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Research Article

Convolutions with the Continuous Primitive Integral

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If F is a continuous function on the real line and $f = F'$ is its distributional derivative, then the continuous primitive integral of distribution f is $\int_a^b f = F(b) - F(a)$. This integral contains the Lebesgue, Henstock-Kurzweil, and wide Denjoy integrals. Under the Alexiewicz norm, the space of integrable distributions is a Banach space. We define the convolution $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ for f an integrable distribution and g a function of bounded variation or an L^1 function. Usual properties of convolutions are shown to hold: commutativity, associativity, commutation with translation. For g of bounded variation, $f * g$ is uniformly continuous and we have the estimate $\|f * g\|_\infty \leq \|f\| \|g\|_{\mathcal{BV}}$, where $\|f\| = \sup_I |\int_I f|$ is the Alexiewicz norm. This supremum is taken over all intervals $I \subset \mathbb{R}$. When $g \in L^1$, the estimate is $\|f * g\| \leq \|f\| \|g\|_1$. There are results on differentiation and integration of convolutions. A type of Fubini theorem is proved for the continuous primitive integral.

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1. Introduction and Notation

The convolution of two functions f and g on the real line is $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$. Convolutions play an important role in pure and applied mathematics in Fourier analysis, approximation theory, differential equations, integral equations, and many other areas. In this paper, we consider convolutions for the continuous primitive integral. This integral extends the Lebesgue, Henstock-Kurzweil, and wide Denjoy integrals on the real line and has a very simple definition in terms of distributional derivatives.

Some of the main results for Lebesgue integral convolutions are that the convolution defines a Banach algebra on L^1 and $* : L^1 \times L^1 \to L^1$ such that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. The convolution is commutative, associative, and commutes with translations. If $f \in L^1$ and $g \in C^n$, then $f * g \in C^n$ and $(f * g)^{(n)}(x) = f * g^{(n)}(x)$. Convolutions also have the approximation property that if $f \in L^p$ $(1 \leq p < \infty)$ and $g \in L^1$, then $\|f * g_t - af\|_p \to 0$ as $t \to 0$, where

$g_t(x) = g(x/t)/t$ and $a = \int_{-\infty}^{\infty} g$. When f is bounded and continuous, there is a similar result for $p = \infty$. For these results see, for example, [1]; see [2] for related results with the Henstock-Kurzweil integral. Using the Alexiewicz norm, all of these results have generalizations to continuous primitive integrals that are proven in what follows.

We now define the continuous primitive integral. For this, we need some notation for distributions. The space of *test functions* is $\mathfrak{D} = C_c^\infty(\mathbb{R}) = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi \in C^\infty(\mathbb{R})$ and $\operatorname{supp}(\phi)$ is compact$\}$. The *support* of function ϕ is the closure of the set on which ϕ does not vanish and is denoted $\operatorname{supp}(\phi)$. Under usual pointwise operations, $\mathfrak{D}$ is a linear space over field $\mathbb{R}$. In $\mathfrak{D}$, we have a notion of convergence. If $\{\phi_n\} \subset \mathfrak{D}$, then $\phi_n \to 0$ as $n \to \infty$ if there is a compact set $K \subset \mathbb{R}$ such that for each n, $\operatorname{supp}(\phi_n) \subset K$, and for each $m \geq 0$, we have $\phi_n^{(m)} \to 0$ uniformly on K as $n \to \infty$. The *distributions* are denoted $\mathfrak{D}'$ and are the continuous linear functionals on $\mathfrak{D}$. For $T \in \mathfrak{D}'$ and $\phi \in \mathfrak{D}$, we write $\langle T, \phi\rangle \in \mathbb{R}$. For $\phi, \psi \in \mathfrak{D}$ and $a, b \in \mathbb{R}$, we have $\langle T, a\phi + b\psi\rangle = a\langle T, \phi\rangle + b\langle T, \psi\rangle$. Moreover, if $\phi_n \to 0$ in $\mathfrak{D}$, then $\langle T, \phi_n\rangle \to 0$ in $\mathbb{R}$. Linear operations are defined in $\mathfrak{D}'$ by $\langle aS + bT, \phi\rangle = a\langle S, \phi\rangle + b\langle T, \phi\rangle$ for $S, T \in \mathfrak{D}'$; $a, b \in \mathbb{R}$ and $\phi \in \mathfrak{D}$. If $f \in L^1_{\mathrm{loc}}$, then $\langle T_f, \phi\rangle = \int_{-\infty}^{\infty} f(x)\phi(x)dx$ defines a distribution $T_f \in \mathfrak{D}'$. The integral exists as a Lebesgue integral. All distributions have derivatives of all orders that are themselves distributions. For $T \in \mathfrak{D}'$ and $\phi \in \mathfrak{D}$, the *distributional derivative* of T is T' where $\langle T', \phi\rangle = -\langle T, \phi'\rangle$. This is also called the *weak derivative*. If $p : \mathbb{R} \to \mathbb{R}$ is a function that is differentiable in the pointwise sense at $x \in \mathbb{R}$, then we write its derivative as $p'(x)$. If p is a C^∞ bijection such that $p'(x) \neq 0$ for any $x \in \mathbb{R}$, then the composition with distribution T is defined by $\langle T \circ p, \phi\rangle = \langle T, (\phi \circ p^{-1})/(p' \circ p^{-1})\rangle$ for all $\phi \in \mathfrak{D}$. Translations are a special case. For $x \in \mathbb{R}$, define the *translation* τ_x on distribution $T \in \mathfrak{D}'$ by $\langle \tau_x T, \phi\rangle = \langle T, \tau_{-x}\phi\rangle$ for test function $\phi \in \mathfrak{D}$, where $\tau_x\phi(y) = \phi(y - x)$. All of the results on distributions we use can be found in [3].

The following Banach space will be of importance: $\mathcal{B}_C = \{F : \mathbb{R} \to \mathbb{R} \mid F \in C^0(\mathbb{R}), F(-\infty) = 0, F(\infty) \in \mathbb{R}\}$. We use the notation $F(-\infty) = \lim_{x\to-\infty} F(x)$ and $F(\infty) = \lim_{x\to\infty} F(x)$. The extended real line is denoted by $\overline{\mathbb{R}} = [-\infty, \infty]$. The space $\mathcal{B}_C$ then consists of functions continuous on $\overline{\mathbb{R}}$ with a limit of 0 at $-\infty$. We denote the functions that are continuous on $\mathbb{R}$ that have real limits at $\pm\infty$ by $C^0(\overline{\mathbb{R}})$. Hence, $\mathcal{B}_C$ is properly contained in $C^0(\overline{\mathbb{R}})$, which is itself properly contained in the space of uniformly continuous functions on $\mathbb{R}$. The space $\mathcal{B}_C$ is a Banach space under the uniform norm; $\|F\|_\infty = \sup_{x\in\mathbb{R}} |F(x)| = \max_{x\in\overline{\mathbb{R}}} |F(x)|$ for $F \in \mathcal{B}_C$. The *continuous primitive integral* is defined by taking $\mathcal{B}_C$ as the space of primitives. The space of integrable distributions is $\mathcal{A}_C = \{f \in \mathfrak{D}' \mid f = F'$ for $F \in \mathcal{B}_C\}$. If $f \in \mathcal{A}_C$, then $\int_a^b f = F(b) - F(a)$ for $a, b \in \overline{\mathbb{R}}$. The distributional differential equation $T' = 0$ has only constant solutions so the primitive $F \in \mathcal{B}_C$ satisfying $F' = f$ is unique. Integrable distributions are then tempered and of order one. This integral, including a discussion of extensions to $\mathbb{R}^n$, is described in [4]. A more general integral is obtained by taking the primitives to be regulated functions, that is, functions with a left and right limit at each point, see [5].

Examples of distributions in $\mathcal{A}_C$ are T_f for functions f that have a finite Lebesgue, Henstock-Kurzweil, or wide Denjoy integral. We identify function f with the distribution T_f. Pointwise function values can be recovered from T_f at points of continuity of f by evaluating the limit $\langle T_f, \phi_n\rangle$ for a *delta sequence* converging to $x \in \mathbb{R}$. This is a sequence of test functions $\{\phi_n\} \subset \mathfrak{D}$ such that for each n, $\phi_n \geq 0$, $\int_{-\infty}^{\infty} \phi_n = 1$, and the support of ϕ_n tends to $\{x\}$ as $n \to \infty$. Note that if $F \in C^0(\overline{\mathbb{R}})$ is an increasing function with $F'(x) = 0$ for almost all $x \in \mathbb{R}$, then the Lebesgue integral $\int_a^b F'(x)dx = 0$ but $F' \in \mathcal{A}_C$ and $\int_a^b F' = F(b) - F(a)$. For another example of a distribution in $\mathcal{A}_C$, let $F \in C^0(\overline{\mathbb{R}})$ be continuous and nowhere differentiable in the pointwise sense. Then $F' \in \mathcal{A}_C$ and $\int_a^b F' = F(b) - F(a)$ for all $a, b \in \overline{\mathbb{R}}$.

The space $\mathcal{A}_C$ is a Banach space under the *Alexiewicz norm*; $\|f\| = \sup_{I\subset\mathbb{R}}|\int_I f|$, where the supremum is taken over all intervals $I \subset \mathbb{R}$. An equivalent norm is $\|f\|' = \sup_{x\in\mathbb{R}}|\int_{-\infty}^{x} f|$. The continuous primitive integral contains the Lebesgue, Henstock-Kurzweil, and wide Denjoy integrals since their primitives are continuous functions. These three spaces of functions are not complete under the Alexiewicz norm and in fact $\mathcal{A}_C$ is their completion. The lack of a Banach space has hampered application of the Henstock-Kurzweil integral to problems outside of real analysis. As we will see in what follows, the Banach space $\mathcal{A}_C$ is a suitable setting for applications of nonabsolute integration.

We will also need to use functions of bounded variation. Let $g : \mathbb{R} \to \mathbb{R}$. The *variation* of g is $Vg = \sup \sum |g(x_i) - g(y_i)|$ where the supremum is taken over all disjoint intervals $\{(x_i, y_i)\}$. The functions of *bounded variation* are denoted $\mathcal{BV} = \{g : \mathbb{R} \to \mathbb{R} \mid Vg < \infty\}$. This is a Banach space under the norm $\|g\|_{\mathcal{BV}} = |g(-\infty)| + Vg$. Equivalent norms are $\|g\|_\infty + Vg$ and $|g(a)| + Vg$ for each $a \in \overline{\mathbb{R}}$. Functions of bounded variation have a left and right limit at each point in $\mathbb{R}$ and limits at $\pm\infty$, so, as above, we will define $g(\pm\infty) = \lim_{x\to\pm\infty} g(x)$.

If $g \in L^1_{\text{loc}}$, then the *essential variation* of g is ess var $g = \sup \int_{-\infty}^{\infty} g\phi'$, where the supremum is taken over all $\phi \in \mathfrak{D}$ with $\|\phi\|_\infty \le 1$. Then $\mathcal{EBV} = \{g \in L^1_{\text{loc}} \mid \text{ess var } g < \infty\}$. This is a Banach space under the norm $\|g\|_{\mathcal{EBV}} = \text{ess sup}|g| + \text{ess var } g$. Let $0 \le \gamma \le 1$. For $g : \mathbb{R} \to \mathbb{R}$, define $g_\gamma(x) = (1-\gamma)g(x-) + \gamma g(x+)$. For left continuity, $\gamma = 0$ and for right continuity $\gamma = 1$. The functions of *normalized bounded variation* are $\mathcal{NBV}_\gamma = \{g_\gamma \mid g \in \mathcal{BV}\}$. If $g \in \mathcal{EBV}$, then ess var $g = \inf Vh$ such that $h = g$ almost everywhere. For each $0 \le \gamma \le 1$, there is exactly one function $h \in \mathcal{NBV}_\gamma$ such that $g = h$ almost everywhere. In this case, ess var $g = Vh$. Changing g on a set of measure zero does not affect its essential variation. Each function of essential bounded variation has a distributional derivative that is a signed Radon measure. This will be denoted μ_g where $\langle g', \phi\rangle = -\langle g, \phi'\rangle = -\int_{-\infty}^{\infty} g\phi' = \int_{-\infty}^{\infty}\phi\, d\mu_g$ for all $\phi \in \mathfrak{D}$.

We will see that $* : \mathcal{A}_C \times \mathcal{BV} \to C^0(\overline{\mathbb{R}})$ and that $\|f * g\|_\infty \le \|f\|\|g\|_{\mathcal{BV}}$. Similarly for $g \in \mathcal{EBV}$. Convolutions for $f \in \mathcal{A}_C$ and $g \in L^1$ will be defined using sequences in $\mathcal{BV} \cap L^1$ that converge to g in the L^1 norm. It will be shown that $* : \mathcal{A}_C \times L^1 \to \mathcal{A}_C$ and that $\|f * g\| \le \|f\|\|g\|_1$.

Convolutions can be defined for distributions in several different ways.

Definition 1.1. Let $S, T \in \mathfrak{D}'$ and $\phi, \psi \in \mathfrak{D}$. Define $\widetilde{\phi}(x) = \phi(-x)$: (i) $\langle T * \psi, \phi\rangle = \langle T, \phi * \widetilde{\psi}\rangle$, (ii) for each $x \in \mathbb{R}$, let $T * \psi(x) = \langle T, \tau_x\widetilde{\psi}\rangle$; (iii) $\langle S * T, \phi\rangle = \langle S(x), \langle T(y), \phi(x+y)\rangle\rangle$.

In (i), $* : \mathfrak{D}' \times \mathfrak{D} \to \mathfrak{D}'$. This definition also applies to other spaces of test functions and their duals, such as the Schwartz space of rapidly decreasing functions or the compactly supported distributions. In (ii), $* : \mathfrak{D}' \times \mathfrak{D} \to C^\infty$. In [1], it is shown that definitions (i) and (ii) are equivalent. In (iii), $* : \mathfrak{D}' \times \mathfrak{D}' \to \mathfrak{D}'$. However, this definition requires restrictions on the supports of S and T. It suffices that one of these distributions has compact support. Other conditions on the supports can be imposed (see [3, 6]). This definition is an instance of the tensor product, $\langle S \otimes T, \Phi\rangle = \langle S(x), \langle T(y), \Phi(x, y)\rangle\rangle$, where now $\Phi \in \mathfrak{D}(\mathbb{R}^2)$.

Under (i), $T*\psi$ is in C^∞. It satisfies $(T*\psi)*\phi = T*(\psi*\phi)$, $\tau_x(T*\psi) = (\tau_x T)*\psi = T*(\tau_x\psi)$, and $(T * \psi)^{(n)} = T*\psi^{(n)} = T^{(n)}*\psi$. Under (iii), with appropriate support restrictions, $S*T$ is in $\mathfrak{D}'$. It is commutative and associative, commutes with translations, and satisfies $(S * T)^{(n)} = S^{(n)} * T = S * T^{(n)}$. It is weakly continuous in $\mathfrak{D}'$, that is, if $T_n \to T$ in $\mathfrak{D}'$, then $T_n * \psi \to T * \psi$ in $\mathfrak{D}'$ see [1, 3, 6, 7] for additional properties of convolutions of distributions.

Although elements of $\mathcal{A}_C$ are distributions, we show in this paper that their behavior as convolutions is more like that of integrable functions.

An appendix contains the proof of a type of Fubini theorem.

2. Convolution in $\mathcal{A}_C \times \mathcal{BV}$

In this section, we prove basic results for the convolution when $f \in \mathcal{A}_C$ and $g \in \mathcal{BV}$. Under these conditions, $f * g$ is commutative, continuous on $\overline{\mathbb{R}}$, and commutes with translations. It can be estimated in the uniform norm in terms of the Alexiewicz and $\mathcal{BV}$ norms. There is also an associative property. We first need the result that $\mathcal{BV}$ forms the space of multipliers for $\mathcal{A}_C$, that is, if $f \in \mathcal{A}_C$, then $fg \in \mathcal{A}_C$ for all $g \in \mathcal{BV}$. The integral $\int_I fg$ is defined using the integration by parts formula in the appendix. The Hölder inequality (A.5) shows that $\mathcal{BV}$ is the dual space of $\mathcal{A}_C$.

We define the convolution of $f \in \mathcal{A}_C$ and $g \in \mathcal{BV}$ as $f * g(x) = \int_{-\infty}^{\infty}(f \circ r_x)g$, where $r_x(t) = x - t$. We write this as $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$.

Theorem 2.1. *Let $f \in \mathcal{A}_C$ and let $g \in \mathcal{BV}$. Then* (a) *$f * g$ exists on $\mathbb{R}$.* (b) *Let $f * g = g * f$.* (c) *Let $\|f * g\|_\infty \le |\int_{-\infty}^{\infty} f|\inf_{\mathbb{R}}|g| + \|f\|Vg \le \|f\|\|g\|_{\mathcal{BV}}$.* (d) *Assume $f * g \in C^0(\overline{\mathbb{R}})$, $\lim_{x\to\pm\infty} f * g(x) = g(\pm\infty)\int_{-\infty}^{\infty} f$.* (e) *If $h \in L^1$, then $f * (g * h) = (f * g) * h \in C^0(\overline{\mathbb{R}})$.* (f) *Let $x, z \in \mathbb{R}$, then $\tau_z(f * g)(x) = (\tau_z f) * g(x) = (f * \tau_z g)(x)$.* (g) *For each $f \in \mathcal{A}_C$, define $\Phi_f : \mathcal{BV} \to C^0(\overline{\mathbb{R}})$ by $\Phi_f[g] = f * g$. Then Φ_f is a bounded linear operator and $\|\Phi_f\| \le \|f\|$. There exists a nonzero distribution $f \in \mathcal{A}_C$ such that $\|\Phi_f\| = \|f\|$. For each $g \in \mathcal{BV}$, define $\Psi_g : \mathcal{A}_C \to C^0(\overline{\mathbb{R}})$ by $\Psi_g[f] = f*g$. Then Ψ_g is a bounded linear operator and $\|\Psi_g\| \le \|g\|_{\mathcal{BV}}$. There exists a nonzero function $g \in \mathcal{BV}$ such that $\|\Psi_g\| = \|g\|_{\mathcal{BV}}$.* (h) $\operatorname{supp}(f*g) \subset \operatorname{supp}(f) + \operatorname{supp}(g)$.

Proof. (a) Existence is given via the integration by parts formula (A.1) in the appendix. (b) See [4, Theorem 11] for a change of variables theorem that can be used with $y \mapsto x - y$. (c) This inequality follows from the Hölder inequality (A.5). (d) Let $x, t \in \mathbb{R}$. From (c), we have

$$\begin{aligned} \left|f * g(t) - f * g(x)\right| &\le \left\|f(t-\cdot) - f(x-\cdot)\right\| \|g\|_{\mathcal{BV}} \\ &= \left\|f(t-x-\cdot) - f(\cdot)\right\| \|g\|_{\mathcal{BV}} \\ &\longrightarrow 0 \quad \text{as } t \longrightarrow x. \end{aligned} \tag{2.1}$$

The last line follows from continuity in the Alexiewicz norm [4, Theorem 22]. Hence, $f * g$ is uniformly continuous on $\mathbb{R}$. Also, it follows that $\lim_{x\to\infty}\int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(y)\lim_{x\to\infty} g(x-y)dy = g(\infty)\int_{-\infty}^{\infty} f$. The limit $x \to \infty$ can be taken under the integral sign since $g(x-y)$ is of uniform bounded variation, that is, $V_{y\in\mathbb{R}}g(x-y) = Vg$. Theorem 22 in [4] then applies. Similarly, as $x \to -\infty$. (e) First show $g * h \in \mathcal{BV}$. Let $\{(s_i, t_i)\}$ be disjoint intervals in $\mathbb{R}$. Then

$$\begin{aligned} \sum\left|g * h(s_i) - g * h(t_i)\right| &\le \sum\int_{-\infty}^{\infty} \left|g(s_i - y) - g(t_i - y)\right|\left|h(y)\right|dy \\ &= \int_{-\infty}^{\infty} \sum\left|g(s_i - y) - g(t_i - y)\right|\left|h(y)\right|dy. \end{aligned} \tag{2.2}$$

Hence, $V(g*h) \leq Vg\,\|h\|_1$. The interchange of sum and integral follows from the Fubini-Tonelli theorem. Now (d) shows $f*(g*h) \in C^0(\overline{\mathbb{R}})$. Write

$$\begin{aligned} f*(g*h)(x) &= \int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} g(x-y-z)h(z)dz\,dy \\ &= \int_{-\infty}^{\infty} h(z)\int_{-\infty}^{\infty} f(y)g(x-y-z)dy\,dz \\ &= (f*g)*h(x). \end{aligned} \tag{2.3}$$

We can interchange orders of integration using Proposition A.3. For (ii) in Proposition A.3, the function $z \mapsto V_{y\in\mathbb{R}}g(x-y-z)h(z) = Vg\,h(z)$ is in L^1 for each fixed $x \in \mathbb{R}$. Since g is of bounded variation, it is bounded so $|g(x-y-z)h(z)| \leq \|g\|_\infty|h(z)|$ and condition (iii) is satisfied. (f) This follows from a linear change of variables as in (a). (g) From (c), we have $\|\Phi_f\| = \sup_{\|g\|_{\mathcal{BV}}=1}\|f*g\|_\infty \leq \sup_{\|g\|_{\mathcal{BV}}=1}\|f\|\|g\|_{\mathcal{BV}} = \|f\|$. Let $f > 0$ be in L^1. If $g = 1$, then $\|g\|_{\mathcal{BV}} = 1$ and $f*g(x) = \int_{-\infty}^{\infty} f$ so $\|\Phi_f\| = \|f\| = \|f\|_1$. To prove $\|\Psi_g\| \leq \|g\|_{\mathcal{BV}}$, note that $\|\Psi_g\| = \sup_{\|f\|=1}\|f*g\|_\infty \leq \sup_{\|f\|=1}\|f\|\|g\|_{\mathcal{BV}} = \|g\|_{\mathcal{BV}}$. Let $g = \chi_{(0,\infty)}$. Then $\|\Psi_g\| = \sup_{\|f\|=1}\|f*g\|_\infty = \sup_{\|f\|=1}\sup_{x\in\mathbb{R}}|\int_{-\infty}^{x} f| = 1 = \|g\|_{\mathcal{BV}}$. (h) Suppose $x \notin \operatorname{supp}(f) + \operatorname{supp}(g)$. Note that we can write $f*g(x) = \int_{-\infty}^{\infty} g(x-y)dF(y)$ in terms of a Henstock-Stieltjes integral, see [4] for details. This integral is approximated by Riemann sums $\sum_{n=1}^{N} g(x-z_n)[F(t_n) - F(t_{n-1})]$ where $z_n \in [t_{n-1}, t_n]$, $-\infty = t_0 < t_1 < \cdots < t_N = \infty$ and there is a gauge function γ mapping $\overline{\mathbb{R}}$ to the open intervals in $\overline{\mathbb{R}}$ such that $[t_{n-1}, t_n] \subset \gamma(z_n)$. If $z_n \notin \operatorname{supp}(f)$, then since $\mathbb{R} \setminus \operatorname{supp}(f)$ is open, there is an open interval $z_n \subset I \subset \mathbb{R} \setminus \operatorname{supp}(f)$. We can take γ such that $[t_{n-1}, t_n] \subset I$ for all $1 \leq n \leq N$. Also, F is constant on each interval in $\mathbb{R} \setminus \operatorname{supp}(f)$. Therefore, $g(x-z_n)[F(t_n) - F(t_{n-1})] = 0$ and only tags $z_n \in \operatorname{supp}(f)$ can contribute to the Riemann sum. However, for all $z_n \in \operatorname{supp}(f)$, we have $x - z_n \notin \operatorname{supp}(g)$ so $g(x-z_n)[F(t_n) - F(t_{n-1})] = 0$. It follows that $f*g(x) = 0$. □

Similar results are proven for $f \in L^p$ in [1, Section 8.2].

If we use the equivalent norm $\|f\|' = \sup_{x\in\mathbb{R}}|\int_{-\infty}^{x} f|$, then $\|\Phi_f\| = \|f\|'$. Also, integration by parts gives $\|\Phi_f\| \leq \|f\|'$. Now, given $f \in \mathcal{A}_C$, let $g = \chi_{(0,\infty)}$. Then $\|g\|_{\mathcal{BV}} = 1$, and $f*g(x) = \int_{-\infty}^{x} f$. Hence, $\|f*g\|_\infty = \|f\|'$ and $\|\Phi_f\| = \|f\|'$. We can have strict inequality in $\|\Psi_g\| \leq \|g\|_{\mathcal{BV}}$. For example, let $g = \chi_{\{0\}}$, then $\|g\|_{\mathcal{BV}} = 2$ but integration by parts shows $f*g = 0$ for each $f \in \mathcal{A}_C$.

Remark 2.2. If $f \in \mathcal{A}_C$ and $g \in \mathcal{EBV}$, one can use Definition A.2 to define $f*g(x) = f*g_\gamma(x)$ where $g_\gamma = g$ almost everywhere and $g_\gamma \in \mathcal{NBV}_\gamma$. All of the results in Theorem 2.1 and the rest of this paper have analogues. Note that $f*g(x) = F(\infty)g_\gamma(-\infty) + F*\mu_g$.

Proposition 2.3. *The three definitions of convolution for distributions in Definition 1.1 are compatible with $f*g$ for $f \in \mathcal{A}_C$ and $g \in \mathcal{BV}$.*

Proof. Let $f \in \mathcal{A}_C$, $g \in \mathcal{BV}$, and $\phi, \psi \in \mathfrak{D}$. Definition 1.1(i) gives

$$\langle f, \widetilde{\psi}*\phi\rangle = \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty}\psi(y-x)\phi(y)dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\psi(y-x)\phi(y)dx\,dy = \langle f*\psi, \phi\rangle. \tag{2.4}$$

Since $\psi \in \mathcal{BV}$ and $\phi \in L^1$, Proposition A.3 justifies the interchange of integrals. Definition 1.1(ii) gives

$$\langle f, \tau_x \widetilde{\psi} \rangle = \int_{-\infty}^{\infty} f(y)\psi(x-y)dy = f * \psi(x). \tag{2.5}$$

Definition 1.1(iii) gives

$$\begin{aligned}
\langle f(y), \langle g(x), \phi(x+y) \rangle \rangle &= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(x)\phi(x+y)dx\,dy \\
&= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(x-y)\phi(x)dx\,dy \\
&= \int_{-\infty}^{\infty} \phi(x) \int_{-\infty}^{\infty} f(y)g(x-y)dx\,dx \\
&= \langle f * g, \phi \rangle.
\end{aligned} \tag{2.6}$$

The interchange of integrals is accomplished using Proposition A.3 since $g \in \mathcal{BV}$ and $\phi \in L^1$. □

The locally integrable distributions are defined as $\mathcal{A}_C(\mathrm{loc}) = \{f \in \mathcal{D}' \mid f = F'$ for some $F \in C^0(\mathbb{R})\}$. Let $f \in \mathcal{A}_C(\mathrm{loc})$ and let $g \in \mathcal{BV}$ with support in the compact interval $[a,b]$. By the Hake theorem [4, Theorem 25], $f * g(x)$ exists if and only if the limits of $\int_\alpha^\beta f(x-y)g(y)dy$ exist as $\alpha \to -\infty$ and $\beta \to \infty$. This gives

$$f * g(x) = \int_a^b f(x-y)g(y)dy = \int_{x-b}^{x-a} f(y)g(x-y)dy. \tag{2.7}$$

There are analogues of the results in Theorem 2.1. For example, $|f * g(x)| \le |\int_{x-b}^{x-a} f| \inf_{[a,b]} |g| + \|f\chi_{[x-b,x-a]}\| V_{[a,b]} g$. There are also versions where the supports are taken to be semi-infinite intervals.

We can also define the distributions with bounded primitive as $\mathcal{A}_C(bd) = \{f \in \mathcal{D}' \mid f = F'$ for some bounded $F \in C^0(\mathbb{R})$ with $F(0) = 0\}$. Let $f \in \mathcal{A}_C(bd)$ and let F be its unique primitive. If $g \in \mathcal{BV}$ such that $g(\pm\infty) = 0$, then

$$\begin{aligned}
f * g(x) &= \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \int_\alpha^\beta f(x-y)g(y)dy \\
&= \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \left[F(x-\alpha)g(\alpha) - F(x-\beta)g(\beta) + \int_\alpha^\beta F(x-y)dg(y) \right] \\
&= \int_{-\infty}^{\infty} F(x-y)dg(y) = \int_{-\infty}^{\infty} F(y)dg(x-y).
\end{aligned} \tag{2.8}$$

It follows that $\|f * g\|_\infty \le \|F\|_\infty Vg$.

It is possible to formulate other existence criteria. For example, if $f(x) = \log|x|\sin(x)$ and $g(x) = |x|^{-\alpha}$ for some $0 < \alpha < 1$, then f and g are not in $\mathcal{A}_C$, $\mathcal{BV}$ or L^p for any $1 \le p \le \infty$ but $f * g$ exists on $\mathbb{R}$ because $f, g \in L^1_{\mathrm{loc}}$ and if $F(x) = \int_0^x f$, then $\lim_{|x|\to\infty} F(x)g(x) = 0$.

The following example shows that $f * g$ needs not to be of bounded variation and hence not absolutely continuous. Let $g = \chi_{(0,\infty)}$. For $f \in \mathcal{A}_C$, we have $f * g(x) = \int_{-\infty}^x f = F(x)$, where $F \in \mathcal{B}_C$ is the primitive of f. However, F needs not to be of bounded variation or even of local bounded variation. For example, let $f(x) = \sin(x^{-2}) - 2x^{-2}\cos(x^{-2})$ and let F be its primitive in $\mathcal{B}_C$. Finally, although $f * g$ is continuous, it needs not to be integrable over $\mathbb{R}$. For example, let $g = 1$, then $f * g(x) = \int_{-\infty}^{\infty} f$ and $\int_{-\infty}^{\infty} f * g$ only exists if $\int_{-\infty}^{\infty} f = 0$.

3. Convolution in $\mathcal{A}_C \times L^1$

We now extend the convolution $f * g$ to $f \in \mathcal{A}_C$ and $g \in L^1$. Since there are functions in L^1 that are not of bounded variation, there are distributions $f \in \mathcal{A}_C$ and functions $g \in L^1$ such that the integral $\int_{-\infty}^{\infty} f(x-y)g(y)dy$ does not exist. The convolution is then defined as the limit in $\|\cdot\|$ of a sequence $f * g_n$ for $g_n \in \mathcal{BV} \cap L^1$ such that $g_n \to g$ in the L^1 norm. This is possible since $\mathcal{BV} \cap L^1$ is dense in L^1. We also give an equivalent definition using the fact that L^1 is dense in $\mathcal{A}_C$. Take a sequence $\{f_n\} \subset L^1$ such that $\|f_n - f\| \to 0$. Then $f * g$ is the limit in $\|\cdot\|$ of $f_n * g$. In this more general setting of convolution defined in $\mathcal{A}_C \times L^1$, we now have an Alexiewicz norm estimate for $f * g$ in terms of estimates of f in the Alexiewicz norm and g in the L^1 norm. There is associativity with L^1 functions and commutativity with translations.

Definition 3.1. Let $f \in \mathcal{A}_C$ and let $g \in L^1$. Let $\{g_n\} \subset \mathcal{BV} \cap L^1$ such that $\|g_n - g\|_1 \to 0$. Define $f * g$ as the unique element in $\mathcal{A}_C$ such that $\|f * g_n - f * g\| \to 0$.

To see that the definition makes sense, first note that $\mathcal{BV} \cap L^1$ is dense in L^1 since step functions are dense in L^1. Hence, the required sequence $\{g_n\}$ exists. Let $[\alpha, \beta] \subset \mathbb{R}$ be a compact interval. Let $F \in \mathcal{B}_C$ be the primitive of f. Then

$$\begin{aligned}
\int_\alpha^\beta f * g_n(x)dx &= \int_\alpha^\beta \int_{-\infty}^{\infty} f(y) g_n(x-y)dy\,dx \\
&= \int_{-\infty}^{\infty} f(y) \int_\alpha^\beta g_n(x-y)dx\,dy
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} f(y) \int_{\alpha-y}^{\beta-y} g_n(x)dx\,dy \\
&= -\int_{-\infty}^{\infty} F(y)\ d\left[\int_{\alpha-y}^{\beta-y} g_n\right]
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} F(y)\left[g_n(\beta-y) - g_n(\alpha-y)\right]dy \\
&= \int_{-\infty}^{\infty} \left(\int_{\alpha-y}^{\beta-y} f\right) g_n(y)dy.
\end{aligned} \tag{3.3}$$

The interchange of orders of integration in (3.1) is accomplished with Proposition A.3 using $g(x,y) = g_n(x-y)\chi_{[\alpha,\beta]}(x)$. Integration by parts gives (3.2) since $\lim_{y\to\infty}\int_{\alpha-y}^{\beta-y} g_n = 0$. As F is continuous and the function $y \mapsto \int_{\alpha-y}^{\beta-y} g_n$ is absolutely continuous, we get (3.3). Taking the supremum over $\alpha, \beta \in \mathbb{R}$ gives

$$\|f * g_n\| \leq \|f\| \|g_n\|_1. \tag{3.4}$$

We now have

$$\|f * g_m - f * g_n\| = \|f * (g_m - g_n)\| \leq \|f\| \|g_m - g_n\|_1 \tag{3.5}$$

and $\{f * g_n\}$ is a Cauchy sequence in $\mathcal{A}_C$. Since $\mathcal{A}_C$ is complete, this sequence has a limit in $\mathcal{A}_C$ which we denote $f * g$. The definition does not depend on the choice of sequence $\{g_n\}$, thus if $\{h_n\} \subset \mathcal{BV} \cap L^1$ such that $\|h_n - g\|_1 \to 0$, then $\|f * g_n - f * h_n\| \leq \|f\|(\|g_n - g\|_1 + \|h_n - g\|_1) \to 0$ as $n \to \infty$. The previous calculation also shows that if $g \in \mathcal{BV} \cap L^1$, then the integral definition $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ and the limit definition agree.

Definition 3.2. Let $f \in \mathcal{A}_C$ and let $g \in L^1$. Let $\{f_n\} \subset L^1$ such that $\|f_n - f\| \to 0$. Define $f * g$ as the unique element in $\mathcal{A}_C$ such that $\|f_n * g - f * g\| \to 0$.

To show this definition makes sense, first show L^1 is dense in $\mathcal{A}_C$.

Proposition 3.3. *L^1 is dense in $\mathcal{A}_C$.*

Proof. Let $AC(\overline{\mathbb{R}})$ be the functions that are absolutely continuous on each compact interval and which are of bounded variation on the real line. Then, $f \in L^1$ if and only if there exists $F \in AC(\overline{\mathbb{R}})$ such that $F'(x) = f(x)$ for almost all $x \in \mathbb{R}$. Let $f \in \mathcal{A}_C$ be given. Let $F \in \mathcal{B}_C$ be its primitive. For $\epsilon > 0$, take $M > 0$ such that $|\mathrm{F}(x)| < \epsilon$ for $x < -M$ and $|F(x) - F(\infty)| < \epsilon$ for $x > M$. Due to the Weierstrass approximation theorem, there is a continuous function $P : \mathbb{R} \to \mathbb{R}$ such that $P(x) = F(-M)$ for $x \leq -M$, $P(x) = F(M)$ for $x \geq M$, $|P(x) - F(x)| < \epsilon$ for $|x| \leq M$ and P is a polynomial on $[-M, M]$. Hence, $P \in AC(\overline{\mathbb{R}})$ and $\|P' - f\| < 3\epsilon$. □

In Definition 3.2, the required sequence $\{f_n\} \subset L^1$ exists. Let $[\alpha, \beta] \subset \mathbb{R}$ be a compact interval. Then, by the usual Fubini-Tonelli theorem in L^1,

$$\begin{aligned} \int_{\alpha}^{\beta} f_n * g(x)dx &= \int_{\alpha}^{\beta}\int_{-\infty}^{\infty} f_n(x-y)g(y)dy\,dx \\ &= \int_{-\infty}^{\infty} g(y)\int_{\alpha}^{\beta} f_n(x-y)dx\,dy. \end{aligned} \tag{3.6}$$

Take the supremum over $\alpha, \beta \in \mathbb{R}$ and use the $L^1 - L^\infty$ Hölder inequality to get

$$\|f_n * g\| \leq \|f_n\| \|g\|_1. \tag{3.7}$$

It now follows that $\{f_n * g\}$ is a Cauchy sequence. It then converges to an element of $\mathcal{A}_C$. However, (3.7) also shows that this limit is independent of the choice of $\{f_n\}$. To see that

Definitions 3.1 and 3.2 agree, take $\{f_n\} \subset L^1$ with $\|f_n - f\| \to 0$ and $\{g_n\} \subset \mathcal{BV} \cap L^1$ with $\|g_n - g\|_1 \to 0$. Then

$$\begin{aligned}\|f_n * g - f * g_n\| &= \|(f_n - f) * g - f * (g_n - g)\| \\ &\le \|(f_n - f) * g\| + \|f * (g_n - g)\| \\ &\le \|f_n - f\|\|g\|_1 + \|f\|\|g_n - g\|_1\end{aligned} \tag{3.8}$$

Letting $n \to \infty$ shows that the limits of $f_n * g$ in Definition 3.2 and $f * g_n$ in Definition 3.1 are the same.

Theorem 3.4. *Let $f \in \mathcal{A}_C$ and $g \in L^1$. Define $f * g$ as in Definition 3.1. Then* (a)$\|f * g\| \le \|f\|\|g\|_1$. (b) *Let $h \in L^1$. Then $(f * g) * h = f * (g * h) \in \mathcal{A}_C$.* (c) *For each $z \in \mathbb{R}$, $\tau_z(f * g) = (\tau_z f) * g = (f * \tau_z g)$.* (d) *For each $f \in \mathcal{A}_C$, define $\Phi_f : L^1 \to \mathcal{A}_C$ by $\Phi_f[g] = f * g$. Then Φ_f is a bounded linear operator and $\|\Phi_f\| \le \|f\|$. There exists a nonzero distribution $f \in \mathcal{A}_C$ such that $\|\Phi_f\| = \|f\|$. For each $g \in L^1$, define $\Psi_g : \mathcal{A}_C \to \mathcal{A}_C$ by $\Psi_g[f] = f * g$. Then Ψ_g is a bounded linear operator and $\|\Psi_g\| \le \|g\|_1$. There exists a nonzero function $g \in L^1$ such that $\|\Psi_g\| = \|g\|_{\mathcal{BV}}$.* (e) *Define $g_t(x) = g(x/t)/t$ for $t > 0$. We have $a = \int_{-\infty}^{\infty} g_t(x)dx = \int_{-\infty}^{\infty} g$. Then $\|f * g_t - af\| \to 0$ as $t \to 0$.* (f) *Let* $\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g)$.

Proof. Let $\{g_n\}$ be as in Definition 3.1. (a) Since $\|f * g_n\| \to \|f * g\|$, (3.4) shows $\|f * g\| \le \|f\|\|g\|_1$. (b) Let $\{h_n\} \subset \mathcal{BV} \cap L^1$ such that $\|h_n - h\|_1 \to 0$. Then $(f * g) * h := \xi \in \mathcal{A}_C$ such that $\|(f * g) * h_n - \xi\| \to 0$. Since $g * h \in L^1$, there is $\{p_n\} \subset \mathcal{BV} \cap L^1$ such that $\|p_n - g * h\|_1 \to 0$. Then $f * (g * h) := \eta \in \mathcal{A}_C$ such that $\|f * p_n - \eta\| \to 0$. Now,

$$\begin{aligned}\|\xi - \eta\| \le{}& \|(f * g) * h_n - \xi\| + \|f * p_n - \eta\| \\ &+ \|(f * g) * h_n - (f * g_n) * h_n\| + \|(f * g_n) * h_n - f * p_n\|.\end{aligned} \tag{3.9}$$

Using (3.4),

$$\begin{aligned}&\|(f * g) * h_n - (f * g_n) * h_n\| \\ &\quad = \|[f * (g - g_n)] * h_n\| \le \|f\|\|g_n - g\|_1\|h_n\|_1 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.10}$$

Finally, use Theorem 2.1(e) and (3.4) to write

$$\begin{aligned}&\|(f * g_n) * h_n - f * p_n\| \\ &\quad = \|f * (g_n * h_n - p_n)\| \\ &\quad \le \|f\|(\|g_n - g\|_1\|h_n\|_1 + \|g\|_1\|h_n - h\|_1 + \|p_n - g * h\|_1) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.11}$$

(c) The Alexiewicz norm is invariant under translation [4, Theorem 28] so $\tau_z(f * g) \in \mathcal{A}_C$. Use Theorem 2.1(f) to write $\|\tau_z(f * g) - \tau_z(f * g_n)\| = \|f * g - f * g_n\| = \|\tau_z(f * g) - (\tau_z f) * g_n)\| = \|\tau_z(f * g) - f * (\tau_z g_n)\|$. Translation invariance of the L^1 norm completes the proof. (d) From (a), we have $\|\Phi_f\| = \sup_{\|g\|_1 = 1}\|f * g\| \le \sup_{\|g\|_1 = 1}\|f\|\|g\|_1 = \|f\|$. We get equality by considering f

and g to be positive functions in L^1. To prove $\|\Psi_g\| \leq \|g\|_1$, note that $\|\Psi_g\| = \sup_{\|f\|=1}\|f * g\| \leq \sup_{\|f\|=1}\|f\|\|g\|_1 = \|g\|_1$. We get equality by considering f and g to be positive functions in L^1. (e) First consider $g \in \mathcal{BV} \cap L^1$. We have

$$f * g_t(x) = \int_{-\infty}^{\infty} f(x-y)g\left(\frac{y}{t}\right)\frac{dy}{t} = \int_{-\infty}^{\infty} f(x-ty)g(y)dy. \tag{3.12}$$

For $-\infty < \alpha < \beta < \infty$,

$$\begin{aligned}\left|\int_{\alpha}^{\beta}[f * g_t(x) - a\ f(x)]dx\right| &= \left|\int_{\alpha}^{\beta}\int_{-\infty}^{\infty}[f(x-ty) - f(x)]g(y)dy\,dx\right| \\ &= \left|\int_{-\infty}^{\infty}\int_{\alpha}^{\beta}[f(x-ty) - f(x)]g(y)dx\,dy\right|\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\leq \int_{-\infty}^{\infty}\|\tau_{ty}f - f\||g(y)|dy \\ &\leq 2\|f\|\|g\|_1.\end{aligned} \tag{3.14}$$

By dominated convergence, we can take the limit $t \to 0$ inside the integral (3.14). Continuity of f in the Alexiewicz norm then shows $\|f * g_t - af\| \to 0$ as $t \to 0$.

Now take a sequence $\{g^{(n)}\} \subset \mathcal{BV} \cap L^1$ such that $\|g^{(n)} - g\|_1 \to 0$. Define $g_t^{(n)}(x) = g^{(n)}(x/t)/t$ and $a^{(n)} = \int_{-\infty}^{\infty} g^{(n)}(x)dx$. We have

$$\|f * g_t - af\| \leq \left\|f * g_t^{(n)} - a^{(n)}f\right\| + \left\|f * g_t^{(n)} - f * g_t\right\| + \left\|a^{(n)}f - af\right\|. \tag{3.15}$$

By the inequality in (a), $\|f * g_t^{(n)} - f * g_t\| \leq \|f\|\|g_t^{(n)} - g_t\|_1$. Whereas,

$$\left\|g_t^{(n)} - g_t\right\|_1 = \int_{-\infty}^{\infty}\left|g^{(n)}\left(\frac{x}{t}\right) - g\left(\frac{x}{t}\right)\right|\frac{dx}{t} = \left\|g^{(n)} - g\right\|_1 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.16}$$

and $\|a^{(n)}f - af\| = |a^{(n)} - a|\|f\| = \|g^{(n)} - g\|_1\|f\|$. Given $\epsilon > 0$ fix n large enough so that $\|f * g_t^{(n)} - f * g_t\| + \|a^{(n)}f - af\| < \epsilon$. Now let $t \to 0$ in (3.15).

The interchange of order of integration in (3.13) is justified as follows. A change of variables and Proposition A.3 give

$$\begin{aligned}\int_{\alpha}^{\beta}\int_{-\infty}^{\infty} f(x-ty)g(y)dy\,dx &= \int_{\alpha}^{\beta}\int_{-\infty}^{\infty} f(y)g\left(\frac{x-y}{t}\right)\frac{dy}{t}dx \\ &= \int_{-\infty}^{\infty}\int_{\alpha}^{\beta} f(y)g\left(\frac{x-y}{t}\right)dx\frac{dy}{t},\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{\alpha}^{\beta} f(x-ty)g(y)\,dx\,dy &= \iint_{-\infty}^{\infty} f(x)\ g\left(\frac{y}{t}\right)\chi_{(\alpha-y,\beta-y)}(x)\,dx\frac{dy}{t} \\
&= \iint_{-\infty}^{\infty} f(x)\ g\left(\frac{y}{t}\right)\chi_{(\alpha-y,\beta-y)}(x)\frac{dy}{t}\,dx \\
&= \int_{-\infty}^{\infty}\int_{\alpha}^{\beta} f(x)\ g\left(\frac{y-x}{t}\right)\frac{dy}{t}\,dx.
\end{aligned} \tag{3.17}$$

Note that $\int_{\alpha}^{\beta}\int_{-\infty}^{\infty} f(x)g(y)\,dy\,dx = \int_{-\infty}^{\infty}\int_{\alpha}^{\beta} f(x)g(y)\,dx\,dy$ by Corollary A.4. (f) This follows from the equivalence of Definitions 1.1 and 3.1, proved in Proposition 3.5, see [6, Theorems 5.4-2 and 5.3-1]. □

Young's inequality states that $\|f * g\|_p \le \|f\|_p\|g\|_1$ when $f \in L^p$ for some $1 \le p \le \infty$ and $g \in L^1$. Part (a) of Theorem 3.4 extends this to $f \in \mathcal{A}_C$. see [1] for other results when $f \in L^p$.

The fact that convolution is linear in both arguments, together with (b), shows that $\mathcal{A}_C$ is an L^1-module over the L^1 convolution algebra, see [8] for the definition. It does not appear that $\mathcal{A}_C$ is a Banach algebra under convolution.

We now show that Definition 1.1(iii) and the aforementioned definitions agree.

Proposition 3.5. *Let $f \in \mathcal{A}_C$, $g \in L^1$, and $\phi \in \mathfrak{D}$. Define $F(y) = \int_{-\infty}^{y} f$ and $G(x) = \int_{-\infty}^{x} g$. Definitions 1.1 and 3.1 both give*

$$\begin{aligned}
\langle f * g, \phi\rangle &= \int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} g(x)\phi(x+y)\,dx\,dy \\
&= \iint_{-\infty}^{\infty} F(y)G(x)\phi'(x+y)\,dx\,dy.
\end{aligned} \tag{3.18}$$

Proof. Let $\Phi(y) = \int_{-\infty}^{\infty} g(x)\phi(x+y)\,dx$. Then $\Phi \in C^{\infty}(\mathbb{R})$ and $\Phi'(y) = \int_{-\infty}^{\infty} g(x)\phi'(x+y)\,dx$. Also, $\int_{-\infty}^{\infty}|\Phi'(y)|\,dy \le \int_{-\infty}^{\infty}|g(x)|\int_{-\infty}^{\infty}|\phi'(x+y)|\,dy\,dx \le \|g\|_1\|\phi'\|_1$, so $\Phi \in AC(\overline{\mathbb{R}})$. Dominated convergence then shows $\lim_{|y|\to\infty}\Phi(y) = 0$. Integration by parts now gives (3.18).

Let $\{g_n\} \subset \mathcal{BV} \cap L^1$ such that $\|g_n - g\|_1 \to 0$. Since convergence in $\|\cdot\|$ implies convergence in $\mathfrak{D}'$, we have

$$\begin{aligned}
\langle f * g, \phi\rangle &= \lim_{n\to\infty}\langle f * g_n, \phi\rangle \\
&= \lim_{n\to\infty}\iint_{-\infty}^{\infty} f(y)g_n(x-y)\phi(x)\,dy\,dx \\
&= \lim_{n\to\infty}\int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} g_n(x-y)\phi(x)\,dx\,dy.
\end{aligned} \tag{3.19}$$

Proposition A.3 allows interchange of the iterated integrals. Define $\Phi_n(y) = \int_{-\infty}^{\infty} g_n(x)\phi(x+y)\,dx$. Then, $V\Phi_n \le \|g_n\|_1\|\phi'\|_1 \le (\|g\|_1 + 1)\|\phi'\|_1$ for large enough n. Hence, Φ_n is of uniform bounded variation. Theorem 22 in [4], then gives $\langle f * g, \phi\rangle = \int_{-\infty}^{\infty} f(y)\lim_{n\to\infty}\Phi_n(y)\,dy = \int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} g(x-y)\phi(x)dx\,dy$. The last step follows since $\|g_n - g\|_1 \to 0$. □

If $g \in L^1 \setminus \mathcal{BV}$, then $f * g$ needs not to be continuous or bounded. For example, take $1/2 \le \alpha < 1$ and let $f(x) = g(x) = x^{-\alpha}\chi_{(0,1)}(x)$. Then, $f \in L^1 \subset \mathcal{A}_C$ and $g \in L^1 \setminus \mathcal{BV}$. We have $f * g(x) = 0$ for $x \le 0$. For $0 < x \le 1$, we have $f * g(x) = \int_0^x y^{-\alpha}(x-y)^{-\alpha}dy = x^{1-2\alpha}\int_0^1 y^{-\alpha}(1-y)^{-\alpha}dy = x^{1-2\alpha}\Gamma^2(1-\alpha)/\Gamma(2-2\alpha)$. Hence, $f * g$ is not continuous at 0. If $1/2 < \alpha < 1$, then $f * g$ is unbounded at 0.

As another example, consider $f(x) = \sin(\pi x)/\log|x|$ and $g(x) = \chi_{(0,1)}(x)$. Then $f \in \mathcal{A}_C$ and for each $1 \le p \le \infty$, we have $g \in \mathcal{BV} \cap L^p$. And,

$$\begin{aligned} f * g(x) &= \int_{x-1}^{x} \frac{\sin(\pi y)}{\log(y)}dy \quad \text{for } x \ge 2 \\ &= \frac{\cos(\pi(x-1))}{\pi\log(x-1)} - \frac{\cos(\pi x)}{\pi\log(x)} - \frac{1}{\pi}\int_{x-1}^{x}\frac{\cos(\pi y)}{y\log^2(y)}dy \sim -\frac{2\cos(\pi x)}{\pi\log(x)} \quad \text{as } x \longrightarrow \infty. \end{aligned} \tag{3.20}$$

Therefore, by Theorem 2.1(d), $f * g \in C^0(\overline{\mathbb{R}})$, and $\lim_{|x|\to\infty} f * g(x) = 0$ but for each $1 \le p < \infty$, we have $f * g \notin L^p$.

4. Differentiation and Integration

If g is sufficiently smooth, then the pointwise derivative is $(f * g)'(x) = f * g'(x)$. Recall the definition $AC(\overline{\mathbb{R}})$ of primitives of L^1 functions given in the proof of Proposition 3.3. In the following theorem, we require pointwise derivatives of g to exist at each point in $\mathbb{R}$.

Theorem 4.1. *Let $f \in \mathcal{A}_C$, $n \in \mathbb{N}$, and $g^{(k)} \in AC(\overline{\mathbb{R}})$ for each $0 \le k \le n$. Then $f * g \in C^n(\mathbb{R})$ and $(f * g)^{(n)}(x) = f * g^{(n)}(x)$ for each $x \in \mathbb{R}$.*

Proof. First consider $n = 1$. Let $x \in \mathbb{R}$. Then

$$(f * g)'(x) = \lim_{h\to 0}\int_{-\infty}^{\infty} f(y)\left[\frac{g(x+h-y) - g(x-y)}{h}\right]dy. \tag{4.1}$$

To take the limit inside the integral we can show that the bracketed term in the integrand is of uniform bounded variation for $0 < |h| \le 1$. Let $h \ne 0$. Since $g \in AC(\overline{\mathbb{R}})$ it follows that the variation is given by the Lebesgue integrals

$$\begin{aligned} &V_{y\in\mathbb{R}}\left[\frac{g(x+h-y) - g(x-y)}{h}\right] \\ &\quad = \int_{-\infty}^{\infty}\left|\frac{g'(x+h-y) - g'(x-y)}{h}\right|dy \\ &\quad \le \int_{-\infty}^{\infty}|g''(y)|dy + \int_{-\infty}^{\infty}\left|\frac{g'(x+h-y) - g'(x-y)}{h} - g''(x-y)\right|dy. \end{aligned} \tag{4.2}$$

Since $g' \in AC(\overline{\mathbb{R}})$, we have $g'' \in L^1$. The second integral on the right of (4.2) gives the L^1 derivative of g' in the limit $h \to 0$; see [1, page 246]. Hence, in (4.1), we can use [4, Theorem 22] to take the limit under the integral sign. This then gives $(f * g)'(x) = f * g'(x)$. Theorem 2.1(d) now shows $(f * g)' \in C^0(\overline{\mathbb{R}})$. Induction on n completes the proof. □

For similar results when $f \in L^1$, see [1, Proposition 8.10].

Note that $g' \in AC(\overline{\mathbb{R}})$ does not imply $g \in AC(\overline{\mathbb{R}})$. For example, $g(x) = x$. The conditions $g^{(k)} \in \mathcal{BV}$ for $0 \le k \le n+1$ imply those in Theorem 4.1. To see this, it suffices to consider $n = 1$. If $g', g'' \in \mathcal{BV}$, then g'' exists at each point and is bounded. Hence, the Lebesgue integral $g'(x) = g'(0) + \int_0^x g''(y)dy$ exists for each $x \in \mathbb{R}$ and g' is absolutely continuous. Since $g' \in \mathcal{BV}$, we then have $g' \in AC(\overline{\mathbb{R}})$. Similarly, for $n > 1$. The example $g(x) = |x|^{1.5}\sin(1/[1+x^2])$ shows that the $AC(\overline{\mathbb{R}})$ condition in the theorem is weaker than the aforementioned $\mathcal{BV}$ condition since $g, g' \in AC(\overline{\mathbb{R}})$ but $g''(0)$ does not exist so $g'' \notin \mathcal{BV}$.

We found that when $g \in \mathcal{BV} \cap L^1$, then $f * g \in \mathcal{A}_C$. We can compute the distributional derivative $(F * g)' = f * g$, where F is a primitive of f.

Proposition 4.2. *Let $F \in C^0(\overline{\mathbb{R}})$ and write $f = F' \in \mathcal{A}_C$. Let $g \in \mathcal{BV} \cap L^1$. Then $F * g \in C^0(\overline{\mathbb{R}})$ and $(F * g)' = f * g \in \mathcal{A}_C$.*

Proof. Let $x, t \in \mathbb{R}$. Then by the usual Hölder inequality,

$$\begin{aligned}
&\left|F * g(x) - F * g(t)\right| \\
&\quad = \left|\int_{-\infty}^{\infty} [F(x-y) - F(t-y)]g(y)dy\right| \\
&\quad \le \|F(x-\cdot) - F(t-\cdot)\|_\infty \|g\|_1 \longrightarrow 0 \quad \text{as } t \longrightarrow x \text{ since } F \text{ is uniformly continuous on } \overline{\mathbb{R}}.
\end{aligned} \tag{4.3}$$

Hence, $F * g$ is continuous on $\mathbb{R}$. Dominated convergence shows that $\lim_{x\to\pm\infty} F * g(x) = F(\pm\infty)\int_{-\infty}^{\infty} g$. Therefore, $F * g \in C^0(\overline{\mathbb{R}})$.

Let $\phi \in \mathfrak{D}$. Then

$$\begin{aligned}
\left\langle (F * g)', \phi \right\rangle &= -\langle F * g, \phi' \rangle \\
&= -\iint_{-\infty}^{\infty} F(x-y)g(y)\phi'(x)dy\,dx \\
&= -\int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} F(x-y)\phi'(x)dx\,dy \quad \text{(Fubini-Tonelli theorem).}
\end{aligned} \tag{4.4}$$

Integrate by parts and use the change of variables $x \mapsto x + y$ to get

$$\begin{aligned}
\left\langle (F * g)', \phi \right\rangle &= \int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} f(x)\phi(x+y)dx\,dy \\
&= \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} g(y)\phi(x+y)dy\,dx \quad \text{(by Proposition A.3)}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} g(y-x)\phi(y)\,dy\,dx \\
&= \int_{-\infty}^{\infty} \phi(y) \int_{-\infty}^{\infty} f(x)g(y-x)\,dx\,dy \quad \text{(by Proposition A.3)} \\
&= \langle f * g, \phi \rangle.
\end{aligned}
\tag{4.5}
$$

□

This gives an alternate definition of $f * g$ for $f \in \mathcal{A}_C$ and $g \in L^1$.

Theorem 4.3. *Let $f \in \mathcal{A}_C$, let $F \in \mathcal{B}_C$ be the primitive of f and let $g \in L^1$. Define $f * g$ as in Definition 3.1. Then $(F * g)' = f * g \in \mathcal{A}_C$.*

Proof. Let $-\infty < \alpha < \beta < \infty$. Let $\{g_n\} \subset \mathcal{BV} \cap L^1$ such that $\|g_n - g\|_1 \to 0$. By Proposition 4.2, we have

$$
\int_{\alpha}^{\beta} (F * g)' = F * g(\beta) - F * g(\alpha) = \int_{-\infty}^{\infty} F(y)\,[g(\beta - y) - g(\alpha - y)]\,dy. \tag{4.6}
$$

As in (3.3), $\int_{\alpha}^{\beta} f * g_n = \int_{-\infty}^{\infty} F(y)[g_n(\beta - y) - g_n(\alpha - y)]\,dy$. Hence,

$$
\begin{aligned}
&\left| \int_{\alpha}^{\beta} \left[(F * g)' - f * g_n \right] \right| \\
&\quad = \left| \int_{-\infty}^{\infty} F(y)\,[(g(\beta - y) - g_n(\beta - y)) - (g(\alpha - y) - g_n(\alpha - y))]\,dy \right| \\
&\quad \le \|F\|_{\infty} \left(\|g(\beta - \cdot) - g_n(\beta - \cdot)\|_1 + \|g(\alpha - \cdot) - g_n(\alpha - \cdot)\|_1 \right) \\
&\quad = 2\|f\|\,\|g_n - g\|_1.
\end{aligned}
\tag{4.7}
$$

Therefore, $\|(F * g)' - f * g_n\| \le 2\|f\|\|g_n - g\|_1 \to 0$ as $n \to \infty$. □

The following theorem and its corollary give results on integrating convolutions.

Theorem 4.4. *Let $f \in \mathcal{A}_C$ and let $g \in L^1$. Define $F(x) = \int_{-\infty}^{x} f$ and $G(x) = \int_{-\infty}^{x} g$. Then, $f * G \in C^0(\overline{\mathbb{R}})$ and $f * G(x) = F * g(x)$ for all $x \in \mathbb{R}$.*

Proof. Since $G \in AC(\overline{\mathbb{R}})$, Theorem 2.1(d) shows $f * G \in C^0(\overline{\mathbb{R}})$. We have

$$
\begin{aligned}
f * G(x) &= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{x-y} g(z)\,dz\,dy \\
&= \iint_{-\infty}^{\infty} f(y)\chi_{(-\infty, x-y)}(z)g(z)\,dz\,dy
\end{aligned}
$$

$$
\begin{aligned}
&= \iint_{-\infty}^{\infty} f(y)\chi_{(-\infty,x-y)}(z)g(z)\,dy\,dz \\
&= \int_{-\infty}^{\infty} g(z)\int_{-\infty}^{x-z} f(y)\,dy\,dz \\
&= F * g(x).
\end{aligned}
\tag{4.8}
$$

Proposition A.3 justifies the interchange of orders of integration. □

Corollary 4.5. *Note that the following hold:* (a) $f * g = (F * g)' = (f * G)'$, (b) *for all* $-\infty \le \alpha < \beta \le \infty$, *one has* $\int_\alpha^\beta f * g = F * g(\beta) - F * g(\alpha) = f * G(\beta) - f * G(\beta)$.

Hence, the convolution $f * g$ can be evaluated by taking the distributional derivative of the Lebesgue integral $F * g$. Since $f * G \in C^0(\overline{\mathbb{R}})$, when $f \in \mathcal{A}_C$ and $G \in \mathcal{BV}$, we can use the equation $f * g = (f * G)'$ to define $f * g$ for $f \in \mathcal{A}_C$ and $g = G'$ for $G \in \mathcal{BV}$. In this case, g will be a signed Radon measure. As $G(x) = \int_{-\infty}^{x} g$ and this integral is a regulated primitive integral [5], we will save this case for discussion elsewhere.

Appendix

The integration by parts formula is as follows. If $f \in \mathcal{A}_C$ and $g \in \mathcal{BV}$, it gives the integral of fg in terms of a Henstock-Stieltjes integral:

$$
\int_{-\infty}^{\infty} fg = F(\infty)g(\infty) - \int_{-\infty}^{\infty} F\,dg,
\tag{A.1}
$$

see [4] and [9, page 199].

We have the following corollary for functions of essential bounded variation.

Corollary A.1. *Let* $F \in C^0(\overline{\mathbb{R}})$. *Let* $g \in \mathcal{EBV}$. *Fix* $0 \le \gamma \le 1$. *Take* $g_\gamma \in \mathcal{NBV}_\gamma$ *such that* $g_\gamma = g$ *almost everywhere. Let* μ_g *be the signed Radon measure given by* g'. *Then* $\int_{-\infty}^{\infty} F\,dg_\gamma = \int_{-\infty}^{\infty} F\,d\mu_g$.

Proof. The distributional derivative of g is $\langle g', \phi\rangle = -\langle g, \phi'\rangle = -\int_{-\infty}^{\infty} g\phi' = \int_{-\infty}^{\infty} \phi\,d\mu_g$ for all $\phi \in \mathfrak{D}$. Note that g_γ is unique and $\mu_g = \mu_{g_\gamma}$. Suppose $\phi \in \mathfrak{D}$ with $\mathrm{supp}(\phi) \subset [A, B] \subset \mathbb{R}$. Then, using integration by parts for the Henstock-Stieltjes integral:

$$
\begin{aligned}
\langle g_\gamma, \phi'\rangle &= \int_A^B g_\gamma\phi' = g_\gamma(B)\phi(B) - g_\gamma(A)\phi(A) - \int_A^B \phi\,dg_\gamma = -\int_{-\infty}^{\infty} \phi\,dg_\gamma \\
&= -\langle g_\gamma', \phi\rangle = -\int_{-\infty}^{\infty} \phi\,d\mu_{g_\gamma} = -\int_{-\infty}^{\infty} \phi\,d\mu_g.
\end{aligned}
\tag{A.2}
$$

Let $F \in C^0(\overline{\mathbb{R}})$. There is a uniformly bounded sequence $\{\phi_n\} \subset \mathfrak{D}$ such that $\phi_n \to F$ pointwise on $\mathbb{R}$. By dominated convergence,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\phi_n\, dg_\gamma = \int_{-\infty}^{\infty} F\, dg_\gamma = \lim_{n\to\infty}\int_{-\infty}^{\infty}\phi_n\, d\mu_g = \int_{-\infty}^{\infty} F\, d\mu_g. \tag{A.3}$$

□

Corollary A.1 now justifies the following definition.

Definition A.2. Let $f \in \mathcal{A}_C$ and let $F \in \mathcal{B}_C$ be its primitive. Let $g \in \mathcal{EBV}$. Fix $0 \le \gamma \le 1$ and take $g_\gamma \in \mathcal{NBV}_\gamma$ such that $g_\gamma = g$ almost everywhere. Define

$$\int_{-\infty}^{\infty} fg = g_\gamma(\infty)F(\infty) - \int_{-\infty}^{\infty} F\, d\mu_g = \int_{-\infty}^{\infty} fg_\gamma. \tag{A.4}$$

Since limits at infinity are not affected by the choice of γ, the definition is independent of γ.

The Hölder inequality is

$$\left|\int_{-\infty}^{\infty} fg\right| \le \left|\int_{-\infty}^{\infty} f\right| \inf_{\mathbb{R}}|g| + \|f\| Vg \le \|f\| \|g\|_{\mathcal{BV}}, \tag{A.5}$$

and is valid for all $f \in \mathcal{A}_C$ and $g \in \mathcal{BV}$. For $g \in \mathcal{EBV}$, we replace g with g_γ. This gives

$$\left|\int_{-\infty}^{\infty} fg\right| \le \left|\int_{-\infty}^{\infty} f\right| \inf_{\mathbb{R}}|g_\gamma| + \|f\| Vg_\gamma \le \|f\| \|g\|_{\mathcal{EBV}}, \tag{A.6}$$

see [2, Lemma 24] for a proof using the Henstock-Kurzweil integral. The same proof works for the continuous primitive integral.

Fubini theorem has been established in [10] for the continuous primitive integral on compact intervals. This says that if a double integral exists in the plane, then the two iterated integrals exist and are equal. Of more utility for the case at hand is to show directly that iterated integrals are equal without resorting to the double integral. The following theorem extends a type of Fubini theorem proved in [11, page 58] for the wide Denjoy integral on compact intervals.

Proposition A.3. *Let $f \in \mathcal{A}_C$. Let $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be measurable. Assume* (i) *for each $x \in \mathbb{R}$ the function $y \mapsto g(x,y)$ is in $\mathcal{BV}$;* (ii) *the function $x \mapsto V_{y\in\mathbb{R}}\, g(x,y)$ is in L^1;* (iii) *there is $M \in L^1$ such that for each $y \in \mathbb{R}$, one has $|g(x,y)| \le M(x)$. Then the iterated integrals exist and are equal, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)g(x,y)dy\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)g(x,y)dx\, dy$.*

Proof. Let $F \in \mathcal{B}_C$ be the primitive of f. For each $x \in \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} f(y)g(x,y)dy = F(\infty)g(x,\infty) - \int_{-\infty}^{\infty} F(y)d_2 g(x,y), \tag{A.7}$$

where $d_2(x,y)$ indicates a Henstock-Stieltjes integral with respect to y. Then,

$$\iint_{-\infty}^{\infty} f(y)g(x,y)dy\,dx = F(\infty)\int_{-\infty}^{\infty} g(x,\infty)dx - \iint_{-\infty}^{\infty} F(y)d_2 g(x,y)dx. \tag{A.8}$$

The integral $\int_{-\infty}^{\infty} g(x,\infty)\,dx$ exists due to condition (iii). The iterated integral in (A.8) converges absolutely since

$$\left|\iint_{-\infty}^{\infty} F(y)d_2 g(x,y)dx\right| \le \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} F(y)\; d_2 g(x,y)\right|dx \le \|F\|_\infty \int_{-\infty}^{\infty} V_{y\in\mathbb{R}}\, g(x,y)dx. \tag{A.9}$$

Now, show the function $y \mapsto \int_{-\infty}^{\infty} g(x,y)dx$ is in $\mathcal{BV}$. Let $\{(s_i,t_i)\}_{i=1}^n$ be disjoint intervals in $\mathbb{R}$. Then

$$\begin{aligned}\sum_{i=1}^n \left|\int_{-\infty}^{\infty} g(x,s_i)dx - \int_{-\infty}^{\infty} g(x,t_i)dx\right| &\le \sum_{i=1}^n \int_{-\infty}^{\infty} |g(x,s_i) - g(x,t_i)|dx \\ &= \int_{-\infty}^{\infty} \sum_{i=1}^n |g(x,s_i) - g(x,t_i)|dx \\ &\le \int_{-\infty}^{\infty} V_{y\in\mathbb{R}}\; g(x,y)dx.\end{aligned} \tag{A.10}$$

The interchange of summation and integration follows from condition (ii) and the usual Fubini-Tonelli theorem. Hence, the function $y \mapsto \int_{-\infty}^{\infty} g(x,y)dx$ is in $\mathcal{BV}$ and the iterated integral $\int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} g(x,y)dx\,dy$ exists.

Integrate by parts

$$\int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} g(x,y)dx\,dy \tag{A.11}$$

$$= F(\infty)\int_{-\infty}^{\infty} g(x,\infty)dx - \int_{-\infty}^{\infty} F(y)d\left[\int_{-\infty}^{\infty} g(x,y)dx\right]. \tag{A.12}$$

In (A.11), we have $\lim_{y\to\infty}\int_{-\infty}^{\infty} g(x,y)dx = \int_{-\infty}^{\infty} g(x,\infty)dx$ due to dominated convergence and condition (iii). To complete the proof, we need to show that the integrals in (A.8) and (A.12) are equal. First, consider the case when $F = \chi_{(a,b)}$ for an interval $(a,b) \subset \mathbb{R}$. Then (A.8) becomes $\int_{-\infty}^{\infty}\int_a^b d_2 g(x,y)dx = \int_{-\infty}^{\infty}[g(x,b) - g(x,a)]dx$. Moreover, now (A.12) becomes $\int_a^b d[\int_{-\infty}^{\infty} g(x,y)dx] = \int_{-\infty}^{\infty} g(x,b)dx - \int_{-\infty}^{\infty} g(x,a)dx$. Hence, when F is a step function, $F(y) = \sum_{i=1}^n c_i\chi_{I_i}(y)$ for some $n \in \mathbb{N}$, disjoint intervals $\{I_i\}_{i=1}^n$ and real numbers $\{c_i\}_{i=1}^n$, we have the desired equality of (A.8) and (A.12). However, $F \in \mathcal{B}_C$ is uniformly continuous on $\overline{\mathbb{R}}$, that is, for each $\epsilon > 0$, there is $\delta > 0$ such that for all $0 \le |x-y| < \delta$, we have $|F(x)-F(y)| < \epsilon$, for all $x < -1/\delta$, we have $|F(x)| < \epsilon$ and for all $x > 1/\delta$, we have $|F(x) - F(\infty)| < \epsilon$. It then follows from the compactness of $\overline{\mathbb{R}}$ that the step functions are dense in $\mathcal{B}_C$. Hence, there is a sequence of step functions $\{\sigma_N\}$ such that $\|F - \sigma_N\|_\infty \to 0$. In (A.8), we have

$$\lim_{N\to\infty}\iint_{-\infty}^{\infty}\sigma_N(y)d_2g(x,y)dx = \iint_{-\infty}^{\infty}F(y)d_2g(x,y)dx. \tag{A.13}$$

The N limit can be brought inside the x integral using dominated convergence and (ii) since $|\int_{-\infty}^{\infty}\sigma_N(y)d_2g(x,y)| \le (\|F\|_\infty+1)V_{y\in\mathbb{R}}\, g(x,y)$ for large enough N. The N limit can be brought inside the y integral using dominated convergence since $|\sigma_N(y)| \le (\|F\|_\infty+1)$ for large enough N. In (A.12), we have

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\sigma_N(y)d\left[\int_{-\infty}^{\infty}g(x,y)dx\right] = \int_{-\infty}^{\infty}F(y)d\left[\int_{-\infty}^{\infty}g(x,y)dx\right]. \tag{A.14}$$

The N limit can be brought inside the y integral since $\{\sigma_N\}$ converges to F uniformly on $\overline{\mathbb{R}}$ and $d[\int_{-\infty}^{\infty}g(x,y)dx]$ is a finite-signed measure. □

Corollary A.4. *If f has compact support, one can replace (iii) with (iv): for each $y \in \operatorname{supp}(f)$ the function $x \mapsto g(x,y)$ is in L^1.*

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