pi}\int_{C_r} \log\left|\frac{r^2 - \overline{\xi}z}{r(z - \xi)}\right| \Delta \log\|f(\xi)\|\, dx \wedge dy.\end{aligned} \tag{3.4}$$

Here $z_j(\mathbf{0})$ and $z_j(\widehat{\infty})$ are all the zeros and poles counting their multiplies of f in D.

In order to obtain the relationship between $T(r, f)$ and $T(r, f')$, we should first establish the following two lemmas.

Lemma 3.3. *Let $f : \mathbb{C} \to E$ be a nonzero E-valued meromorphic mapping, and $f(0) \neq \widehat{\infty}$. If R and R' are two positive numbers, and $R < R'$, then there exists a $\theta_0 \in [0, 2\pi)$, such that for any $0 \leq r \leq R$ one has*

$$\log^+\left\|f\left(re^{i\theta_0}\right)\right\| \leq \frac{R' + R}{R' - R} m(R', f) + n(R', f)\log 4 + N(R', f). \tag{3.5}$$

Proof. For $z = re^{i\theta}, 0 \leq r \leq R$. By Lemma 3.2 we have

$$\begin{aligned}\log\|f(z)\| \leq {} & \frac{1}{2\pi}\int_0^{2\pi} \log\left\|f\left(R'e^{i\theta}\right)\right\| \frac{R'^2 - r^2}{R'^2 - 2R'r\cos(\theta - \phi) + r^2}\, d\phi \\ & + \sum_{j=1}^{n} \log\left|\frac{R'^2 - \overline{b_j}z}{R'(z - b_j)}\right|,\end{aligned} \tag{3.6}$$

where $\{b_j\}_{j=1}^n$ are the poles of $f(z)$ in $|z| \leq R'$. Then

$$\begin{aligned}\log^+\|f(z)\| & \leq \frac{R' + r}{R' - r} m(R', f) + \sum_{j=1}^{n} \log\frac{2R'}{|z - b_j|} \\ & \leq \frac{R' + r}{R' - r} m(R', f) + \log\frac{(2R')^n}{\prod_{j=1}^{n}|z - b_j|}.\end{aligned} \tag{3.7}$$

Writing $b_j = |b_j|e^{i\phi_j}$, we have

$$\left|re^{i\theta} - \left|b_j\right|e^{i\phi_j}\right| \geq \left|b_j\right|\left|\sin(\theta - \phi_j)\right|. \tag{3.8}$$

Thus

$$\prod_{j=1}^{n}|z-b_j| \geq \left(\prod_{j=1}^{n}|b_j|\right)\left(\prod_{j=1}^{n}|\sin(\theta-\phi_j)|\right). \tag{3.9}$$

However,

$$\int_0^{\pi}\log\left|\prod_{j=1}^{n}\right|\sin(\theta-\phi_j)\ \ | d\theta = n\int_0^{\pi}\log|\sin\theta| d\theta = -n\pi\log 2. \tag{3.10}$$

Hence there exists a real number θ_0 such that

$$\left|\prod_{j=1}^{n}\sin(\theta_0-\phi_j)\right| > \frac{1}{2^n}. \tag{3.11}$$

Combining (3.7) and (3.9) with (3.11), we have

$$\begin{aligned}\log^+\|f\left(re^{i\theta_0}\right)\| &\leq \frac{R'+R}{R'-R}m(R',f) + n\log 4 + \sum_{j=1}^{n}\log\frac{R'}{|b_j|}\\ &\leq \frac{R'+R}{R'-R}m(R',f) + n\log 4 + N(R',f).\end{aligned} \tag{3.12}$$

□

Lemma 3.4. *Let $f:\mathbb{C}\to E$ be a nonzero E-valued meromorphic mapping, and let $R<R'<R''$ be three positive numbers. Then there exists a positive number $R\leq\rho\leq R'$, and for $|z|=\rho$, one has*

$$\log^+\|f(z)\| \leq \frac{R''+R'}{R''-R'}m(R'',f) + n(R'',f)\log\frac{8eR''}{R'-R}. \tag{3.13}$$

Proof. Let $\{b_j\}_{j=1}^{n}$ be the poles of $f(z)$ in $|z|\leq R''$. By Boutroux-Cartan Theorem, we have

$$\prod_{j=1}^{n}|z-b_j| \geq \left(\frac{R'-R}{4e}\right)^n, \tag{3.14}$$

except for some points contained in a pack of disks whose radius does not exceed $(R'-R)/2$. Then there exists a circle $\{z:|z|=\rho\}$ such that $R\leq\rho\leq R'$ and $\{|z|=\rho\}\bigcap(\gamma)=\emptyset$.

Thus (3.14) holds on $\{|z| = \rho\}$. For any $z \in \{z : |z| = \rho\}$, we have

$$\begin{aligned}\log^+\|f\left(re^{i\theta_0}\right)\| &\le \frac{R'' + \rho}{R'' - \rho} m(R'', f) + \sum_{j=1}^{n} \log\left|\frac{R''2 - \overline{b_j} z}{R''(z - b_j)}\right| \\ &\le \frac{R'' + R'}{R'' - R'} m(R'', f) + n \log \frac{8eR''}{R' - R}.\end{aligned} \tag{3.15}$$

□

Now we are in the position to establish the following Chuang's inequality.

Theorem 3.5. *Let $f : \mathbb{C} \to E$ be a nonzero E-valued meromorphic mapping and $f(0) \ne \widehat{\infty}$. Then for $\tau > 1$ and $0 < r < R$, one has*

$$T(r, f) < C_\tau T(\tau r, f') + \log^+ \tau r + 4 + \log^+\|f(0)\|, \tag{3.16}$$

where C_τ is a positive constant.

Proof. Take a σ such that $\sigma^3 = \tau$ and denote $r_1 = \sigma r, r_2 = \sigma r_1, r_3 = \sigma r_2$. Applying Lemma 3.3 to $f'(z)$, we can find a real number θ_0 such that $0 \le t \le r_1$, and we have

$$\log^+\|f'\left(te^{i\theta_0}\right)\| \le \frac{r_2 + r_1}{r_2 - r_1} m(r_2, f') + n(r_2, f') \log 4 + N(r_2, f'). \tag{3.17}$$

In view of Lemma 3.4, for a fixed $\rho \in [r, r_1]$ we have

$$\log^+\|f'(z)\| \le \frac{r_2 + r_1}{r_2 - r_1} m(r_2, f') + n(r_2, f') \log \frac{8er_2}{r_1 - r}, \tag{3.18}$$

on $\{z : |z| = \rho\}$.

From the origin along the segment $\arg z = \theta_0$ to $\rho e^{i\theta_0}$, and along $\{z : |z| = \rho\}$ turn a rotation to $\rho e^{i\theta_0}$. We denote this curve by L, and its length is $(2\pi + 1)\rho$.

We notice that $\varphi(z) = \|f(z)\|$ is continuous on L. As in [5], E_n is an n-dimensional projective space of E with a basis $\{\mathbf{e}_i\}_{i=1}^n$. The projection operator $P_n : E \to E_n$ is a realization of E_n associated to the basis and $P_n(f(z)) = (f_1(z), f_2(z), \ldots, f_n(z))$. We have $P_n(f'(z)) = (P_n(f(z)))' = \sum_{i=1}^n f_i'(z)\mathbf{e}_i$ and $P_n(f(z)) = P_n(f(0)) + \sum_{i=1}^n (\int_0^z f_i'(\zeta) d\zeta)\mathbf{e}_i$. Therefore, since E_n is finite dimensional, there exists $K > 0$ (appearing in the inequality $\|\cdot\|_1 \leq K\|\cdot\|_2$, where $\|\cdot\|_1$ and $\|\cdot\|_2$ are any two norms on E_n) such that

$$\begin{aligned}
\|P_n(f(z))\| &\leq \|P_n(f(0))\| + \left\| \sum_{i=1}^{n} \left(\int_0^z f_i'(\xi) d\xi \right) e_i \right\| \\
&\leq \|P_n(f(0))\| + \frac{1}{K} \left(\sum_{i=1}^{n} \left| \int_0^z f_i'(\xi) d\xi \right|^2 \right)^{1/2} \\
&\leq \|P_n(f(0))\| + \frac{1}{K} \left(\sum_{i=1}^{n} \max_{\xi \in L} \left| f_i'(\xi) \right|^2 \right)^{1/2} (2\pi + 1)\rho \\
&\leq \|P_n(f(0))\| + \frac{K'}{K} M_n (2\pi + 1)\rho,
\end{aligned} \tag{3.19}$$

where $M_n = \max_{z \in L} \|P_n(f'(z))\|$. Thus, we have

$$\|P_n(f(z))\| \leq \|P_n(f(0))\| + M_n(2\pi + 1)\rho + O(1), \quad |z| = \rho. \tag{3.20}$$

In virtue of [6–8], every meromorphic mapping $f(z)$ with values in a Banach space E with a Schauder basis and the projections $P_n(f)$ are convergent in its natural topology; that is, they converge uniformly to f in any compact subset W of $\mathbb{C} \setminus \mathrm{P}_f$ (P_f being the set of poles the f in $\mathbb{C}$). Thus for n large enough, we have

$$\|P_n(f(z))\| = \|f(z)\| + O(1), \quad \text{for any } z \in W \subseteq \mathbb{C} \setminus P_f. \tag{3.21}$$

A similar argument to $f'(z)$ implies that for n large enough

$$\|P_n(f'(z))\| = \|f'(z)\| + O(1), \quad M_n \leq M + O(1) \quad \text{for any } z \in W \subseteq \mathbb{C} \setminus P_{f'}, \tag{3.22}$$

where $M = \max_{z \in L} \|f'(z)\|$.

Combining (3.20), (3.21), and (3.22) and the fact that the compact set $\{z : |z| = \rho\} \subseteq L \subseteq \mathbb{C} \setminus P_f$, we get

$$\|f(z)\| \leq \|f(0)\| + M(2\pi + 1)\rho + O(1). \tag{3.23}$$

Then

$$\log^+\|f(z)\| \leq \log^+\|f(0)\| + \log^+ M + \log^+\rho + \log 8e\pi + O(1). \tag{3.24}$$

In virtue of (3.13) and (3.17), we have

$$\begin{aligned}\log^+ M &\le \frac{r_2+r_1}{r_2-r_1} m(r_2, f') + n(r_2, f') \log \frac{8er_2}{r_1 - r} + N(r_2, f') \\ &\le \left\{ \frac{\log(8er_2/(r_1-r))}{\log(r_3/r_2)} + \frac{r_2+r_1}{r_2-r_1} \right\} T(r_3, f') = C'_\tau T(r_3, f').\end{aligned} \tag{3.25}$$

Therefore,

$$m(\rho, f) < C'_\tau T(r_3, f') + \log^+(\tau r) + 4 + \log^+\|f(0)\|. \tag{3.26}$$

Thus we have

$$\begin{aligned}T(r, f) \le T(\rho, f) &< (C'_\tau + 1) T(r_3, f') + \log^+(\tau r) + 4 + \log^+\|f(0)\| \\ &= C_\tau T(\tau r, f') + \log^+(\tau r) + 4 + \log^+\|f(0)\|.\end{aligned} \tag{3.27}$$

□

The following result says that we can also control the $T(r, f')$ by $T(r, f)$.

Theorem 3.6. *Let $f(z)(z \in \mathbb{C})$ be a nonconstant E-valued meromorphic mapping. Then one has*

$$T(r, f') \le 2T(r, f) + O(\log r + \log^+ T(r, f)). \tag{3.28}$$

Proof. One has

$$\begin{aligned}T(r, f') &= m(r, f') + N(r, f') \\ &\le m(r, f) + N(r, f') + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{\|f'(re^{i\phi})\|}{\|f(re^{i\phi})\|} d\phi \\ &= m(r, f) + N(r, f) + \overline{N}(r, f) + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{\|f'(re^{i\phi})\|}{\|f(re^{i\phi})\|} d\phi \\ &\le 2T(r, f) + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{\|f'(re^{i\phi})\|}{\|f(re^{i\phi})\|} d\phi \\ &= 2T(r, f) + O(\log r + \log^+ T(r, f)).\end{aligned} \tag{3.29}$$

From Theorems 3.5 and 3.6, we have the following. □

Corollary 3.7. *For a nonconstant E-valued meromorphic mapping $f(z)(z \in \mathbb{C})$, One has $\lambda(f) = \lambda(f')$, $\mu(f) = \mu(f')$.*

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