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Research Article

Global Bifurcation for Second-Order Neumann Problem with a Set-Valued Term

Ruyun Ma and Jiemei Li

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

Correspondence should be addressed to Ruyun Ma, mary@nwnu.edu.cn

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We study the global bifurcation of the differential inclusion of the form $-(ku')' + g(\cdot,u) \in \mu F(\cdot,u)$, $u'(0) = 0 = u'(1)$, where F is a "set-valued representation" of a function with jump discontinuities along the line segment $[0,1] \times \{0\}$. The proof relies on a Sturm-Liouville version of Rabinowitz's bifurcation theorem and an approximation procedure.

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1. Introduction

We are concerned with the following differential inclusion which arises from a Budyko-North type energy balance climate models:

$$\begin{gathered} -\left(ku'\right)'(x) + g(x,u(x)) \in \mu F(x,u(x)), \quad x \in (0,1) \text{ a.e.} \\ u'(0) = 0, \qquad u'(1) = 0; \end{gathered} \tag{1.1}$$

see [1–6] and the references therein. In particular, the set-valued right-hand side arises from a jump discontinuity of the albedo at the ice-edge in these models. By filling in such a gap, one arrives at the set-valued problem (1.1). As in [6], we are here interested in a considerably simplified version as compared to the situation from climate modeling; for example, a one-dimensional regular Sturm-Liouville differential operator substitutes for a two-dimensional Laplace-Beltrami operator or a singular Legendre-type operator, and the jump discontinuity is transformed to $u = 0$ in a way, which resembles only locally the climatological problem.

Assume that

(H1) $k \in C^1([0,1])$, $\inf k > 0$;

(H2) $g \in C([0,1] \times \mathbb{R})$, $g(x,\cdot)$ strictly increasing for $x \in [0,1]$,

$$g_1(x) := \lim_{|y| \to 0} \frac{g(x,y)}{y} \tag{1.2}$$

exists uniformly for $x \in [0,1]$, and $g_1(x) > 0$ on $[0,1]$,

(H2′) g satisfies that

$$g_2(x) := \lim_{|y| \to \infty} \frac{g(x,y)}{y} \tag{1.3}$$

exists uniformly for $x \in [0,1]$;

(H3) $f_+ \in C([0,1] \times \mathbb{R}_+, (0,\infty))$, $\inf f_+ > 0$, $f_- \in C([0,1] \times \mathbb{R}_-, (-\infty,0))$, $\sup f_- < 0$.

Let F in (1.1) be given by

$$F(x,y) := \begin{cases} \{f_+(x,y)\}, & x \in [0,1],\ y > 0, \\ [f_-(x,0), f_+(x,0)], & x \in [0,1], \\ \{f_-(x,y)\} & x \in [0,1],\ y < 0, \end{cases} \tag{1.4}$$

and set

$$\mathcal{S} := \Big\{ (\mu, w) \in \mathbb{R} \times C^1([0,1]) \mid (\mu, w) \text{ solves } (1.1) \Big\}. \tag{1.5}$$

Throughout $\mathcal{S}$ will be considered as subset of the Banach space $Y := \mathbb{R} \times C^1[0,1]$ under the norm

$$\|(\mu, w)\|_Y := \max\{|\mu|, \|w\|_\infty, \|w'\|_\infty\}. \tag{1.6}$$

Let

$$\mathbb{Z}_+ := \{0, 1, 2, \ldots\}. \tag{1.7}$$

Using a Sturm-Liouville version of Rabinowitz's bifurcation theorem and an approximation procedure, Hetzer [6] proved the following.

Theorem A (see [6, Theorem]). *Let (H1)–(H3) be fulfilled. Then there exist sequences $\{C_n^\pm\}_{n \in \mathbb{Z}_+}$ of unbounded, closed, connected subsets of $\mathcal{S}$ with $(0, \mathbf{0}) \in C_n^\pm$ and the property that u has exactly n zeroes, which are all simple, if $(\mu, u) \in C_n^\pm \setminus \{(0, \mathbf{0})\}$. Moreover, u is positive (negative) on an interval $(0, \tilde{x})$ for some $\tilde{x} \in (0,1]$, if $(\mu, u) \in C_n^+$ ($(\mu, u) \in C_n^-$) and $u \not\equiv 0$.*

It is easy to see from Theorem A that the effect of the discontinuity at zero is a solution branch which consists of infinitely many subbranches all meeting in $(0,\mathbf{0})$. Two subbranches are distinguished by the number of zeroes of the respective solutions. However, Theorem A provides no any information about the asymptotic behavior of $C_n^{\pm}$ at infinity.

It is the purpose of this paper to study the asymptotic behavior of $C_n^{\pm}$ at infinity, and accordingly, to determine values of μ, for which there exist infinitely many *nodal solutions* of (1.1) (here and after, a function $u \in AC^1[0,1]$ is a *nodal solution* of (1.1) if all of zeroes of u are simple). To wit, we have the following.

Theorem 1.1. *Let (H1)–(H3) and (H2′) be fulfilled. Assume that*

(H4)

$$(f_+)_\infty(x) = (f_-)_\infty(x) =: b(x) \in C([0,1],(0,\infty)), \tag{1.8}$$

where

$$(f_+)_\infty(x) := \lim_{s\to+\infty} \frac{f_+(x,s)}{s}, \qquad (f_-)_\infty(x) := \lim_{s\to-\infty} \frac{f_-(x,s)}{s}. \tag{1.9}$$

Then for each $n \in \mathbb{Z}_+$, C_n^+ joins $(0,\mathbf{0})$ with (η_n,∞), C_n^- joins $(0,\mathbf{0})$ with (η_n,∞), where η_n, $(n \in \mathbb{Z}_+)$, is the n-th eigenvalue of the linear problem:

$$\begin{gathered} -(ku')'(x) + g_2(x)u(x) = \eta b(x)u(x), \quad x \in [0,1], \\ u'(0) = 0, \qquad u'(1) = 0. \end{gathered} \tag{1.10}$$

Corollary 1.2. *Let (H1)–(H4) and (H2′) be fulfilled. Let $k \in \mathbb{N}$ be fixed. Then*

(1) for each $\mu \in [\eta_{k-1},\eta_k)$, (1.1) has infinitely many solutions:

$$u_j^\nu, \quad \nu \in \{+,-\}, j \in \{k,k+1\ldots\}, \tag{1.11}$$

which satisfies that u_j^+ has exactly j simple zeroes and u_j^+ is positive on an interval $(0,\tilde{x})$ for some $\tilde{x} \in (0,1]$, u_j^- has exactly j simple zeroes and u_j^- is negative on an interval $(0,\tilde{x})$ for some $\tilde{x} \in (0,1)$;

(2) for each $\mu \in (0,\eta_0)$, (1.1) has infinitely many solutions:

$$u_j^\nu, \quad \nu \in \{+,-\}, \quad j \in \{0,1,2\ldots\} \tag{1.12}$$

which satisfies that u_j^+ has exactly j simple zeroes, and u_j^+ is positive on an interval $(0,\tilde{x})$ for some $\tilde{x} \in (0,1]$, u_j^- has exactly j simple zeroes, and u_j^- is negative on an interval $(0,\tilde{x})$ for some $\tilde{x} \in (0,1)$.

2. Notations and Preliminary Results

Recall Kuratowski's notion of lower and upper limits of sequences of sets.

Definition 2.1 (see [7]). Let X be a metric space and let $\{Z_l\}_{l\in\mathbb{N}}$ be a sequence of subsets of X. The set

$$\limsup_{l\to\infty} Z_l := \left\{x \in X : \liminf_{l\to\infty} \operatorname{dist}(x, Z_l) = 0\right\} \tag{2.1}$$

is called the upper limit of the sequence $\{Z_l\}$, whereas

$$\liminf_{l\to\infty} Z_l := \left\{x \in X : \lim_{l\to\infty} \operatorname{dist}(x, Z_l) = 0\right\} \tag{2.2}$$

is called the lower limit of the sequence $\{Z_l\}$.

Definition 2.2 (see [7]). A *component* of a set M is meant a maximal connected subset of M.

Lemma 2.3 (see [7]). *Suppose that Y is a compact metric space, A and B are nonintersecting closed subsets of Y, and no component of Y intersects both A and B. Then there exist two disjoint compact subsets Y_A and Y_B, such that $Y = Y_A \cup Y_B$, $A \subset Y_A$, $B \subset Y_B$.*

Using the above Whyburn Lemma, Ma and An [8] proved the following.

Lemma 2.4 (see [8, Lemma 2.1]). *Let Z be a Banach space and let $\{A_n\}$ be a family of closed connected subsets of Z. Assume that*

(i) *there exist $z_n \in A_n$, $n = 1, 2, \ldots$, and $z^* \in Z$, such that $z_n \to z^*$;*

(ii) $r_n = \sup\{\|x\| \mid x \in A_n\} = \infty$;

(iii) *for every $R > 0$, $(\bigcup_{n=1}^{\infty} A_n) \cap B_R$ is a relatively compact set of Z, where*

$$B_R = \{x \in Z \mid \|x\| \le R\}. \tag{2.3}$$

Then there exists an unbounded component $\mathcal{C}$ in $\limsup_{l\to\infty} A_l$ and $z^* \in \mathcal{C}$.

Remark 2.5. The limiting processes for sets go back at least to the work of Kuratowski [9]. Lemma 2.4 will play an important role in the proof of Theorem 1.1. It is a slight generalization of the following well-known results due to Whyburn [7].

Proposition 2.6 (Whyburn [7, page 12]). *Let Z be a Banach space and let $\{A_n\}$ be a family of closed connected subsets of Z. Let $\liminf_{l\to\infty} A_l \neq \emptyset$ and $\bigcup_{l\in\mathbb{N}} A_l$ is relatively compact. Then $\limsup_{l\to\infty} A_l$ is nonempty, compact, and connected.*

Lemma 2.7. *Let $q \in C([0,1], (0,\infty))$. Let $p_m \in C([0,1], (0,\infty))$ be such that*

$$p_m(t) \ge \rho, \quad t \in [0,1] \tag{2.4}$$

for some $\rho > 0$. Suppose that the sequence $\{(\mu_m, y_m)\}$ satisfies

$$-(ky_m')' + q(t)y_m = \mu_m p_m(t) y_m, \qquad y_m'(0) = y_m'(1) = 0 \tag{2.5}$$

with either

$$(y_m|_I)(t) > 0 \quad \forall m \text{ sufficiently large} \tag{2.6}$$

or

$$(y_m|_I)(t) < 0 \quad \forall m \text{ sufficiently large,} \tag{2.7}$$

where $I := [\alpha, \beta]$ with $\alpha < \beta$ being a given closed subinterval of $(0,1)$. Then

$$|\mu_m| \leq M_0 \tag{2.8}$$

for some positive constant M_0.

Proof. We only deal with the case that $(y_m|_I)(t) > 0$ for all m sufficiently large. The other case can be treated by the similar way. We may assume that $(y_m|_I)(t) > 0$ for all $m \in \mathbb{N}$.

We divide the proof into three cases.

Case 1. Let (α_m, β_m) be a subinterval of $[0,1]$ satisfying

(i) $I \subset (\alpha_m, \beta_m)$;

(ii) $y_m(\alpha_m) = y_m(\beta_m) = 0$;

(iii) $y_m(t) > 0$ for all $t \in (\alpha_m, \beta_m)$.

Let $\psi_m(t)$ and $\varphi_m(t)$ be the unique solution of the problems:

$$\begin{gathered}
-(ky')' + q(t)y = 0, \quad t \in (\alpha_m, \beta_m), \\
y(\alpha_m) = 0, \qquad y'(\alpha_m) = 1, \\
-(ky')' + q(t)y = 0, \quad t \in (\alpha_m, \beta_m), \\
y(\beta_m) = 0, \qquad y'(\beta_m) = -1,
\end{gathered} \tag{2.9}$$

respectively. Then it is easy to check $\psi_m(\cdot)$ is nondecreasing on (α_m, β_m), $\varphi_m(\cdot)$ is nonincreasing on (α_m, β_m), and that Green's function $G_m(t,s)$ of

$$\begin{gathered}
-(ky')' + q(t)y = 0, \quad t \in (\alpha_m, \beta_m), \\
y(\alpha_m) = y(\beta_m) = 0
\end{gathered} \tag{2.10}$$

is explicitly given by

$$G_m(t,s) = \frac{1}{\varphi_m(\alpha_m)} \begin{cases} \psi_m(t)\varphi_m(s), & \alpha_m \le t \le s \le \beta_m, \\ \varphi_m(t)\psi_m(s), & \alpha_m \le s \le t \le \beta_m. \end{cases} \tag{2.11}$$

Let $\Psi(t)$ and $\Phi(t)$ be the unique solution of the problems:

$$\begin{gathered} -(ky')' + q(t)y = 0, \quad t \in (0,1), \\ y(0) = 0, \qquad y'(0) = 1, \\ -(ky')' + q(t)y = 0, \quad t \in (0,1), \\ y(1) = 0, \qquad y'(1) = -1, \end{gathered} \tag{2.12}$$

respectively. Then it is easy to check that $\Psi(\cdot)$ is nondecreasing on $(0,1)$ and $\Phi(\cdot)$ is nonincreasing on $(0,1)$, and

$$\Phi(0) \ge \varphi_m(\alpha_m), \qquad \Psi(1) \ge \psi_m(\beta_m). \tag{2.13}$$

Let $\psi_I(t)$ and $\varphi_I(t)$ be the unique solution of the problems

$$\begin{gathered} -(ky')' + q(t)y = 0, \quad t \in (\alpha,\beta), \\ y(\alpha) = 0, \qquad y'(\alpha) = 1, \\ -(ky')' + q(t)y = 0, \quad t \in (\alpha,\beta), \\ y(\beta) = 0, \qquad y'(\beta) = -1, \end{gathered} \tag{2.14}$$

respectively. Then, for $(t,s) \in [\alpha + (\beta-\alpha)/4, \beta - (\beta-\alpha)/4] \times [\alpha + (\beta-\alpha)/4, \beta - (\beta-\alpha)/4]$,

$$G_m(t,s) \ge \frac{1}{\Phi(0)} \psi_I\left(\alpha + \frac{\beta-\alpha}{4}\right) \varphi_I\left(\beta - \frac{\beta-\alpha}{4}\right). \tag{2.15}$$

Since

$$\frac{G_m(t,s)}{G_m(s,s)} \geq \begin{cases} \dfrac{\psi_m(t)}{\psi_m(s)}, & \alpha_m \leq t \leq s \leq \beta_m, \\ \dfrac{\varphi_m(t)}{\varphi_m(s)}, & \alpha_m \leq s \leq t \leq \beta_m, \end{cases}$$
$$\geq \begin{cases} \dfrac{\psi_m(t)}{\Psi(1)}, & \alpha_m \leq t \leq s \leq \beta_m, \\ \dfrac{\varphi_m(t)}{\Phi(0)}, & \alpha_m \leq s \leq t \leq \beta_m, \end{cases} \tag{2.16}$$
$$\geq \min\left\{\frac{\psi_m(t)}{\Psi(1)}, \frac{\varphi_m(t)}{\Phi(0)}\right\} =: \delta_m(t),$$

it follows that for $t \in [\alpha + (\beta-\alpha)/4, \beta - (\beta-\alpha)/4]$,

$$\begin{aligned} y_m(t) &= \mu_m \int_{\alpha_m}^{\beta_m} G_m(t,s) p_m(s) y_m(s) ds \\ &\geq \delta_m(t) \mu_m \int_{\alpha_m}^{\beta_m} G_m(s,s) p_m(s) y_m(s) ds \\ &\geq \delta_m(t) \left\| \left(y_m|_{[\alpha_m,\beta_m]} \right) \right\|_\infty \\ &\geq \delta_m(t) \left\| \left(y_m|_{[\alpha+(\beta-\alpha)/4, \beta-(\beta-\alpha)/4]} \right) \right\|_\infty \\ &\geq \delta_I(t) \left\| \left(y_m|_{[\alpha+(\beta-\alpha)/4, \beta-(\beta-\alpha)/4]} \right) \right\|_\infty, \end{aligned} \tag{2.17}$$

where

$$\delta_I(t) := \min\left\{\frac{\psi_I(t)}{\Psi(1)}, \frac{\varphi_I(t)}{\Phi(0)}\right\}. \tag{2.18}$$

Set

$$\delta_0 := \min\left\{\delta_I(t) \mid t \in \left[\alpha + \frac{\beta-\alpha}{4}, \beta - \frac{\beta-\alpha}{4}\right]\right\}. \tag{2.19}$$

Then

$$\min_{t\in[\alpha+(\beta-\alpha)/4, \beta-(\beta-\alpha)/4]} y_m(t) \geq \delta_0 \left\| \left(y_m|_{[\alpha+(\beta-\alpha)/4, \beta-(\beta-\alpha)/4]} \right) \right\|_\infty. \tag{2.20}$$

By (2.5), we have that

$$y_m(t) = \mu_m \int_{\alpha_m}^{\beta_m} G_m(t,s) p_m(s) y_m(s) ds, \tag{2.21}$$

which together with (2.15) and (2.20) imply that for $t \in [\alpha + (\beta-\alpha)/4, \beta - (\beta-\alpha)/4]$,

$$\begin{aligned}
&y_m(t) \\
&\quad \geq \mu_m \int_I G_m(t,s) \rho y_m(s)\, ds \\
&\quad \geq \mu_m \int_{\alpha+(\beta-\alpha)/4}^{\beta-(\beta-\alpha)/4} G_m(t,s) \rho y_m(s)\, ds \\
&\quad \geq \delta_0 \mu_m \int_{\alpha+(\beta-\alpha)/4}^{\beta-(\beta-\alpha)/4} G_m(t,s) \rho\, ds \cdot \left\| \left(y_m \big|_{[\alpha+(\beta-\alpha)/4, \beta-(\beta-\alpha)/4]} \right) \right\|_\infty \\
&\quad \geq \delta_0 \frac{\mu_m}{\Phi(0)} \varphi_I \left(\alpha + \frac{\beta-\alpha}{4} \right) \varphi_I \left(\beta - \frac{\beta-\alpha}{4} \right) \rho \int_{\alpha+(\beta-\alpha)/4}^{\beta-(\beta-\alpha)/4} ds \cdot \left\| \left(y_m \big|_{[\alpha+(\beta-\alpha)/4, \beta-(\beta-\alpha)/4]} \right) \right\|_\infty .
\end{aligned} \tag{2.22}$$

Therefore

$$|\mu_m| \leq \left(\frac{\delta_0\ \rho}{\Phi(0)} \varphi_I \left(\alpha + \frac{\beta-\alpha}{4} \right) \varphi_I \left(\beta - \frac{\beta-\alpha}{4} \right) \cdot \frac{\beta-\alpha}{2} \right)^{-1}. \tag{2.23}$$

Case 2. Let $(0, \beta_m)$ be a subinterval of $[0,1]$ satisfying

(i) $I \subset (0, \beta_m)$;

(ii) $y_m'(0) = 0$, $y_m(\beta_m) = 0$;

(iii) $y_m(t) > 0$ for all $t \in (0, \beta_m)$.

Let $\overline{\varphi}_m(t)$ and $\overline{\varphi}_m(t)$ be the unique solution of the problems:

$$\begin{gathered}
-(ky')' + q(t)y = 0, \quad t \in (0, \beta_m), \\
y'(0) = 0, \qquad y(\beta_m) = 1, \\
-(ky')' + q(t)y = 0, \quad t \in (0, \beta_m), \\
y(\beta_m) = 0, \qquad y'(\beta_m) = -1,
\end{gathered} \tag{2.24}$$

respectively. Then it is easy to check that $\overline{\varphi}_m(\cdot)$ is nondecreasing on $(0, \beta_m)$, $\overline{\varphi}_m(\cdot)$ is nonincreasing on $(0, \beta_m)$, and Green's function $G^*(t,s)$ of

$$\begin{gathered}
-(ky')' + q(t)y = 0, \quad t \in (0, \beta_m), \\
y'(0) = y(\beta_m) = 0
\end{gathered} \tag{2.25}$$

is explicitly given by

$$G^*(t,s) = \frac{1}{\overline{\varphi}_m(0)} \begin{cases} \overline{\varphi}_m(t)\overline{\varphi}_m(s), & 0 \le t \le s \le \beta_m, \\ \overline{\varphi}_m(t)\overline{\psi}_m(s), & 0 \le s \le t \le \beta_m. \end{cases} \tag{2.26}$$

By the similar method to prove Case 1, we may get the desired results.

Case 3. Let $(\alpha_m, 1)$ be a subinterval of $[0,1]$ satisfying

(i) $I \subset (\alpha_m, 1)$;

(ii) $y_m(\alpha_m) = 0$, $y'_m(1) = 0$;

(iii) $y_m(t) > 0$ for all $t \in (\alpha_m, 1)$.

Using the same method to prove Case 2, with obvious changes, we may show that (2.8) is true.

Case 4. Let $(\alpha_m, \beta_m) = (0,1)$. We may assume that $y_m(t) > 0$ for all $(0,1)$.

Let $\psi(t)$ and $\varphi(t)$ be the unique solution of the problems

$$\begin{gathered} -(ky')' + q(t)y = 0, \quad t \in (0,1), \\ y(0) = 0, \qquad y'(0) = 1, \\ -(ky')' + q(t)y = 0, \quad t \in (0,1), \\ y(1) = 0, \qquad y'(1) = -1, \end{gathered} \tag{2.27}$$

respectively. Then, it is easy to verify that ψ is strictly increasing on $[0,1]$ and φ is strictly decreasing on $[0,1]$. Using the same method to deal with Case 1, we may get the desired results. □

3. Proof of the Results

Recall the proof of Theorem A.

By [6, Remark 1], the hypotheses (H1)–(H3) imply that

$$\mathcal{S} \cap \left((-\infty, 0] \times C^1([0,1]) \right) = (-\infty, 0] \times \{\mathbf{0}\}. \tag{3.1}$$

Actually, such continua can be obtained as upper limits in the sense of Kuratowski of sequences of solution continua from associated continuous problems. To this end one sets

$$d_f := \min\{\inf f_+, \inf |f_-|\} \tag{3.2}$$

and selects an approximation sequence $\{f_l\} \in C([0,1] \times \mathbb{R}, \mathbb{R})^{\mathbb{N}}$ of F satisfying

(A1) $f_l(x,y) = ly$ for $x \in [0,1]$ and $y \in [-d_f/2l, d_f/2l]$;

(A2) $f_l(x,y) \times \mathrm{sgn}(y) \geq d_f/2$ for $x \in [0,1]$ and $|y| \geq d_f/2l$; $f_l \leq f_+$ on $[0,1] \times [d_f/2l, d_f/l]$; $f_l \geq f_-$ on $[0,1] \times [-d_f/l, -d_f/2l]$;

(A3) $f_l(x,y) = f_+(x,y)$ for $x \in [0,1]$ and $y \geq d_f/l$; $f_l(x,y) = f_-(x,y)$ for $x \in [0,1]$ and $y \leq -d_f/l$;

(A4) $\{f_l(x,y)\}_{l\in\mathbb{N}}$ is nondecreasing in l for $(x,y) \in [0,1] \times (0,\infty)$; $\{f_l(x,y)\}_{l\in\mathbb{N}}$ is nonincreasing in l for $(x,y) \in [0,1] \times (-\infty, 0)$.

Remark 3.1. Let

$$\xi(x,u) := g(x,u) - g_1(x)u. \tag{3.3}$$

We may show that there exists a positive constant $\overline{\gamma}$, independent of l, such that for each $l \in \mathbb{N}$,

$$\frac{f_l(x,u)}{u} - \frac{\xi(x,u)}{\gamma u} \geq \rho_0, \quad \forall \gamma \geq \overline{\gamma} \tag{3.4}$$

for some constant $\rho_0 > 0$.

In fact, it is easy to see from the definition of f_l that

$$\frac{f_l(x,u)}{u} \geq \rho_1, \quad u \neq 0 \tag{3.5}$$

for some positive constant ρ_1, independent of l.

Applying (H2) and (H2′), it concludes that

$$0 \leq \left|\frac{\xi(x,u)}{u}\right| \leq \rho_2 \tag{3.6}$$

for some positive constant ρ_2. Therefore, if we take

$$\overline{\gamma} := \frac{2\rho_2}{\rho_1}, \qquad \rho_0 = \frac{\rho_1}{2}, \tag{3.7}$$

then (3.4) holds.

It is easy to see thanks to (H2) and (A1) that

$$\begin{gathered} -(kv')'(x) + g(x,v(x)) = \mu f_l(x,v(x)), \qquad x \in [0,1], \\ v'(0) = 0, \qquad v'(1) = 0 \end{gathered} \tag{3.8_l}$$

falls into the scope of the Sturm-Liouville version of the celebrated Rabinowitz bifurcation theorem (cf. [10] for a more general, but somewhat different setting).

Indeed, denote the strictly increasing sequence of simple eigenvalues of

$$\begin{gathered} -\left(k\psi'\right)'(x) + g_1(x)\psi(x) = \lambda\psi(x), \quad x \in [0,1], \\ \psi'(0) = 0, \qquad \psi'(1) = 0, \end{gathered} \tag{3.9}$$

by $\{\lambda_n\}_{n\in\mathbb{Z}_+}$ and set

$$\mu_{n,l} := \frac{\lambda_n}{l}. \tag{3.10}$$

Then $(\mu_{n,l}, \mathbf{0})$ is a bifurcation point of the solution set of (3.8_l) for every $n \in \mathbb{Z}_+$, and for each $(n,l) \in \mathbb{Z}_+ \times \mathbb{N}$, there exist two unbounded closed connected subsets $C^{\pm}_{n,l}$ of the solution set of (3.8_l) with the following.

(a) $C^+_{n,l} \cap C^-_{n,l} = \{(\mu_{n,l}, \mathbf{0})\}$. Moreover, $(\mu_{n,l}, \mathbf{0})$ is the only bifurcation point contained in $C^{\pm}_{n,l}$.

(b) If $(\mu, \vartheta) \in C^+_{n,l}$ and $\vartheta \not\equiv 0$, then ϑ possesses exactly n simple zeroes (and no multiple zeroes) in $(0,1)$ and is positive on $(0,\delta)$ for some $\delta > 0$.

(c) If $(\mu, \vartheta) \in C^-_{n,l}$ and $\vartheta \not\equiv 0$, then ϑ possesses exactly n simple zeroes (and no multiple zeroes) in $(0,1)$ and is negative on $(0,\delta)$ for some $\delta > 0$.

Combining the above with the fact

$$\lim_{l\to\infty} \left(\mu_{n,l}, \mathbf{0}\right) = (0, \mathbf{0}) \tag{3.11}$$

and utilizing Lemma 2.4, it concludes that there exists an unbounded component C^{ν}_n with

$$\begin{gathered} (0, \mathbf{0}) \in C^{\nu}_n, \\ C^{\nu}_n \subseteq \limsup_{l\to\infty} C^{\nu}_{n,l}, \quad \nu \in \{+,-\}. \end{gathered} \tag{3.12}$$

As an immediate consequence of [6, Lemma 4-6], we have the following

Lemma 3.2. *If $(\mu, u) \in C^{\pm}_n$, then (μ, u) is a solution of (1.1) and $u \in W^{2,\infty}([0,1])$. Moreover, if $(\mu, u) \in C^+_n$ with $u \not\equiv 0$, u has exactly n simple zeroes in $[0,1]$, and u is positive on an interval $(0, \widetilde{x})$ for some $\widetilde{x} \in (0,1]$; if $(\mu, u) \in C^-_n$ with $u \not\equiv 0$, u has exactly n simple zeroes in $[0,1]$, and u is negative on an interval $(0, \widetilde{x})$ for some $\widetilde{x} \in (0,1]$.*

Lemma 3.3. *Let (H1)–(H4), (H2′) and (A1)–(A4) be fulfilled. Then for each $(n,l) \in \mathbb{Z}_+ \times \mathbb{N}$, the connected component $C^{\pm}_{n,l}$ joins $(\mu_{n,l}, \mathbf{0})$ with (η_n, ∞).*

Proof. Assume that $\{(r_k, y_k)\} \subset C^+_{n,l}$ for some fixed $(n, l) \in \mathbb{Z}_+ \times \mathbb{N}$ with

$$|r_k| + \|y_k\|_{C^1} \longrightarrow \infty. \tag{3.13}$$

The case $\{(r_k, y_k)\} \subset C^-_{n,l}$ can be treated by the same way.

We divide the proof into two steps.

Step 1. We show that if there exists a constant number $M > 0$ such that

$$r_k \in (0, M], \tag{3.14}$$

then $C^+_{n,l}$ joins $(\mu_{n,l}, \mathbf{0})$ with (η_n, ∞). In this case it follows that

$$\|y_k\|_{C^1} \longrightarrow \infty. \tag{3.15}$$

Define

$$\zeta_l(r, x, u) := r\left[f_l(x, u) - b(x)u\right] - \left[g(x, u) - g_2(x)u\right]. \tag{3.16}$$

Then $\{(r_k, y_k)\}$ satisfies the problem:

$$\begin{gathered} -\left(ky_k'\right)'(x) + g_2(x)y_k(x) = r_k b(x)y_k(x) + \zeta_l\left(r_k, x, y_k(x)\right), \quad x \in [0,1], \\ y_k'(0) = 0, \qquad y_k'(1) = 0. \end{gathered} \tag{3.17$_l$}$$

Set

$$\tilde{\zeta}_l(u) = \max_{0 \le |s| \le u, x \in [0,1], r \in [0,M]} |\zeta_l(r, x, s)| \tag{3.18}$$

then $\tilde{\zeta}$ is nondecreasing, and (H4) and (H2′) yields

$$\lim_{u \to \infty} \frac{\tilde{\zeta}(u)}{u} = 0. \tag{3.19}$$

Now, we divide (3.17$_l$) by $\|y_k\|_{C^1}$ and set $\overline{y}_k = (y_k / \|y_k\|_{C^1})$. Since $\overline{y}_k$ is bounded in $C^2[0,1]$, after taking a subsequence if necessary, we have that $\overline{y}_k \to \overline{y}$ for some $\overline{y} \in C^1[0,1]$ with $\|\overline{y}\|_{C^1} = 1$. Moreover, from the definition of f_l and (3.19) and the fact that $\tilde{\zeta}$ is nondecreasing, we have that

$$\lim_{k \to \infty} \frac{\left|\zeta_l\left(r_k, x, y_k(x)\right)\right|}{\|y_k\|_{C_1}} = 0 \tag{3.20}$$

since

$$\frac{|\zeta_l(r_k, x, y_k(x))|}{\|y_k\|_{C_1}} \le \frac{\widetilde{\zeta}(|y_k(x)|)}{\|y_k\|_{C_1}} \le \frac{\widetilde{\zeta}(\|y_k\|_\infty)}{\|y_k\|_{C_1}} \le \frac{\widetilde{\zeta}\left(\|y_k\|_{C_1}\right)}{\|y_k\|_{C_1}}. \tag{3.21}$$

By standard limit procedure, we get

$$\begin{gathered} -\left(k\overline{y}'\right)'(x) + g_2(x)\overline{y}(x) = \overline{r}b(x)\overline{y}(x), \qquad x \in [0,1], \\ \overline{y}'(0) = 0, \qquad \overline{y}'(1) = 0, \end{gathered} \tag{3.22}$$

where $\overline{r} := \lim_{k\to\infty} r_k$, again choosing a subsequence and relabeling if necessary. Moreover, the fact that y_k, $k \in \mathbb{Z}_+$, has exactly n simple zeroes in $[0,1]$ implies that $\overline{y}$ has exactly n simple zeroes in $[0,1]$, too. Therefore $\overline{r} = \eta_n$.

Step 2. We show that there exists a constant M such that $r_k \in (0, M]$, for all n. Suppose there is no such M, choosing a subsequence and relabeling if necessary, it follows that

$$\lim_{k\to\infty} r_k = \infty. \tag{3.23}$$

Let

$$\tau(1,k) < \cdots < \tau(n,k) \tag{3.24}$$

denote the zeroes of y_k, and set

$$0 = \tau(0,k), \qquad \tau(n+1,k) = 1. \tag{3.25}$$

Then, after taking a subsequence if necessary,

$$\lim_{k\to\infty} \tau(l,k) := \tau(l,\infty), \quad l \in \{0,1,\ldots,n+1\}. \tag{3.26}$$

We claim that for all $l \in \{0,1,\ldots,n\}$

$$\tau(l+1,\infty) - \tau(l,\infty) = 0. \tag{3.27}$$

Suppose on the contrary that there exists $l_0 \in \{0,1,\ldots,n\}$ such that

$$\tau(l_0,\infty) < \tau(l_0+1,\infty). \tag{3.28}$$

Define a function $p : [0,1] \to \mathbb{R}$ by

$$p_l(x) := \begin{cases} \dfrac{f_l(x, y_k(x))}{y_k(x)} - \dfrac{\xi(x, y_k(x))}{r_k y_k(x)}, & x \in [0,1],\ y_k(x) \neq 0, \\ l, & y_k(x) = 0. \end{cases} \tag{3.29}$$

Then by Remark 3.1, there exists ρ_0, such that

$$p_l(x) \geq \rho_0, \quad x \in [0,1]. \tag{3.30}$$

Now we choose a closed interval $I \subset (\tau(l_0, \infty), \tau(l_0 + 1, \infty))$ with positive length, then we know from Lemma 2.7 that y_k (after taking a subsequence if necessary) must change sign on I. However, this contradicts the fact that for all k sufficiently large, we have $I \subset (\tau(l_0, k), \tau(l_0 + 1, k))$ and

$$(-1)^{l_0} \nu y_k(x) > 0, \quad x \in (\tau(l_0, k),\ \tau(l_0 + 1, k)). \tag{3.31}$$

Therefore, (3.27) holds.

On the other hand, it follows

$$1 = \tau(n+1, k) - \tau(0, k) = \sum_{l=0}^{n} (\tau(l+1, k) - \tau(l, k)) \tag{3.32}$$

that

$$1 = \sum_{l=0}^{n} (\tau(l+1, \infty) - \tau(l, \infty)) \tag{3.33}$$

which contradicts (3.27).

Therefore

$$|r_k| \leq M \tag{3.34}$$

for some constant number $M > 0$, independent of $k \in \mathbb{N}$. □

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. We only prove that C_n^+ has the desired property, the case of C_n^- can be treated by the same way.

Assume that $\{(\mu_k, z_k)\} \subset C_n^+$ is a sequence with

$$|\mu_k| + \|z_k\|_{C^1} \longrightarrow \infty. \tag{3.35}$$

We claim that

$$\lim_{k\to\infty}(\mu_k, z_k) = (\eta_n, \infty). \tag{3.36}$$

Assume on the contrary that (3.36) is not true. We divide the proof into two cases.

Case 1. $\lim_{k\to\infty}\mu_k \neq \eta_n$.In this case, we may take a subsequence of $\{\mu_k\}$, denote it by $\{\mu_k\}$ again, with the property that there exists $\varepsilon_0 > 0$, such that for each $k \in \mathbb{N}$,

$$|\mu_k - \eta_n| \geq \epsilon_0. \tag{3.37}$$

Since $\{(\mu_k, z_k)\} \subset C_n^+$, it follows that for each $k \in \mathbb{Z}_+$, there exists a sequence $\{(\gamma_{k_j}, z_{k_j})\} \subset C^+_{n,k_j}$, such that

$$\lim_{j\to\infty}\gamma_{k_j} = \mu_k, \qquad \lim_{j\to\infty} z_{k_j} = z_k. \tag{3.38}$$

Now let us consider the sequence $\{(\gamma_{k_k}, z_{k_k})\}$. Obviously, we have that

$$\begin{gathered}(\gamma_{k_k}, z_{k_k}) \in C^+_{n,k_k},\\ |\gamma_{k_k}| + \|z_{k_k}\|_{C^1} \longrightarrow \infty.\end{gathered} \tag{3.39}$$

Equation (3.39) implies that

$$\begin{gathered}-\left(kz'_{k_k}\right)'(x) + g_2(x)z_{k_k}(x) = \gamma_{k_k}b(x)z_{k_k}(x) + \zeta_{k_k}(\gamma_{k_k}, x, z_{k_k}(x)), \quad x \in [0,1],\\ z'_{k_k}(0) = 0, \qquad z'_{k_k}(1) = 0,\end{gathered} \tag{3.40}$$

Noticing that ρ_0 in (3.30) is independent of l and using Remark 3.1 and the method to prove Lemma 3.3 and with obvious changes, we may show that $\{\gamma_{k_k}\}$ is bounded, and subsequently

$$\lim_{k\to\infty}\gamma_{k_k} = \eta_n. \tag{3.41}$$

However, this contradicts (3.37).

Case 2. $\lim_{k\to\infty}\|z_k\|_{C^1} \neq \infty$. In this case, after taking a subsequence of $\{z_k\}$ and relabeling if necessary, we may assume that

$$\|z_k\|_{C^1} \leq M_0 \tag{3.42}$$

for some constant $M_0 > 0$. Equation (3.35) together with (3.42) implies

$$\lim_{k\to\infty}\mu_k = +\infty. \tag{3.43}$$

Using the same notations as those in Case 1, we have from (3.43) that

$$\lim_{k\to\infty}\gamma_{k_k}=+\infty. \tag{3.44}$$

Combining this with (3.40) and using Remark 3.1 and the similar method to prove Step 2 of Lemma 3.3 and noticing that ρ_0 in (3.30) is independent of l, it concludes that $\{\gamma_{k_k}\}$ is bounded. This is a contradiction. □

Remark 3.4. It is easy to see from Theorem 1.1 and its proof that the "jumping" of F at $u=0$: $f_+(x,0)-f_-(x,0)(=:\Delta(x))$ does not affect the asymptotic behavior of $C_n^{\pm}$ at infinity. In other words, for any nonnegative function $\Delta(x)$, the asymptotic behavior of $C_n^{\pm}$ at infinity is the same.

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