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Research Article

Some Identities of Symmetry for the Generalized Bernoulli Numbers and Polynomials

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By the properties of p-adic invariant integral on $\mathbb{Z}_p$, we establish various identities concerning the generalized Bernoulli numbers and polynomials. From the symmetric properties of p-adic invariant integral on $\mathbb{Z}_p$, we give some interesting relationship between the power sums and the generalized Bernoulli polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x)\,dx = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{1.1}$$

(see [1]). From the definition (1.1), we have

$$I(f_1) = I(f) + f'(0), \quad \text{where } f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0},\ f_1(x) = f(x+1). \tag{1.2}$$

Let $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$. Then we can derive the following equation from (1.2):

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i), \tag{1.3}$$

(see [1]). It is well known that the ordinary Bernoulli polynomials $B_n(x)$ are defined as

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.4}$$

(see [1–25]), and the Bernoulli number B_n are defined as $B_n = B_n(0)$.

Let d be a fixed positive integer. For $n \in \mathbb{N}$, we set

$$\begin{gathered} X = X_d = \lim_{\overleftarrow{N}} \left(\mathbb{Z}/dp^N\mathbb{Z}\right), \qquad X_1 = \mathbb{Z}_p; \\ X^* = \bigcup_{\substack{0<a<dp,\\ (a,p)=1}} (a + dp\mathbb{Z}_p); \\ a + dp^N\mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \left(\operatorname{mod} dp^N\right) \right\}, \end{gathered} \tag{1.5}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. It is easy to see that

$$\int_X f(x)dx = \int_{\mathbb{Z}_p} f(x)dx, \quad \text{for } f \in \mathrm{UD}(\mathbb{Z}_p). \tag{1.6}$$

In [14], the Witt's formula for the Bernoulli numbers are given by

$$\int_{\mathbb{Z}_p} x^n dx = B_n, \quad n \in \mathbb{Z}_+. \tag{1.7}$$

Let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$. Then the generalized Bernoulli polynomials attached to χ are defined as

$$\sum_{a=1}^{d} \frac{\chi(a) t e^{at}}{e^{dt} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}, \tag{1.8}$$

(see [22]), and the generalized Bernoulli numbers attached to χ, $B_{n,\chi}$ are defined as $B_{n,\chi} = B_{n,\chi}(0)$.

In this paper, we investigate the interesting identities of symmetry for the generalized Bernoulli numbers and polynomials attached to χ by using the properties of p-adic invariant integral on $\mathbb{Z}_p$. Finally, we will give relationship between the power sum polynomials and the generalized Bernoulli numbers attached to χ.

2. Symmetry of Power Sum and the Generalized Bernoulli Polynomials

Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$. From (1.3), we note that

$$\int_X \chi(x) e^{xt} dx = \frac{t \sum_{i=0}^{d-1} \chi(i) e^{it}}{e^{dt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}, \tag{2.1}$$

where $B_{n,\chi}(x)$ are the nth generalized Bernoulli numbers attached to χ. Now, we also see that the generalized Bernoulli polynomials attached to χ are given by

$$\int_X \chi(y) e^{(x+y)t} dy = \frac{t \sum_{i=0}^{d-1} \chi(i) e^{it}}{e^{dt} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}. \tag{2.2}$$

By (2.1) and (2.2), we easily see that

$$\int_X \chi(x) x^n dx = B_{n,\chi}, \qquad \int_X \chi(y)(x+y)^n dy = B_{n,\chi}(x). \tag{2.3}$$

From (2.2), we have

$$B_{n,\chi}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} B_{\ell,\chi} x^{n-\ell}. \tag{2.4}$$

From (2.2), we can also derive

$$\int_X \chi(x) e^{xt} dx = \sum_{i=0}^{d-1} \chi(i) \frac{t}{e^{dt} - 1} e^{(i/d)dt} = \sum_{n=0}^{\infty} \left(d^{n-1} \sum_{i=0}^{d-1} \chi(i) B_n\left(\frac{i}{d}\right) \right) \frac{t^n}{n!}. \tag{2.5}$$

Therefore, we obtain the following lemma.

Lemma 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$\int_X \chi(x) x^n dx = B_{n,\chi} = d^{n-1} \sum_{i=0}^{d-1} \chi(i) B_i\left(\frac{i}{d}\right). \tag{2.6}$$

We observe that

$$\frac{1}{t}\left(\int_X \chi(x) e^{(nd+x)t} dx - \int_X e^{xt} \chi(x) dx \right) = \frac{nd \int_X \chi(x) e^{xt} dx}{\int_X e^{ndxt} dx} = \frac{e^{ndt} - 1}{e^{dt} - 1} \left(\sum_{i=0}^{d-1} \chi(i) e^{it} \right). \tag{2.7}$$

Thus, we have

$$\frac{1}{t}\left(\int_X \chi(x) e^{(nd+x)t} dx - \int_X \chi(x) e^{xt} dx \right) = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{nd-1} \chi(\ell) \ell^k \right) \frac{t^k}{k!}. \tag{2.8}$$

Let us define the p-adic functional $T_k(\chi, n)$ as follows:

$$T_k(\chi, n) = \sum_{\ell=0}^{n} \chi(\ell)\ell^k, \quad \text{for } k \in \mathbb{Z}_+. \tag{2.9}$$

By (2.8) and (2.9), we see that

$$\frac{1}{t}\left(\int_X \chi(x) e^{(nd+x)t} dx - \int_X \chi(x) e^{xt} dx\right) = \sum_{n=0}^{\infty} (T_k(\chi, nd-1)) \frac{t^k}{k!}. \tag{2.10}$$

By using Taylor expansion in (2.10), we have

$$\int_X \chi(x)(dn+x)^k dx - \int_X \chi(x) x^k dx = kT_{k-1}(\chi, nd-1), \quad \text{for } k, n, d \in \mathbb{N}\ . \tag{2.11}$$

That is,

$$B_{k,\chi}(nd) - B_{k,\chi} = kT_{k-1}(\chi, nd-1). \tag{2.12}$$

Let $w_1, w_2, d \in \mathbb{N}$. Then we consider the following integral equation:

$$\frac{d\iint_X \chi(x_1)\chi(x_2) e^{(w_1x_1+w_2x_2)t} dx_1 dx_2}{\int_X e^{dw_1w_2xt} dx} = \frac{t(e^{dw_1w_2t}-1)}{(e^{w_1dt}-1)(e^{w_2dt}-1)} \left(\sum_{a=0}^{d-1} \chi(a) e^{w_1at}\right)\left(\sum_{b=0}^{d-1} \chi(b) e^{w_2bt}\right). \tag{2.13}$$

From (2.7) and (2.10), we note that

$$\frac{dw_1 \int_X \chi(x) e^{xt} dx}{\int_X e^{dw_1xt} dx} = \sum_{k=0}^{\infty} (T_k(\chi, dw_1-1)) \frac{t^k}{k!}. \tag{2.14}$$

Let us consider the p-adic functional $T_\chi(w_1, w_2)$ as follows:

$$T_\chi(w_1, w_2) = \frac{d\iint_X \chi(x_1)\chi(x_2) e^{(w_1x_1+w_2x_2+w_1w_2x)t} dx_1 dx_2}{\int_X e^{dw_1w_2x_3t} dx_3}. \tag{2.15}$$

Then we see that $T_\chi(w_1, w_2)$ is symmetric in w_1 and w_2, and

$$T_\chi(w_1, w_2) = \frac{t(e^{dw_1w_2t}-1) e^{w_1w_2xt}}{(e^{w_1dt}-1)(e^{w_2dt}-1)} \left(\sum_{a=0}^{d-1} \chi(a) e^{w_1at}\right)\left(\sum_{b=0}^{d-1} \chi(b) e^{w_2bt}\right). \tag{2.16}$$

By (2.15) and (2.16), we have

$$
\begin{aligned}
T_\chi(w_1,w_2) &= \left(\frac{1}{w_1}\int_X \chi(x_1)e^{w_1(x_1+w_2x)t}dx_1\right)\left(\frac{dw_1\int_X\chi(x_2)e^{w_2x_2t}dx_2}{\int_X e^{dw_1w_2xt}dx}\right)\\
&= \left(\frac{1}{w_1}\sum_{i=0}^{\infty}B_{i,\chi}(w_2x)\frac{w_1^it^i}{i!}\right)\left(\sum_{k=0}^{\infty}T_k(\chi,dw_1-1)\frac{w_2^kt^k}{k!}\right)\\
&= \frac{1}{w_1}\left(\sum_{\ell=0}^{\infty}\left(\sum_{i=0}^{\ell}\frac{B_{i,\chi}(w_2x)T_{\ell-i}(\chi,dw_1-1)w_1^iw_2^{\ell-i}\ell!}{i!(\ell-i)!}\right)\frac{t^\ell}{\ell!}\right)\\
&= \sum_{\ell=0}^{\infty}\left(\sum_{i=0}^{\ell}\binom{\ell}{i}B_{i,\chi}(w_2x)T_{\ell-i}(\chi,dw_1-1)w_1^{i-1}w_2^{\ell-i}\right)\frac{t^\ell}{\ell!}.
\end{aligned}
\tag{2.17}
$$

From the symmetric property of $T_\chi(w_1,w_2)$ in w_1 and w_2, we note that

$$
\begin{aligned}
T_\chi(w_1,w_2) &= \left(\frac{1}{w_2}\int_X \chi(x_2)e^{w_2(x_2+w_1x)t}dx_2\right)\left(\frac{dw_2\int_X\chi(x_1)e^{w_1x_1t}dx_1}{\int_X e^{dw_1w_2xt}dx}\right)\\
&= \left(\frac{1}{w_2}\sum_{i=0}^{\infty}B_{i,\chi}(w_1x)\frac{w_2^it^i}{i!}\right)\left(\sum_{k=0}^{\infty}T_k(\chi,dw_2-1)\frac{w_1^kt^k}{k!}\right)\\
&= \frac{1}{w_2}\left(\sum_{\ell=0}^{\infty}\left(\sum_{i=0}^{\ell}\frac{B_{i,\chi}(w_1x)w_2^iT_{\ell-i}(\chi,dw_2-1)w_1^{\ell-i}\ell!}{i!(\ell-i)!}\right)\frac{t^\ell}{\ell!}\right)\\
&= \sum_{\ell=0}^{\infty}\left(\sum_{i=0}^{\ell}\binom{\ell}{i}w_2^{i-1}w_1^{\ell-i}B_{i,\chi}(w_1x)T_{\ell-i}(\chi,dw_2-1)\right)\frac{t^\ell}{\ell!}.
\end{aligned}
\tag{2.18}
$$

By comparing the coefficients on the both sides of (2.17) and (2.18), we obtain the following theorem.

Theorem 2.2. *For $w_1,w_2,d\in\mathbb{N}$, one has*

$$
\sum_{i=0}^{\ell}\binom{\ell}{i}B_{i,\chi}(w_2x)T_{\ell-i}(\chi,dw_1-1)w_1^{i-1}w_2^{\ell-i} = \sum_{i=0}^{\ell}\binom{\ell}{i}B_{i,\chi}(w_1x)T_{\ell-i}(\chi,dw_2-1)w_2^{i-1}w_1^{\ell-i}.
\tag{2.19}
$$

Let $x=0$ in Theorem 2.2. Then we have

$$
\sum_{i=0}^{\ell}\binom{\ell}{i}B_{i,\chi}T_{\ell-i}(\chi,dw_1-1)w_1^{i-1}w_2^{\ell-i} = \sum_{i=0}^{\ell}\binom{\ell}{i}B_{i,\chi}T_{\ell-i}(\chi,dw_2-1)w_2^{i-1}w_1^{\ell-i}.
\tag{2.20}
$$

By (2.14) and (2.16), we also see that

$$
\begin{aligned}
T_\chi(w_1,w_2) &= \left(\frac{e^{w_1w_2xt}}{w_1}\int_X \chi(x_1)e^{w_1x_1t}dx_1\right)\left(\frac{dw_1\int_X \chi(x_2)e^{w_2x_2t}dx_2}{\int_X e^{dw_1w_2xt}dx}\right)\\
&= \left(\frac{e^{w_1w_2xt}}{w_1}\int_X \chi(x_1)e^{w_1x_1t}dx_1\right)\left(\frac{e^{dw_1w_2t}-1}{e^{w_2dt}-1}\right)\left(\sum_{i=0}^{d-1}\chi(i)e^{w_2it}\right)\\
&= \left(\frac{e^{w_1w_2xt}}{w_1}\int_X \chi(x_1)e^{w_1x_1t}dx_1\right)\left(\sum_{\ell=0}^{w_1-1}\sum_{i=0}^{d-1}e^{w_2(i+\ell d)t}\chi(i+\ell d)\right)\\
&= \left(\frac{e^{w_1w_2xt}}{w_1}\int_X \chi(x_1)e^{w_1x_1t}dx_1\right)\left(\sum_{i=0}^{dw_1-1}e^{w_2it}\chi(i)\right)\\
&= \frac{1}{w_1}\sum_{i=0}^{dw_1-1}\chi(i)\int_X \chi(x_1)e^{w_1(x_1+w_2x+(w_2/w_1)i)t}dx_1\\
&= \frac{1}{w_1}\sum_{i=0}^{dw_1-1}\chi(i)\sum_{k=0}^{\infty}B_{k,\chi}\left(w_2x+\frac{w_2}{w_1}i\right)\frac{w_1^kt^k}{k!}\\
&= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{dw_1-1}\chi(i)B_{k,\chi}\left(w_2x+\frac{w_2}{w_1}i\right)w_1^{k-1}\right)\frac{t^k}{k!}.
\end{aligned}
\tag{2.21}
$$

From the symmetric property of $T_\chi(w_1,w_2)$ in w_1 and w_2, we can also derive the following equation:

$$
\begin{aligned}
T_\chi(w_1,w_2) &= \left(\frac{e^{w_1w_2xt}}{w_2}\int_X \chi(x_2)e^{w_2x_2t}dx_2\right)\left(\frac{dw_2\int_X \chi(x_1)e^{w_1x_1t}dx_1}{\int_X e^{dw_1w_2xt}dx}\right)\\
&= \left(\frac{e^{w_1w_2xt}}{w_2}\int_X \chi(x_2)e^{w_2x_2t}dx_2\right)\left(\frac{e^{dw_1w_2t}-1}{e^{w_1dt}-1}\right)\left(\sum_{i=0}^{d-1}\chi(i)e^{w_1it}\right)\\
&= \left(\frac{e^{w_1w_2xt}}{w_2}\int_X \chi(x_2)e^{w_2x_2t}dx_2\right)\left(\sum_{\ell=0}^{w_2-1}e^{w_1d\ell t}\right)\left(\sum_{i=0}^{d-1}\chi(i)e^{w_1it}\right)\\
&= \frac{1}{w_2}\sum_{i=0}^{dw_2-1}\chi(i)\int_X \chi(x_2)e^{w_2(x_2+w_1x+(w_1/w_2)i)t}dx_2\\
&= \frac{1}{w_2}\sum_{i=0}^{dw_2-1}\chi(i)\sum_{k=0}^{\infty}B_{k,\chi}\left(w_1x+\frac{w_1}{w_2}i\right)\frac{w_2^kt^k}{k!}\\
&= \sum_{k=0}^{\infty}\left\{\sum_{i=0}^{dw_2-1}\chi(i)B_{k,\chi}\left(w_1x+\frac{w_1}{w_2}i\right)w_2^{k-1}\right\}\frac{t^k}{k!}.
\end{aligned}
\tag{2.22}
$$

By comparing the coefficients on the both sides of (2.21) and (2.22), we obtain the following theorem.

Theorem 2.3. *For $w_1, w_2, d \in \mathbb{N}$, one has*

$$\sum_{i=0}^{dw_1-1} \chi(i) B_{k,\chi}\left(w_2 x + \frac{w_2}{w_1} i\right) w_1^{k-1} = \sum_{i=0}^{dw_2-1} \chi(i) B_{k,\chi}\left(w_1 x + \frac{w_1}{w_2} i\right) w_2^{k-1}. \tag{2.23}$$

Remark 2.4. Let $x = 0$ in Theorem 2.3. Then we see that

$$\sum_{i=0}^{dw_1-1} \chi(i) B_{k,\chi}\left(\frac{w_2}{w_1} i\right) w_1^{k-1} = \sum_{i=0}^{dw_2-1} \chi(i) B_{k,\chi}\left(\frac{w_1}{w_2} i\right) w_2^{k-1}. \tag{2.24}$$

If we take $w_2 = 1$, then we have

$$\sum_{i=0}^{dw_1-1} \chi(i) B_{k,\chi}\left(\frac{i}{w_1}\right) w_1^{k-1} = \sum_{i=0}^{d-1} \chi(i) B_{k,\chi}(w_1 i). \tag{2.25}$$

Remark 2.5. Let χ be trivial character. Then we can easily derive the "multiplication theorem for Bernoulli polynomials" from Theorems 2.2 and 2.3 (see [14]).

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