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Research Article

Multiple Solutions of Quasilinear Elliptic Equations in $\mathbb{R}^N$

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Assume that Q is a positive continuous function in $\mathbb{R}^N$ and satisfies some suitable conditions. We prove that the quasilinear elliptic equation $-\Delta_p u + |u|^{p-2}u = Q(z)|u|^{q-2}u$ in $\mathbb{R}^N$ admits at least two solutions in $\mathbb{R}^N$ (one is a positive ground-state solution and the other is a sign-changing solution).

1. Introduction

For $N \geq 3$, $2 \leq p < N$, and $p < q < p^* = Np/(N-p)$, we consider the quasilinear elliptic equations

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u = Q(z)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p}\left(\mathbb{R}^N\right), \end{gathered} \tag{1.1}$$

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u = Q_\infty|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p}\left(\mathbb{R}^N\right), \end{gathered} \tag{1.2}$$

where Δ_p is the p-Laplacian operator, that is,

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial z_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial z_i}\right). \tag{1.3}$$

Let Q be a positive continuous function in $\mathbb{R}^N$ and satisfy

$$Q(z) \geq Q_\infty = \lim_{|z|\to\infty} Q(z) > 0, \ Q(z) > Q_\infty \text{ on a set of positive measure.} \tag{Q1}$$

Associated with (1.1) and (1.2), we define the functionals a, b, b^∞, J, and J^∞, for $u \in W^{1,p}(\mathbb{R}^N)$,

$$\begin{gathered}
a(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^p + |u|^p\right) dz = \|u\|_{1,p}^p, \\
b(u) = \int_{\mathbb{R}^N} Q(z)|u|^q dz, \qquad b^\infty(u) = \int_{\mathbb{R}^N} Q_\infty |u|^q dz, \\
J(u) = \frac{1}{p} a(u) - \frac{1}{q} b(u), \qquad J^\infty(u) = \frac{1}{p} a(u) - \frac{1}{q} b^\infty(u).
\end{gathered} \tag{1.4}$$

It is easy to verify that the functionals a, b, b^∞, J, and J^∞ are C^1.

For the case $p = 2$, Lions [1, 2] proved that if $\lim_{|z|\to\infty} Q(z) = Q_\infty$, and $Q(z) \geq Q_\infty > 0$, then (1.1) has a positive ground-state solution in $\mathbb{R}^N$. Benci and Cerami [3] proved that (1.2) does not have any ground-state solution in an exterior domain. Bahri and Li [4] proved that there is at least one positive solution of (1.1) in $\mathbb{R}^N$ (or an exterior domain) when $\lim_{|z|\to\infty} Q(z) = Q_\infty > 0$ and $Q(z) \geq Q_\infty - C\exp(-\delta|z|)$ for $\delta > 2$. Cao [5] has studied the multiplicity of solutions (one is a positive ground-state solution and the other is a nodal solution) of (1.1) with Neumann condition in an exterior domain as follows. Assume that $\lim_{|z|\to\infty} Q(z) = Q_\infty > 0$, and $Q(z) \geq Q_\infty + C|z|^{-m}\exp(-\delta|z|)$ for $C > 0$, $m < (N-1)/2$, $\delta = q/(q+1)$, then (1.1) has at least two nontrivial solutions (one is a positive ground-state solution and the other is a nodal solution) in an exterior domain.

This article is motivated by the above papers. If Q is a positive continuous function in $\mathbb{R}^N$ and satisfies (Q1), then we prove that (1.1) admits a positive ground-state solution in $\mathbb{R}^N$. Combine it with some ideas of Cerami et al. [6] to show that if Q also satisfies $Q(z) \geq Q_\infty + C\exp(-\delta|z|)$ for $0 < \delta < \theta = (p-1)^{-1/p}$, then a nodal solution of (1.1) exists.

2. Preliminaries

We define the Palais-Smale (denoted by (PS)) sequences and (PS)-conditions in $W^{1,p}(\mathbb{R}^N)$ for J as follows.

Definition 2.1. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $W^{1,p}(\mathbb{R}^N)$ for J if $J(u_n) = \beta + o_n(1)$ and $J'(u_n) = o_n(1)$ strongly in $W^{-1,p'}(\mathbb{R}^N)$ as $n \to \infty$, where $W^{-1,p'}(\mathbb{R}^N)$ is the dual space of $W^{1,p}(\mathbb{R}^N)$ and $1/p + 1/p' = 1$

(ii) J satisfies the $(\mathrm{PS})_\beta$-condition in $W^{1,p}(\mathbb{R}^N)$ if every $(\mathrm{PS})_\beta$-sequence in $W^{1,p}(\mathbb{R}^N)$ for J contains a convergent subsequence.

Lemma 2.2. *Let $\beta \in \mathbb{R}$ and let $\{u_n\}$ be a $(PS)_\beta$-sequence in $W^{1,p}(\mathbb{R}^N)$ for J, then $\{u_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$. Moreover, $a(u_n) = b(u_n) + o_n(1) = (qp/(q-p))\beta + o_n(1)$ as $n \to \infty$ and $\beta \geq 0$.*

Proof. Since $p \geq 2$, we have that $\sqrt[p]{a(u_n)} \leq 1$ if $a(u_n) \leq 1$ and $\sqrt[p]{a(u_n)} \leq \sqrt{a(u_n)}$ if $a(u_n) > 1$. For sufficiently large n, we get

$$\begin{aligned} |\beta| + 2 + \sqrt{a(u_n)} &\geq |\beta| + 1 + \sqrt[p]{a(u_n)} \\ &\geq J(u_n) - \frac{1}{q}\langle J'(u_n), u_n\rangle = \left(\frac{1}{p} - \frac{1}{q}\right)a(u_n). \end{aligned} \tag{2.1}$$

It follows that $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Then $\langle J'(u_n), u_n\rangle = o_n(1)$ as $n \to \infty$. Thus,

$$\beta + o_n(1) = J(u_n) = \left(\frac{1}{p} - \frac{1}{q}\right)a(u_n) + o_n(1) = \left(\frac{1}{p} - \frac{1}{q}\right)b(u_n) + o_n(1), \tag{2.2}$$

that is, $a(u_n) = b(u_n) + o_n(1) = (qp/(q-p))\beta + o_n(1)$ as $n \to \infty$ and $\beta \geq 0$. □

Define

$$\alpha\left(\mathbb{R}^N\right) = \inf_{u \in \mathbf{M}(\mathbb{R}^N)} J(u), \tag{2.3}$$

where $\mathbf{M}(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \mid a(u) = b(u)\}$, and

$$\alpha^\infty\left(\mathbb{R}^N\right) = \inf_{u \in \mathbf{M}^\infty(\mathbb{R}^N)} J^\infty(u), \tag{2.4}$$

where $\mathbf{M}^\infty(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \mid a(u) = b^\infty(u)\}$.

Lemma 2.3. *Let u be a sign-changing solution of* (1.1). *Then $J(u) \geq 2\alpha(\mathbb{R}^N)$.*

Proof. Define $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. Since u is a sign-changing solution of (1.1), then u^- is nonnegative and nonzero. Multiply (1.1) by u^- and integrate it to obtain

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla u^- + |u|^{p-2}uu^-\right)dz = \int_{\mathbb{R}^N} Q(z)|u|^{q-2}uu^- dz, \tag{2.5}$$

that is, $u^- \in \mathbf{M}(\mathbb{R}^N)$ and $J(u^-) \geq \alpha(\Omega)$. Similarly, $J(u^+) \geq \alpha(\mathbb{R}^N)$. Hence,

$$J(u) = J(u^+) + J(u^-) \geq 2\alpha\left(\mathbb{R}^N\right). \tag{2.6}$$

□

Lemma 2.4. (i) *For each $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$, there exists a positive number s_u such that $s_u u \in \mathbf{M}(\mathbb{R}^N)$ and $\sup_{s \geq 0} J(su) = J(s_u u)$.*

(ii) *Let $\beta > 0$ and let $\{u_n\}$ be a sequence in $W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ for J such that $a(u_n) = b(u_n) + o(1)$ and $J(u_n) = \beta + o(1)$. Then there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $s_n = 1 + o(1)$, $\{s_n u_n\} \subset \mathbf{M}(\mathbb{R}^N)$, and $J(s_n u_n) = \beta + o(1)$ as $n \to \infty$.*

Proof. (i) For each $u \in W_0^{1,p}(\mathbb{R}^N) \setminus \{0\}$ and $s \geq 0$, let

$$h_u(s) = J(su) = \frac{s^p}{p} a(u) - \frac{s^q}{q} b(u). \tag{2.7}$$

Thus, $h_u'(s) = s^{p-1} a(u) - s^{q-1} b(u)$. Define $s_u = (a(u)/b(u))^{1/(q-p)} > 0$, then $h_u'(s_u) = 0$, that is, $s_u u \in \mathbf{M}(\mathbb{R}^N)$.

(ii) By (i), there exists a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $\{s_n u_n\} \subset \mathbf{M}(\mathbb{R}^N)$, that is, $s_n^p a(u_n) = s_n^q b(u_n)$ for each n. Since $a(u_n) = b(u_n) + o(1)$ and $J(u_n) = \beta + o(1)$, we have that $s_n = 1 + o(1)$. Hence, $J(s_n u_n) = \beta + o(1)$ as $n \to \infty$. □

Lemma 2.5. *There exists $c > 0$ such that $\|u\|_{1,p} \geq c > 0$ for each $u \in \mathbf{M}(\mathbb{R}^N)$, where c is independent of u.*

Proof. For each $u \in \mathbf{M}(\mathbb{R}^N)$, by the Sobolev inequality, we obtain that

$$\|u\|_{1,p}^p = \int_{\mathbb{R}^N} Q(z)|u|^q dz \leq c_1 \|u\|_{1,p}^q. \tag{2.8}$$

This implies that $\|u\|_{1,p} \geq c_1^{-1/(q-p)} = c > 0$ for each $u \in \mathbf{M}(\mathbb{R}^N)$. □

By Lemma 2.5, $\alpha(\mathbb{R}^N) > 0$.

Lemma 2.6. *Let $u \in \mathbf{M}(\mathbb{R}^N)$ such that*

$$J(u) = \min_{v \in \mathbf{M}(\mathbb{R}^N)} J(v) = \alpha\left(\mathbb{R}^N\right), \tag{2.9}$$

then u is a nonzero solution of (1.1) *in* $\mathbb{R}^N$.

Proof. Suppose that $\psi(v) = \int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) dz - \int_{\mathbb{R}^N} Q(z)|v|^q dz$, then

$$\langle \psi'(v), v \rangle = (p-q) \int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) dz < 0 \quad \text{for each } v \in \mathbf{M}\left(\mathbb{R}^N\right). \tag{2.10}$$

Since $J(u) = \min_{v \in \mathbf{M}(\mathbb{R}^N)} J(v)$, by the Lagrange multiplier theorem, there is a $\lambda \in \mathbb{R}$ such that $J'(u) = \lambda \psi'(u)$ in $W^{-1,p'}(\mathbb{R}^N)$. Then we have

$$0 = \langle J'(u), u \rangle = \lambda \langle \psi'(u), u \rangle. \tag{2.11}$$

Thus, $\lambda = 0$ and $J'(u) = 0$ in $W^{-1,p'}(\mathbb{R}^N)$. Therefore, u is a nonzero solution of (1.1) in $\mathbb{R}^N$ with $J(u) = \alpha(\mathbb{R}^N)$. □

Lemma 2.7. *There is a* $(PS)_{\alpha(\mathbb{R}^N)}$*-sequence in* $W^{1,p}(\mathbb{R}^N)$ *for* J.

Proof. Let $\{u_n\} \subset \mathbf{M}(\mathbb{R}^N)$ be a minimizing sequence of $\alpha(\mathbb{R}^N)$. Applying the Ekeland principle, there exists a sequence $\{v_n\} \subset \mathbf{M}(\mathbb{R}^N)$ such that $\|v_n - u_n\|_{1,p} < 1/n$, $J(v_n) = \alpha(\mathbb{R}^N) + o(1)$, and $J'|_{\mathbf{M}(\mathbb{R}^N)}(v_n) = o(1)$ strongly in $W^{-1,p'}(\mathbb{R}^N)$ as $n \to \infty$. Let $\psi(u) = a(u) - b(u)$ for each $u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$, then

$$\mathbf{M}\left(\mathbb{R}^N\right) = \left\{u \in W^{1,p}\left(\mathbb{R}^N\right) \setminus \{0\} \mid \psi(u) = 0\right\}. \tag{2.12}$$

Thus, there exists a sequence $\{\theta_n\} \subset \mathbb{R}$ such that $J'(v_n) = \theta_n \psi'(v_n) + o_n(1)$, where $o_n(1) \to 0$ as $n \to \infty$. Since $v_n \in \mathbf{M}(\mathbb{R}^N)$, we have that

$$\begin{gathered} 0 = \left\langle J'(v_n), v_n \right\rangle = \theta_n \left\langle \psi'(v_n), v_n \right\rangle + \left\langle o_n(1), v_n \right\rangle, \\ \left\langle \psi'(v_n), v_n \right\rangle = (p - q) a(v_n) \neq 0 \quad \forall n. \end{gathered} \tag{2.13}$$

Hence, $\theta_n \to 0$ as $n \to \infty$. This implies that $J'(v_n) = o(1)$ strongly in $W^{-1,p'}(\mathbb{R}^N)$ as $n \to \infty$, that is, $\{v_n\} \subset \mathbf{M}(\mathbb{R}^N)$ is a $(\mathrm{PS})_{\alpha(\Omega)}$-sequence in $W^{1,p}(\mathbb{R}^N)$ for J. □

Remark 2.8. The above definitions and lemmas also hold for $J^\infty, \mathbf{M}^\infty(\mathbb{R}^N)$, and $\alpha^\infty(\mathbb{R}^N)$.

3. Existence of a Ground-State Solution

Using the arguments by Lions [1, 2], Benci and Cerami [3], Struwe [7], and Alves [8], we have the following decomposition lemma.

Lemma 3.1 (Palais-Smale Decomposition Lemma for J). *Assume that Q is a positive continuous function in $\mathbb{R}^N$ and $\lim_{|z| \to \infty} Q(z) = Q_\infty > 0$. Let $\{u_n\}$ be a $(PS)_\beta$-sequence in $W^{1,p}(\mathbb{R}^N)$ for J. Then there are a subsequence $\{u_n\}$, a positive integer l, sequences $\{z_n^i\}_{n=1}^\infty$ in $\mathbb{R}^N$, functions u in $W^{1,p}(\mathbb{R}^N)$, and $w^i \neq 0$ in $W^{1,p}(\mathbb{R}^N)$ for $1 \le i \le l$ such that*

$$\begin{gathered} \left|z_n^i\right| \longrightarrow \infty \quad \text{for } 1 \le i \le l, \\ -\Delta_p u + |u|^{p-2} u = Q(z) |u|^{q-2} u \quad \text{in } \mathbb{R}^N, \\ -\Delta_p w^i + \left|w^i\right|^{p-2} w^i = Q_\infty \left|w^i\right|^{q-2} w^i \quad \text{in } \mathbb{R}^N, \\ u_n = u + \sum_{i=1}^{l} w^i\left(\cdot - z_n^i\right) + o_n(1) \quad \text{strongly in } W^{1,p}\left(\mathbb{R}^N\right), \\ J(u_n) = J(u) + \sum_{i=1}^{l} J^\infty\left(w^i\right) + o_n(1). \end{gathered} \tag{3.1}$$

In addition, if $u_n \ge 0$, then $u \ge 0$ and $w^i \ge 0$ for $1 \le i \le l$.

Lemma 3.2. *Let $\{u_n\} \subset \mathbf{M}(\mathbb{R}^N)$ be a $(PS)_\beta$-sequence in $W^{1,p}(\mathbb{R}^N)$ for J with $0 < \beta < \alpha^\infty(\mathbb{R}^N)$. Then there exist a subsequence $\{u_n\}$ and a nonzero $u \in W^{1,p}(\mathbb{R}^N)$ such that $u_n \to u$ strongly in $W^{1,p}(\mathbb{R}^N)$ and $J(u) = \beta$, that is, J satisfies the $(PS)_\beta$-condition in $W^{1,p}(\mathbb{R}^N)$.*

Proof. Since $\{u_n\} \subset \mathbf{M}(\mathbb{R}^N)$ is a $(\mathrm{PS})_\beta$-sequence in $W^{1,p}(\mathbb{R}^N)$ for J with $0 < \beta < \alpha^\infty(\mathbb{R}^N)$, by Lemma 2.2, $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Thus, there exist a subsequence $\{u_n\}$ and $u \in W^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\mathbb{R}^N)$. It is easy to check that u is a solution of (1.1) in $\mathbb{R}^N$. Applying Palais-Smale Decomposition Lemma 3.1, we get

$$\alpha^\infty > \beta = J(u_n) \geq l\alpha^\infty. \tag{3.2}$$

Then $l = 0$ and $u \neq 0$. Hence, $u_n \to u$ strongly in $W^{1,p}(\mathbb{R}^N)$ and $J(u) = \beta$. □

Let $w \in W^{1,p}(\mathbb{R}^N)$ be the positive ground-state solution of (1.2) in $\mathbb{R}^N$. Using the same arguments by Li and Yan [9] and Marcos do Ó [10, Lemma 3.8], or see Serrin and Tang [11, page 899] and Li and Zhao [12, Theorem 1.1], we obtain the following results:

(i) $w \in L^\infty(\mathbb{R}^N) \cap C^{1,\gamma_0}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $0 < \gamma_0 < 1$ and $\lim_{|z|\to\infty} w(z) = 0$;

(ii) for any $\varepsilon > 0$, there exist positive numbers C_1 and C_2 such that

$$C_2 \exp(-(\theta + \varepsilon)|z|) \leq w(z) \leq C_1 \exp(-(\theta - \varepsilon)|z|) \quad \forall z \in \mathbb{R}^N, \tag{3.3}$$

where $\theta = (p-1)^{-1/p}$.

Remark 3.3. Similarly, we also show that all positive solutions of (1.1) in $\mathbb{R}^N$ have exponential decay.

By Lemma 3.2, we can prove the following theorem.

Theorem 3.4. *Assume that Q is a positive continuous function in $\mathbb{R}^N$ and satisfies* (Q1). *Then there exists a positive ground-state solution u_0 of* (1.1) *in $\mathbb{R}^N$.*

Proof. Let $w \in W^{1,p}(\mathbb{R}^N)$ be the positive ground-state solution of (1.2) in $\mathbb{R}^N$, then w is a minimizer of $\alpha^\infty(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (|\nabla w|^p + w^p)\,dz = \int_{\mathbb{R}^N} Q_\infty w^q\,dz. \tag{3.4}$$

By Lemma 2.4(i), there exists a positive number s_w such that $s_w w \in \mathbf{M}(\mathbb{R}^N)$, that is, $\int_{\mathbb{R}^N}(|\nabla(s_w w)|^p + (s_w w)^p)dz = \int_{\mathbb{R}^N} Q(z)(s_w w)^q dz$. Since $Q(z) > Q_\infty$ on a set of positive measure, we can deduce that $s_w < 1$. Therefore,

$$\begin{aligned}
\alpha\left(\mathbb{R}^N\right) \le J(s_w w) &= \left(\frac{1}{p} - \frac{1}{q}\right)(s_w)^p \int_{\mathbb{R}^N} \left(|\nabla w|^p + w^p\right) dz \\
&< \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{R}^N} \left(|\nabla w|^p + w^p\right) dz \\
&= \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{R}^N} Q_\infty w^q dz = \alpha^\infty\left(\mathbb{R}^N\right).
\end{aligned} \tag{3.5}$$

Applying Lemma 3.2, there exists $u_0 \in W^{1,p}(\mathbb{R}^N)$ such that $J(u_0) = \alpha(\mathbb{R}^N)$. From the results of Lemmas 2.6 and 2.3, by Maximum Principle, u_0 is a positive ground-state solution of (1.1) in $\mathbb{R}^N$. □

4. Existence of a Nodal Solution

In this section, assume that Q is a positive continuous function in $\mathbb{R}^N$ and satisfies (Q1). In order to prove Lemma 4.8, Q also satisfies the following condition (Q2): there exist some constants $C > 0$ and $0 < \delta < \theta = (p-1)^{-1/p}$ such that

$$Q(z) \ge Q_\infty + C\exp(-\delta|z|) \quad \forall z \in \mathbb{R}^N. \tag{Q2}$$

Let h be a functional in $W^{1,p}(\mathbb{R}^N)$ defined by

$$h(u) = \begin{cases} \dfrac{b(u)}{a(u)} & \text{for } u \neq 0, \\ 0 & \text{for } u = 0. \end{cases} \tag{4.1}$$

We define

$$\begin{aligned}
\mathbf{M}_0 &= \left\{u \in W^{1,p}\left(\mathbb{R}^N\right) \mid h(u^+) = 1,\ h\left(u^-\right) = 1\right\} \subset \mathbf{M}\left(\mathbb{R}^N\right), \\
\mathcal{N} &= \left\{u \in W^{1,p}\left(\mathbb{R}^N\right) \mid \left|h(u^\pm) - 1\right| < \frac{1}{2}\right\} \supset \mathbf{M}_0,
\end{aligned} \tag{4.2}$$

where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

Lemma 4.1. (i) *If $u \in W^{1,p}(\mathbb{R}^N)$ changes sign, then there exist positive numbers $s^\pm(u) = s^\pm$ such that $s^+u^+ \in \mathbf{M}(\mathbb{R}^N)$ and $s^-u^- \in \mathbf{M}(\mathbb{R}^N)$.*

(ii) *There exists $c' > 0$ such that $\|u^\pm\|_{1,p} \ge c' > 0$ for each $u \in \mathcal{N}$.*

Proof. (i) Since u^+ and u^- are nonzero and nonnegative, by Lemma 2.4(i), it is easy to obtain the result.

(ii) For each $u \in \mathcal{N}$, by Lemma 2.4(i), there exists $s^{\pm}(u) = s^{\pm} > 0$ such that $s^{\pm}u^{\pm} \in \mathbf{M}(\mathbb{R}^N)$. Then

$$\frac{1}{2} < (s^{\pm})^{p-q} = \frac{b(u^{\pm})}{a(u^{\pm})} < \frac{3}{2} \quad \text{for each } u \in \mathcal{N}. \tag{4.3}$$

By Lemma 2.5, we have

$$\left\| s^{\pm}u^{\pm} \right\|_{1,p} \geq c \quad \text{for some } c > 0 \text{ and each } u \in \mathcal{N}. \tag{4.4}$$

Hence, $\|u^{\pm}\|_{1,p} \geq c/s^{\pm} \geq c' > 0$ for each $u \in \mathcal{N}$. □

Consider these minimization problem

$$\gamma\left(\mathbb{R}^N\right) = \inf_{u \in \mathbf{M}_0} J(u). \tag{4.5}$$

By Lemma 4.1, $\gamma(\mathbb{R}^N) > 0$.

Lemma 4.2. *There exists a sequence $\{u_n\} \subset \mathcal{N}$ such that $J(u_n) = \gamma(\mathbb{R}^N) + o_n(1)$ and $J'(u_n) = o_n(1)$ strongly in $W^{-1,p}(\mathbb{R}^N)$ as $n \to \infty$.*

Proof. It is similar to the proof of Zhu [13]. □

Lemma 4.3. *Let f and g be real-valued functions in $\mathbb{R}^N$. If $g(z) > 0$ in $\mathbb{R}^N$, then one has the following inequalities:*

(i) $(f + g)^+ \geq f^+$,

(ii) $(f + g)^- \leq f^-$,

(iii) $(f - g)^+ \leq f^+$,

(iv) $(f - g)^- \geq f^-$.

Lemma 4.4. *Let $\{u_n\} \subset \mathcal{N}$ be a $(PS)_{\gamma(\mathbb{R}^N)}$-sequence in $W^{1p}(\mathbb{R}^N)$ for J satisfying*

$$\alpha\left(\mathbb{R}^N\right) < \gamma\left(\mathbb{R}^N\right) < \alpha\left(\mathbb{R}^N\right) + \alpha^{\infty}\left(\mathbb{R}^N\right)\left(< 2\alpha^{\infty}\left(\mathbb{R}^N\right)\right). \tag{4.6}$$

Then there exists $u^* \in \mathbf{M}_0$ such that u_n converges to u^* strongly in $W^{1,p}(\mathbb{R}^N)$ and u^* is a higher-energy solution of (1.1) *such that $J(u^*) = \gamma(\mathbb{R}^N)$.*

Proof. By the definition of the $(\mathrm{PS})_{\gamma(\mathbb{R}^N)}$-sequence in $W^{1,p}(\mathbb{R}^N)$ for J, it is easy to see that $\{u_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ and satisfies

$$\int_{\mathbb{R}^N} \left(\left|\nabla u_n^{\pm}\right|^p + \left|u_n^{\pm}\right|^p\right) dz = \int_{\mathbb{R}^N} Q(z) \left|u_n^{\pm}\right|^q dz + o_n(1). \tag{4.7}$$

By Lemma 4.1(ii), there exists $c' > 0$ such that

$$c' \leq \int_{\mathbb{R}^N} \left(|\nabla u_n^{\pm}|^p + |u_n^{\pm}|^p\right) dz = \int_{\mathbb{R}^N} Q(z)|u_n^{\pm}|^q dz + o_n(1). \tag{4.8}$$

Using the Palais-Smale Decomposition Lemma 3.1, then we have $\gamma(\mathbb{R}^N) = J(u^*) + \sum_{i=1}^{l} J^{\infty}(w_i)$, where u^* is a solution of (1.1) in $\mathbb{R}^N$ and w_i is a solution of (1.2) in $\mathbb{R}^N$. Since $J^{\infty}(w_i) \geq \alpha^{\infty}(\mathbb{R}^N)$ for each $i \in \mathbb{N}$ and $\alpha(\mathbb{R}^N) < \alpha^{\infty}(\mathbb{R}^N)$, we have $l \leq 1$. Now we want to show that $l = 0$. On the contrary, suppose that $l = 1$.

(i) w_1 is a sign-changing solution of (1.2): by Lemma 2.3 and Remark 2.8, we have $\gamma(\mathbb{R}^N) \geq 2\alpha^{\infty}(\mathbb{R}^N)$, which is a contradiction.

(ii) w_1 is a constant-sign solution of (1.2): we may assume that $w_1 > 0$. Applying the Decomposition Lemma 3.1, there exists a sequence $\{z_n^1\}$ in $\mathbb{R}^N$ such that $|z_n^1| \to \infty$, and

$$\left\| u_n - u^* - w_1\left(\cdot - z_n^1\right) \right\|_{1,p} = o_n(1). \tag{4.9}$$

By the Sobolev continuous embedding inequality, we obtain

$$\left\| u_n - u^* - w_1\left(\cdot - z_n^1\right) \right\|_{L^q} = o_n(1). \tag{4.10}$$

Since $w_1 > 0$, by Lemma 4.3, then

$$\left\| (u_n - u^*)^- \right\|_{L^q} = o_n(1) \quad \text{as } n \longrightarrow \infty. \tag{4.11}$$

(a) Suppose that $u^* \equiv 0$; we obtain $\|u_n^-\|_{L^q} = o_n(1)$ as $n \to \infty$. Then

$$0 < c' \leq \int_{\mathbb{R}^N} Q(z)|u_n^-|^q dz = o_n(1), \tag{4.12}$$

which is a contradiction.

(b) Suppose that $u^* \not\equiv 0$. We have $\gamma(\mathbb{R}^N) = J(u^*) + J^{\infty}(w_1) \geq \alpha(\mathbb{R}^N) + \alpha^{\infty}(\mathbb{R}^N)$, which is a contradiction.

By (i) and (ii), then $l = 0$. Thus, $\|u_n - u^*\|_{1,p} = o_n(1)$ as $n \to \infty$ and $J(u^*) = \gamma(\mathbb{R}^N)$. Finally, we claim that u^* is a sign-changing solution of (1.1) in $\mathbb{R}^N$. If $u^* > 0$ (or < 0), by Lemma 4.3, then $\|u_n^-\|_{L^q} = o_n(1)$ (or $\|u_n^-\|_{L^q} = o_n(1)$). Similarly, we have the inequality (4.12), which is a contradiction. Moreover, by Lemma 2.3, $2\alpha(\mathbb{R}^N) \leq \gamma(\mathbb{R}^N)$. □

Recall that w is the positive ground-state solution of (1.2) in $\mathbb{R}^N$. For any $\varepsilon > 0$, there exist positive numbers C_1 and C_2 such that

$$C_2 \exp(-(\theta + \varepsilon)|z|) \leq w(z) \leq C_1 \exp(-(\theta - \varepsilon)|z|) \quad \forall z \in \mathbb{R}^N, \tag{4.13}$$

where $\theta = (p-1)^{-1/p}$. Define

$$w_n(z) = w(z - z_n) \quad \text{where } z_n = (0, \dots, 0, n) \in \mathbb{R}^N. \tag{4.14}$$

Clearly, $w_n(z) \in W^{1,p}(\mathbb{R}^N)$.

Lemma 4.5. *There are $n_0 \in \mathbb{N}$ and real numbers t_1^* and t_2^* such that for $n \geq n_0$*

$$t_1^* u_0 - t_2^* w_n \in \mathbf{M}_0, \qquad \gamma\left(\mathbb{R}^N\right) \leq J\left(t_1^* u_0 - t_2^* w_n\right), \tag{4.15}$$

where $1/p \leq t_1^*, t_2^* \leq p$, and u_0 is the positive ground-state solution of (1.1) *in $\mathbb{R}^N$.*

Proof. Applying the mean value theorem by Miranda [14], the proof is similar to that of Zhu [13] (or see Hsu [15, page 728]). □

We need the following lemmas to prove that $\sup_{1/p \leq t_1^*, t_2^* \leq p} J(t_1^* u_0 - t_2^* w_n) < \alpha(\mathbb{R}^N) + \alpha^\infty(\mathbb{R}^N)$ for sufficiently large n.

Lemma 4.6. *Let E be a domain in $\mathbb{R}^N$. If $f : E \to \mathbb{R}$ satisfies*

$$\int_E \left| f(z) e^{\sigma|z|} \right| dz < \infty \quad \text{for some } \sigma > 0, \tag{4.16}$$

then

$$\left(\int_E f(z) e^{-\sigma|z - \overline{z}|} dz \right) e^{\sigma|\overline{z}|} = \int_E f(z) e^{\sigma\langle z, \overline{z}\rangle / |\overline{z}|} dz + o(1) \quad \text{as } |\overline{z}| \longrightarrow \infty. \tag{4.17}$$

Proof. Since $\sigma|\overline{z}| \leq \sigma|z| + \sigma|z - \overline{z}|$, we have

$$\left| f(z) e^{-\sigma|z - \overline{z}|} e^{\sigma|\overline{z}|} \right| \leq \left| f(z) e^{\sigma|z|} \right|. \tag{4.18}$$

Since $-\sigma|z - \overline{z}| + \sigma|\overline{z}| = \sigma(\langle z, \overline{z}\rangle / |\overline{z}|) + o(1)$ as $|\overline{z}| \to \infty$, then the lemma follows from the Lebesgue-dominated convergence theorem. □

Lemma 4.7. *For all $x, y \in \mathbb{R}^N$, one has the following inequality:*

$$|x - y|^\rho \leq \left(|x|^{\rho-2} x - |y|^{\rho-2} y \right)(x - y), \quad \text{where } \rho \geq 2. \tag{4.19}$$

Proof. See Yang [16, Lemma 4.2.]. □

Lemma 4.8. *There exists an $n_0^* \in \mathbb{N}$ such that for $n \geq n_0^* \geq n_0$*

$$\gamma\left(\mathbb{R}^N\right) \leq \sup_{1/p \leq t_1^*, t_2^* \leq p} J\left(t_1^* u_0 - t_2^* w_n\right) < \alpha\left(\mathbb{R}^N\right) + \alpha^\infty\left(\mathbb{R}^N\right), \tag{4.20}$$

where u_0 is a positive ground-state solution of (1.1) *in $\mathbb{R}^N$.*

Proof. By Lemma 4.7, then

$$
\begin{aligned}
&J\left(t_1^* u_0 - t_2^* w_n\right)\\
&\quad= \frac{1}{p}\left\|t_1^* u_0 - t_2^* w_n\right\|_{1,p}^p - \frac{1}{q} b\left(t_1^* u_0 - t_2^* w_n\right)\\
&\quad\le \frac{1}{p}\left\{\int_{\mathbb{R}^N}\left(\left|\nabla\left(t_1^* u_0\right)\right|^{p-2}\nabla\left(t_1^* u_0\right) - \left|\nabla\left(t_2^* w_n\right)\right|^{p-2}\nabla\left(t_2^* w_n\right)\right)\left(\nabla\left(t_1^* u_0\right) - \nabla\left(t_2^* w_n\right)\right)\right\}\\
&\qquad+ \frac{1}{p}\left\{\int_{\mathbb{R}^N}\left(\left|t_1^* u_0\right|^{p-2}\left(t_1^* u_0\right) - \left|t_2^* w_n\right|^{p-2}\left(t_2^* w_n\right)\right)\left(t_1^* u_0 - t_2^* w_n\right)\right\} - \frac{1}{q} b\left(t_1^* u_0 - t_2^* w_n\right)\\
&\quad\le J\left(t_1^* u_0\right) + J^{\infty}\left(t_2^* w\right) - \frac{\left(t_2^*\right)^q}{q}\int_{\mathbb{R}^N}(Q(z) - Q_{\infty}) w(z - z_n)^q dz\\
&\qquad- \frac{1}{q} b\left(t_1^* u_0 - t_2^* w_n\right) + \frac{1}{q} b\left(t_1^* u_0\right) + \frac{1}{q} b\left(t_2^* w_n\right).
\end{aligned}
\tag{4.21}
$$

Since $\sup_{t\ge 0} J(tu_0) = \alpha(\mathbb{R}^N)$ and $\sup_{t\ge 0} J^{\infty}(tw) = \alpha^{\infty}(\mathbb{R}^N)$, using the inequality

$$
|c_1 - c_2|^q > c_1^q + c_2^q - K\left(c_1^{q-1} c_2 + c_1 c_2^{q-1}\right),
\tag{4.22}
$$

for any $c_1, c_2 > 0$, and some positive constant K, then

$$
\begin{aligned}
\sup_{1/p \le t_1^*, t_2^* \le p} J\left(t_1^* u_0 - t_2^* w_n\right) &\le \alpha\left(\mathbb{R}^N\right) + \alpha^{\infty}\left(\mathbb{R}^N\right) - \frac{1}{p^q q}\int_{\mathbb{R}^N}(Q(z) - Q_{\infty}) w(z - z_n)^q dz\\
&\quad+ K'\left[\int_{\mathbb{R}^N}\left(u_0^{q-1} w_n + w_n^{q-1} u_0\right) dz\right].
\end{aligned}
\tag{4.23}
$$

(i) Since $Q(z) \ge Q_{\infty} + C\exp(-\delta|z|)$ for some constants $C > 0$ and $0 < \delta < \theta$, by Lemma 4.6, we have that there exists an $n_1 \ge n_0$ such that for $n \ge n_1$

$$
\int_{\mathbb{R}^N}(Q(z) - Q_{\infty}) w(z - z_n)^q dz \ge C'\exp\left(-\min\{\delta, q(\theta + \varepsilon)\}|\overline{z}|\right) \ge C'\exp(-\delta n).
\tag{4.24}
$$

(ii) Applying Lemma 4.6, there exists an $n_2 \ge n_1$ such that for $n \ge n_2$

$$
\int_{\mathbb{R}^N} u_0^{q-1} w_n dz \le C_1' \int_{\mathbb{R}^N}\exp(-(q-1)(\theta - \varepsilon)|z|)\exp(-(\theta - \varepsilon)|z - z_n|) dz \le C_1''\exp(-(\theta - \varepsilon) n).
\tag{4.25}
$$

Similarly, we also obtain that there exists an $n_3 \geq n_2$ such that for $n \geq n_3$

$$\int_{\mathbb{R}^N} w_n^{q-1} u_0 \mathrm{d}z \leq C_1''' \exp(-(\theta - \varepsilon)n). \tag{4.26}$$

By (i) and (ii), choosing $0 < \varepsilon < \theta - \delta$, we can find an $n_0^* \geq n_3 \geq n_0$ such that for $n \geq n_0^*$

$$\sup_{1/p \leq t_1, t_2 \leq p} J\left(t_1^* u_0 - t_2^* w_n\right) < \alpha\left(\mathbb{R}^N\right) + \alpha^\infty\left(\mathbb{R}^N\right). \tag{4.27}$$

□

Theorem 4.9. *Assume that Q is a positive continuous function in $\mathbb{R}^N$ and satisfies* (Q1) *and* (Q2), *then* (1.1) *has a positive solution and a nodal solution in $\mathbb{R}^N$.*

Proof. By Lemmas 4.2, 4.4, 4.5, and 4.8 and Theorem 3.4, we obtain the proof. □

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