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Research Article

On Perfectly Homogeneous Bases in Quasi-Banach Spaces

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For $0 < p < \infty$ the unit vector basis of ℓ_p has the property of perfect homogeneity: it is equivalent to all its normalized block basic sequences, that is, perfectly homogeneous bases are a special case of symmetric bases. For Banach spaces, a classical result of Zippin (1966) proved that perfectly homogeneous bases are equivalent to either the canonical c_0-basis or the canonical ℓ_p-basis for some $1 \leq p < \infty$. In this note, we show that (a relaxed form of) perfect homogeneity characterizes the unit vector bases of ℓ_p for $0 < p < 1$ as well amongst bases in nonlocally convex quasi-Banach spaces.

1. Introduction and Background

Let us first review the relevant elementary concepts and definitions. Further details can be found in the books [1, 2] and the paper [3]. A (real) quasi-normed space X is a locally bounded topological vector space. This is equivalent to saying that the topology on X is induced by a *quasi-norm*, that is, a map $\|\cdot\| : X \to [0,\infty)$ satisfying

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$ if $\alpha \in \mathbb{R}$, $x \in X$;

(iii) there is a constant $\kappa \geq 1$ so that for any x_1 and $x_2 \in X$ we have

$$\|x_1 + x_2\| \leq \kappa(\|x_1\| + \|x_2\|). \tag{1.1}$$

The best constant κ in inequality (1.1) is called the *modulus of concavity* of the quasi-norm. If $\kappa = 1$, the quasi-norm is a norm. A quasi-norm on X is *p-subadditive* if

$$\|x_1 + x_2\|^p \leq \|x_1\|^p + \|x_2\|^p, \quad x_1, x_2 \in X. \tag{1.2}$$

A theorem by Aoki [4] and Rolewicz [5] asserts that every quasi-norm has an equivalent p-subadditive quasi-norm, where $0 < p \leq 1$ is given by $\kappa = 2^{1/p-1}$. A p-subadditive quasi-norm $\|\cdot\|$ induces an invariant metric on X by the formula $d(x, y) = \|x - y\|^p$. The space X is called *quasi-Banach space* if X is complete for this metric. A quasi-Banach space is isomorphic to a Banach space if and only if it is locally convex.

A basis $(x_n)_{n=1}^{\infty}$ of a quasi-Banach space X is *symmetric* if $(x_n)_{n=1}^{\infty}$ is equivalent to $(x_{\pi(n)})_{n=1}^{\infty}$ for any permutation π of $\mathbb{N}$. Symmetric bases are unconditional and so there exists a nonnegative constant K such that for all $x = \sum_{n=1}^{\infty} a_n x_n$ the inequality

$$\left\| \sum_{n=1}^{\infty} \theta_n a_n x_n \right\| \leq K \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \tag{1.3}$$

holds for any bounded sequence $(\theta_n)_{n=1}^{\infty} \in B_{\ell_\infty}$. The least such constant K is called the *unconditional constant* of $(x_n)_{n=1}^{\infty}$.

For instance, the canonical basis of the spaces ℓ_p for $0 < p < \infty$ is symmetric and 1-unconditional. What is more, it is the *only* symmetric basis of ℓ_p up to equivalence, that is, whenever $(x_n)_{n=1}^{\infty}$ is another normalized symmetric basis of ℓ_p, there is a constant C such that

$$\frac{1}{C}\left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq C \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}, \tag{1.4}$$

for any finitely nonzero sequence of scalars $(a_n)_{n=1}^{\infty}$ [6, 7].

The spaces ℓ_p for $0 < p < 1$ share the property of uniqueness of symmetric basis with all natural quasi-Banach spaces whose Banach envelope (i.e., the smallest containing Banach space) is isomorphic to ℓ_1, as was recently proved in [8]. For other results on uniqueness of unconditional or symmetric basis in nonlocally convex quasi-Banach spaces the reader can consult the papers [9, 10].

This article illustrates how Zippin's techniques can also be used to characterize the unit vector bases of ℓ_p for $0 < p < 1$ as the only, up to equivalence, perfectly homogeneous bases in nonlocally convex quasi-Banach spaces. We use standard Banach space theory terminology and notation throughout, as may be found in [11, 12].

2. Perfectly Homogeneous Bases in Quasi-Banach Spaces

Let $(x_i)_{i=1}^{\infty}$ be a basis for a quasi-Banach space X. A block basic sequence $(u_n)_{n=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$,

$$u_n = \sum_{p_{n-1}+1}^{p_n} a_i x_i, \tag{2.1}$$

is said to be a *constant coefficient block basic sequence* if for each n there is a constant c_n so that $a_i = c_n$ or $a_i = 0$ for $p_{n-1} + 1 \leq i \leq p_n$.

Definition 2.1. A basis $(x_i)_{i=1}^{\infty}$ of a quasi-Banach space X is *almost perfectly homogeneous* if every normalized constant coefficient block basic sequence of $(x_i)_{i=1}^{\infty}$ is equivalent to $(x_i)_{i=1}^{\infty}$.

Let us notice that using a uniform boundedness argument we obtain that, in fact, if $(x_i)_{i=1}^{\infty}$ is almost perfectly homogeneous then it is *uniformly* equivalent to all its normalized constant coefficient block basic sequences. That is, there is a constant $M \geq 1$ such that for any normalized constant coefficient block basic sequence $(u_n)_{n=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$ we have

$$M^{-1}\left\|\sum_{k=1}^{n} a_k x_k\right\| \leq \left\|\sum_{k=1}^{n} a_k u_k\right\| \leq M\left\|\sum_{k=1}^{n} a_k x_k\right\|, \tag{2.2}$$

for all choices of scalars $(a_k)_{k=1}^{n}$ and $n \in \mathbb{N}$. Equation (2.2) also yields that for any increasing sequence of integers $(k_j)_{j=1}^{\infty}$,

$$M^{-1}\left\|\sum_{j=1}^{n} x_j\right\| \leq \left\|\sum_{j=1}^{n} x_{k_j}\right\| \leq M\left\|\sum_{j=1}^{n} x_j\right\|. \tag{2.3}$$

This is our main result (cf. [13]).

Theorem 2.2. *Let X be a nonlocally convex quasi-Banach space with normalized basis $(x_i)_{i=1}^{\infty}$. Suppose that $(x_i)_{i=1}^{\infty}$ is almost perfectly homogeneous. Then $(x_i)_{i=1}^{\infty}$ is equivalent to the canonical basis of ℓ_q for some $0 < q < 1$.*

Proof. Let κ be the modulus of concavity of the quasi-norm. Since X is nonlocally convex, $\kappa > 1$. By the Aoki-Rolewicz theorem we can assume that the quasi-norm is p-subadditive for $0 < p < 1$ such that $\kappa = 2^{1/p-1}$. We will show that $(x_i)_{i=1}^{\infty}$ is equivalent to the canonical ℓ_q-basis for some $p \leq q < 1$.

By renorming, without loss of generality we can assume $(x_i)_{i=1}^{\infty}$ to be 1-unconditional. For each n put,

$$\lambda(n) = \left\|\sum_{i=1}^{n} x_i\right\|. \tag{2.4}$$

Note that

$$1 \leq \lambda(n) \leq n^{1/p}, \quad n \in \mathbb{N}, \tag{2.5}$$

and that, by the 1-unconditionality of the basis, the sequence $(\lambda(n))_{n=1}^{\infty}$ is nondecreasing.

We are going to construct disjoint blocks of length n of the basis $(x_i)_{i=1}^{\infty}$ as follows:

$$v_1 = \sum_{i=1}^{n} x_i, \qquad v_2 = \sum_{i=n+1}^{2n} x_i, \ldots, \qquad v_j = \sum_{i=(j-1)n+1}^{jn} x_i, \ldots. \tag{2.6}$$

Equation (2.3) says that

$$M^{-1}\lambda(n) \leq \|v_j\| \leq M\lambda(n), \quad j \in \mathbb{N}, \tag{2.7}$$

and so by the 1-unconditionality of $(x_i)_{i=1}^{\infty}$,

$$\frac{1}{M\lambda(n)}\left\|\sum_{j=1}^{m} v_j\right\| \leq \left\|\sum_{j=1}^{m}\|v_j\|^{-1}v_j\right\| \leq \frac{M}{\lambda(n)}\left\|\sum_{j=1}^{m} v_j\right\|, \quad m \in \mathbb{N}. \tag{2.8}$$

On the other hand, by (2.2) we know that

$$\frac{\lambda(m)}{M} \leq \left\|\sum_{j=1}^{m}\|v_j\|^{-1}v_j\right\| \leq M\lambda(m), \quad m \in \mathbb{N}. \tag{2.9}$$

If we put these last two inequalities together we obtain

$$\frac{1}{M^2}\lambda(m)\lambda(n) \leq \lambda(mn) \leq M^2\lambda(m)\lambda(n), \quad m, n \in \mathbb{N}. \tag{2.10}$$

Substituting in (2.10) integers of the form $m = 2^k$ and $n = 2^j$ give

$$\frac{1}{M^2}\lambda\left(2^k\right)\lambda\left(2^j\right) \leq \lambda\left(2^{j+k}\right) \leq M^2\lambda\left(2^k\right)\lambda\left(2^j\right), \quad k, j \in \mathbb{N}. \tag{2.11}$$

For $k = 0, 1, 2, \ldots$, let $h(k) = \log_2\lambda(2^k)$. From (2.11) it follows that

$$\left|h(j) - h(k) - h(j+k)\right| \leq 2\log_2 M. \tag{2.12}$$

We need the following well-known lemma from real analysis.

Lemma 2.3. *Suppose that $(s_n)_{n=1}^{\infty}$ is a sequence of real numbers such that*

$$|s_{m+n} - s_m - s_n| \leq 1 \tag{2.13}$$

for all $m, n \in \mathbb{N}$. Then there is a constant c so that

$$|s_n - cn| \leq 1, \quad n = 1, 2, \ldots. \tag{2.14}$$

Lemma 2.3 yields a constant c so that

$$|h(k) - ck| \leq 2\log_2 M, \quad k = 1, 2, \ldots. \tag{2.15}$$

In turn, using (2.5) we have

$$1 \leq \lambda\left(2^k\right) \leq 2^{k/p}, \quad k = 1, 2, \ldots \tag{2.16}$$

which implies

$$0 \leq h(k) \leq \frac{k}{p}, \tag{2.17}$$

and so, combining with (2.15) we obtain that the range of possible values for c is

$$0 \leq c \leq \frac{1}{p}. \tag{2.18}$$

If $c = 0$ then $(\lambda(n))_{n=1}^{\infty}$ would be (uniformly) bounded and so $(x_i)_{i=1}^{\infty}$ would be equivalent to the canonical basis of c_0, a contradiction with the local nonconvexity of X. Otherwise, if $0 < c \leq 1/p$ there is $q \in [p, \infty)$ such that $c = 1/q$. This way we can rewrite (2.15) in the form

$$\left| h(k) - \frac{k}{q} \right| \leq 2 \log_2 M, \quad k \in \mathbb{N}, \tag{2.19}$$

or equivalently,

$$M^{-2} 2^{k/q} \leq \lambda\left(2^k\right) \leq 2^{k/q} M^2, \quad k \in \mathbb{N}. \tag{2.20}$$

Now, given $n \in \mathbb{N}$ we pick the only integer k so that $2^{k-1} \leq n \leq 2^k$. Then,

$$\lambda\left(2^{k-1}\right) \leq \lambda(n) \leq \lambda\left(2^k\right), \tag{2.21}$$

and so

$$M^{-2} 2^{-1/q} n^{1/q} \leq \lambda(n) \leq M^2 2^{1/q} n^{1/q}. \tag{2.22}$$

If A is any finite subset of $\mathbb{N}$, by (2.3) we have

$$M^{-1} \lambda(|A|) \leq \left\| \sum_{j \in A} x_j \right\| \leq M \lambda(|A|), \tag{2.23}$$

hence

$$C^{-1} |A|^{1/q} \leq \left\| \sum_{j \in A} x_j \right\| \leq C |A|^{1/q}, \tag{2.24}$$

where $C = M^3 2^{1/q}$.

To prove the equivalence of $(x_i)_{i=1}^{\infty}$ with the canonical basis of ℓ_q, given any $n \in \mathbb{N}$ we let $(a_i)_{i=1}^{n}$ be nonnegative scalars such that $a_i^q \in \mathbb{Q}$ and $\sum_{i=1}^{n} a_i^q = 1$. Each a_i^q can be written in the form $a_i^q = m_i/m$ where $m_i \in \mathbb{N}$, m is de common denominator of the a_i^q's, and $\sum_{i=1}^{n} m_i = m$.

Let A_1 be interval of natural numbers $[1, m_1]$ and for $j = 2, \dots, n$ let A_i be the interval of natural numbers $[m_1 + \cdots + m_{i-1} + 1, m_1 + \cdots + m_i]$. The sets $A_1, \dots, A_n$ are disjoint and have cardinality $|A_i| = m_i$ for each $i = 1, \dots, n$. Consider the normalized constant coefficient block basic sequence defined as

$$u_i = c_i^{-1} \sum_{j \in A_i} x_j, \quad 1 \le i \le n, \tag{2.25}$$

where $c_i = \| \sum_{j \in A_i} x_k \|$. Equation (2.24) yields

$$C^{-1} m_i^{1/q} \le c_i \le C m_i^{1/q}, \quad 1 \le i \le n. \tag{2.26}$$

Therefore,

$$\frac{C^{-1}}{m^{1/q}} \left\| \sum_{i=1}^{n} \sum_{j \in A_i} x_j \right\| \le \left\| \sum_{i=1}^{n} a_i u_i \right\| \le \frac{C}{m^{1/q}} \left\| \sum_{i=1}^{n} \sum_{j \in A_i} x_k \right\|, \tag{2.27}$$

that is,

$$C^{-1} \frac{\lambda(m)}{m^{1/q}} \le \left\| \sum_{i=1}^{n} a_i u_i \right\| \le C \frac{\lambda(m)}{m^{1/q}}. \tag{2.28}$$

Thus,

$$\frac{1}{C^2 M} \le \left\| \sum_{i=1}^{n} a_i u_i \right\| \le C^2 M. \tag{2.29}$$

Using (2.2) again, we have

$$\frac{1}{C^2 M^2} \le \left\| \sum_{i=1}^{n} a_i x_i \right\| \le C^2 M^2. \tag{2.30}$$

We note that a simple density argument shows that (2.30) holds whenever $\sum_{i=1}^{n} |a_i|^q = 1$ (i.e., without the assumption that $|a_i|^q$ is rational), and this completes the proof that $(x_i)_{i=1}^{\infty}$ is equivalent to the canonical ℓ_q-basis for some $p \le q < \infty$. Since X is not locally convex, we conclude that $p \le q < 1$. □

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