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Research Article

A New 4-Point C^3 Quaternary Approximating Subdivision Scheme

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A new 4-point C^3 quaternary approximating subdivision scheme with one shape parameter is proposed and analyzed. Its smoothness and approximation order are higher but support is smaller in comparison with the existing binary and ternary 4-point subdivision schemes.

1. Introduction

Computer Aided Geometric Design (CAGD) is a branch of applied mathematics concerned with algorithms for the design of smooth curves/surfaces. One common approach to the design of curves/surfaces related to CAGD is the subdivision scheme. It is an algorithm to generate smooth curves and surfaces as a sequence of successively refined control polygons. At each refinement level, new points are added into the existing polygon and the original points remain existed or discarded in all subsequent sequences of control polygons. The number of points inserted at level $k+1$ between two consecutive points from level k is called arity of the scheme. In the case when a number of points inserted are $2, 3, \ldots, n$ the subdivision schemes are called binary, ternary, $\ldots$, n-ary, respectively. An important review of the different subdivision schemes, which range from binary to any arity, can be found in [1–3]. Due to good properties of the 4-point binary and ternary subdivision schemes [4–9], much attention has been given to extend their ability in modelling curves and surfaces. For example, the 4-point ternary interpolating subdivision scheme [8] can generate higher smoothness than the 4-point binary one [6] by using the same number of control points.

Now a days, the variety of subdivision schemes investigated; our interest is in the direction of quaternary schemes. The goal of this paper is to construct 4-point quaternary subdivision scheme having the higher smoothness and approximation order but smaller support than existing 4-point binary and ternary schemes.

Here we present a 4-point quaternary approximating subdivision scheme. A polygon $f^k = \{f_i^k\}_{i\in\mathbb{Z}}$ is mapped to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbb{Z}}$ by applying the following four subdivision rules:

$$\begin{aligned}
f_{4i}^{k+1} &= \alpha_1 f_{i-1}^k + \alpha_2 f_i^k + \alpha_3 f_{i+1}^k + \alpha_4 f_{i+2}^k, \\
f_{4i+1}^{k+1} &= \alpha_5 f_{i-1}^k + \alpha_6 f_i^k + \alpha_7 f_{i+1}^k + \alpha_8 f_{i+2}^k, \\
f_{4i+2}^{k+1} &= \alpha_8 f_{i-1}^k + \alpha_7 f_i^k + \alpha_6 f_{i+1}^k + \alpha_5 f_{i+2}^k, \\
f_{4i+3}^{k+1} &= \alpha_4 f_{i-1}^k + \alpha_3 f_i^k + \alpha_2 f_{i+1}^k + \alpha_1 f_{i+2}^k,
\end{aligned} \tag{1.1}$$

where the weights $\{\alpha_j\}$ are given by

$$\begin{aligned}
&\alpha_1 = \frac{7}{32} - \frac{7}{64}\omega, \quad \alpha_2 = \frac{29}{64} + \frac{13}{64}\omega, \quad \alpha_3 = \frac{5}{16} - \frac{5}{64}\omega, \quad \alpha_4 = \frac{1}{64} - \frac{1}{64}\omega, \\
&\alpha_5 = \frac{15}{128} - \frac{5}{64}\omega, \quad \alpha_6 = \frac{57}{128} + \frac{7}{64}\omega, \quad \alpha_7 = \frac{49}{128} + \frac{1}{64}\omega, \quad \alpha_8 = \frac{7}{128} - \frac{3}{64}\omega.
\end{aligned} \tag{1.2}$$

The paper is organized as follows. In Section 2 we list all the basic facts about quaternary subdivision schemes needed in the paper. Sections 3 and 4 are devoted for analysis of proposed scheme and its properties, respectively. Finally, in Section 5, comparison of our scheme with other existing 4-point schemes is presented. Some examples reflecting the performance of our scheme by setting the shape parameter to various values are also offered.

2. Preliminaries

A general compact form of univariate quaternary subdivision scheme S which maps a polygon $f^k = \{f_i^k\}_{i\in\mathbb{Z}}$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbb{Z}}$ is defined by

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} \alpha_{i-4j} f_j^k, \quad i \in \mathbb{Z}, \tag{2.1}$$

where the set $\alpha = \{\alpha_i : i \in \mathbb{Z}\}$ of coefficients is called mask of the scheme. A necessary condition for the uniform convergence of the subdivision scheme (2.1) is that

$$\sum_{j\in\mathbb{Z}} \alpha_{4j+p} = 1, \quad p = 0, 1, 2, 3. \tag{2.2}$$

A subdivision scheme is uniformly convergent if for any initial data $f^0 = \{f_i^0 : i \in \mathbb{Z}\}$, there exists a continuous function f, such that for any closed interval $I \subset \mathbb{R}$, that satisfies

$$\lim_{k\to\infty} \sup_{i\in 4^k I} \left| f_i^k - f\left(4^{-k} i\right) \right| = 0. \tag{2.3}$$

Obviously $f = S^\infty f^0$.

For the analysis of subdivision scheme with mask α, it is very practical to consider the z-transform of the mask,

$$\alpha(z) = \sum_{i\in\mathbb{Z}} \alpha_i z^i, \tag{2.4}$$

which is usually called the *symbol* of the scheme. Since the scheme has mask of finite support, the corresponding *symbol* is Laurent polynomial, namely, polynomial in positive and negative powers of the variables. From (2.2) and (2.4) the Laurent polynomial of a convergent subdivision scheme satisfies

$$\alpha(1) = 4, \qquad \alpha\left(e^{2ip\pi/4}\right) = 0, \qquad p = 1,2,3. \tag{2.5}$$

This condition guarantees existence of a related subdivision scheme for the divided differences of the original control points and the existence of associated Laurent polynomial:

$$\alpha^{(1)}(z) = \frac{4z^3}{1+z+z^2+z^3}\alpha(z). \tag{2.6}$$

The subdivision scheme S_1 with *symbol* $\alpha^{(1)}(z)$ is related to scheme S with *symbol* $\alpha(z)$ by the following theorem.

Theorem 2.1 (see [1]). *Let S denote a subdivision scheme with symbol $\alpha(z)$ satisfying* (2.2). *Then there exists a subdivision scheme S_1 with the property*

$$\Delta f^k = S_1 \Delta f^{k-1}, \tag{2.7}$$

where $f^k = S^k f^0$ and $\Delta f^k = \{(\Delta f^k)_i = 4^k(f^k_{i+1} - f^k_i) : \quad i \in \mathbb{Z}\}$. Furthermore, S is a uniformly convergent if and only if $(1/4)S_1$ converges uniformly to the zero function for all initial data f^0, in the sense that

$$\lim_{k\to\infty}\left(\frac{1}{4}S_1\right)^k f^0 = 0. \tag{2.8}$$

Theorem 2.1 indicates that for any given subdivision scheme S, with a mask α satisfying (2.2), we can prove the uniform convergence of S by first deriving the mask of $(1/4)S_1$ and then computing $\|((1/4)S_1)^i\|_\infty$ for $i = 1,2,3,\ldots,L$, where L is the first integer for which $\|((1/4)S_1)^L\|_\infty < 1$. If such an L exists, S converges uniformly. Since there are four rules for computing the values at next refinement level, we define the norm

$$\begin{gathered}\|S\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}}\left|\alpha_{4j+p}\right|,\ p = 0,1,2,3\right\},\\ \left\|\left(\frac{1}{4}S_n\right)^L\right\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}}\left|b^{[n,L]}_{i+4^L j}\right| : i = 0,1,\ldots,4^L-1\right\},\end{gathered} \tag{2.9}$$

where

$$b^{[n,L]}(z) = \frac{1}{4^L}\prod_{j=0}^{L-1}\alpha^{(n)}\left(z^{4^j}\right), \tag{2.10}$$

$$\alpha^{(n)}(z) = \left(\frac{4z^3}{1+z+z^2+z^3}\right)\alpha^{(n-1)}(z) = \left(\frac{4z^3}{1+z+z^2+z^3}\right)^n \alpha(z), \quad n \geq 1. \tag{2.11}$$

Theorem 2.2 (see [1]). *Let S be subdivision scheme with a characteristic $\mho$-polynomial $\alpha(z) = ((1+z+z^2+z^3)/4z^3)^n\ q(z)$, $q \in \mho$. If the subdivision scheme S_n, corresponding to the $\mho$-polynomial $q(z)$, converges uniformly then $S^\infty f^0 \in C^n(\mathbb{R})$ for any initial control polygon f^0.*

Corollary 2.3. *If S is a subdivision scheme of the form above and $(1/4)S_{n+1}$ converges uniformly to the zero function for all initial data f^0 then $S^\infty f^0 \in C^n(\mathbb{R})$ for any initial control polygon f^0.*

Proof. Apply Theorem 2.1 (2nd part) to Theorem 2.2. □

Corollary 2.3 indicates that for any given quaternary subdivision scheme S, we can prove $S^\infty f^0 \in C^n$ by first deriving the mask of $(1/4)S_{n+1}$ and then computing $\|((1/4)S_{n+1})^i\|_\infty$ for $i = 1,2,3,\ldots,L$, where L is the first integer for which $\|((1/4)S_{n+1})^L\|_\infty < 1$. If such an L exists, then $S^\infty f^0 \in C^n$.

Theorem 2.4 (see [10]). *The approximation order of a convergent subdivision scheme S which is exact for P_n (set of polynomials at most degree n) is $n+1$.*

3. Smoothness Analysis of Proposed Scheme

This section is devoted for analysis of 4-point quaternary approximating subdivision scheme by using Laurent polynomial method. The following result shows that scheme is C^3 continuous.

Theorem 3.1. *The 4-point quaternary approximating subdivision scheme* (1.1) *is C^3 for any ω in* $(0, 1.5)$.

Proof. For the given mask of proposed scheme S

$$\begin{aligned}
\mathbf{a} &= \{\alpha_j\} \\
&= \Big\{\ldots,0,0,\frac{1}{64}-\frac{1}{64}\omega,\frac{7}{128}-\frac{3}{64}\omega,\frac{15}{128}-\frac{5}{64}\omega,\frac{7}{32}-\frac{7}{64}\omega,\frac{5}{16}-\frac{5}{64}\omega, \\
&\quad \frac{49}{128}+\frac{1}{64}\omega,\frac{57}{128}+\frac{7}{64}\omega,\frac{29}{64}+\frac{13}{64}\omega,\frac{29}{64}+\frac{13}{64}\omega,\frac{57}{128}+\frac{7}{64}\omega,\frac{49}{128}+\frac{1}{64}\omega, \\
&\quad \frac{5}{16}-\frac{5}{64}\omega,\frac{7}{32}-\frac{7}{64}\omega,\frac{15}{128}-\frac{5}{64}\omega,\frac{7}{128}-\frac{3}{64}\omega,\frac{1}{64}-\frac{1}{64}\omega,0,0,\ldots\Big\},
\end{aligned} \tag{3.1}$$

the Laurent polynomial is

$$\alpha(z) = z^{-3}\left(1 + z + z^2 + z^3\right)\xi_1(z), \tag{3.2}$$

where

$$\begin{aligned}
\xi_1(z) = {} & \left(\frac{1}{64} - \frac{1}{64}\omega\right)z^{-5} + \left(\frac{5}{128} - \frac{1}{32}\omega\right)z^{-4} + \left(\frac{1}{16} - \frac{1}{32}\omega\right)z^{-3} + \left(\frac{13}{128} - \frac{1}{32}\omega\right)z^{-2} \\
& + \left(\frac{7}{64} + \frac{1}{64}\omega\right)z^{-1} + \left(\frac{7}{64} + \frac{1}{16}\omega\right) + \left(\frac{1}{8} + \frac{1}{16}\omega\right)z + \left(\frac{7}{64} + \frac{1}{16}\omega\right)z^2 + \left(\frac{7}{64} + \frac{1}{64}\omega\right)z^3 \\
& + \left(\frac{13}{128} - \frac{1}{32}\omega\right)z^4 + \left(\frac{1}{16} - \frac{1}{32}\omega\right)z^5 + \left(\frac{5}{128} - \frac{1}{32}\omega\right)z^6 + \left(\frac{1}{64} - \frac{1}{64}\omega\right)z^7.
\end{aligned} \tag{3.3}$$

From Laurent polynomial (2.10) for $L = n = 1$ and (3.2), we have

$$b^{[1,1]}(z) = \frac{1}{4}\alpha^{(1)}(z) = \frac{z^3}{1 + z + z^2 + z^3}\alpha(z) = \xi_1(z). \tag{3.4}$$

For C^0 continuity of S we require that the Laurent polynomial $\alpha(z)$ satisfy (2.5), which it does, and $\|(1/4)S_1\|_\infty < 1$. The norm of scheme $(1/4)S_1$ is

$$\left\|\frac{1}{4}S_1\right\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}}\left|b_{i+4j}^{[1,1]}\right| : i = 0, 1, 2, 3\right\}. \tag{3.5}$$

This implies for $-7.75 < \omega < 8.0$

$$\left\|\frac{1}{4}S_1\right\|_\infty = \max\{\aleph_1, \aleph_2, \aleph_3\} < 1, \tag{3.6}$$

where

$$\begin{aligned}
\aleph_1 &= 2\left|\frac{1}{64} - \frac{1}{64}\omega\right| + 2\left|\frac{7}{64} + \frac{1}{64}\omega\right|, \\
\aleph_2 &= \left|\frac{5}{128} - \frac{1}{32}\omega\right| + \left|\frac{7}{64} + \frac{1}{16}\omega\right| + \left|\frac{13}{128} - \frac{1}{32}\omega\right|, \\
\aleph_3 &= 2\left|\frac{1}{16} - \frac{1}{32}\omega\right| + \left|\frac{1}{8} + \frac{1}{16}\omega\right|.
\end{aligned} \tag{3.7}$$

Therefore $(1/4)S_1$ converges uniformly. Hence, by Corollary 2.3, $S^\infty f^0 \in C^0$. By (3.4) the Laurent polynomial of scheme S_1 can be written as

$$\alpha^{(1)}(z) = z^{-3}\left(1 + z + z^2 + z^3\right)\xi_2(z), \tag{3.8}$$

where

$$\xi_2(z) = \left(\frac{1}{16} - \frac{1}{16}\omega\right)z^{-2} + \left(\frac{3}{32} - \frac{1}{16}\omega\right)z^{-1} + \frac{3}{32} + \frac{5}{32}z + \left(\frac{3}{32} + \frac{1}{8}\omega\right)z^2 + \left(\frac{3}{32} + \frac{1}{8}\omega\right)z^3 + \frac{5}{32}z^4 + \frac{3}{32}z^5 + \left(\frac{3}{32} - \frac{1}{16}\omega\right)z^6 + \left(\frac{1}{16} - \frac{1}{16}\omega\right)z^7. \tag{3.9}$$

Utilizing (2.10) for $n = 2$ and $L = 1$ and (3.8) we get

$$b^{[2,1]}(z) = \frac{1}{4}\alpha^{(2)}(z) = \frac{z^3}{1 + z + z^2 + z^3}\alpha^{(1)}(z) = \xi_2(z). \tag{3.10}$$

For C^1 continuity of S it needs that $\alpha^{(1)}(z)$ satisfy (2.5), which it does, and $\|(1/4)S_2\|_\infty < 1$. The norm of scheme $(1/4)S_2$ is

$$\left\|\frac{1}{4}S_2\right\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}}\left|b^{[2,1]}_{i+4j}\right| : i = 0,1,2,3\right\}. \tag{3.11}$$

This implies for $-3.75 < \omega < 4.25$

$$\left\|\frac{1}{4}S_2\right\|_\infty = \max\{\aleph_4, \aleph_5\} < 1, \tag{3.12}$$

where

$$\aleph_4 = \left|\frac{1}{16} - \frac{1}{16}\omega\right| + \left|\frac{3}{32} + \frac{1}{8}\omega\right| + \left|\frac{3}{32} - \frac{1}{16}\omega\right|, \qquad \aleph_5 = \frac{1}{4}. \tag{3.13}$$

Therefore $(1/4)S_2$ is uniformly convergent. Hence, by Corollary 2.3, $S^\infty f^0 \in C^1$. Now from (3.10) Laurent polynomial of scheme S_2 is

$$\alpha^{(2)}(z) = z^{-3}\left(1 + z + z^2 + z^3\right)\xi_3(z), \tag{3.14}$$

where

$$\xi_3(z) = \left(\frac{1}{4} - \frac{1}{4}\omega\right)z + \frac{1}{8}z^2 + \frac{1}{4}\omega z^3 + \frac{1}{4}z^4 + \frac{1}{4}\omega z^5 + \frac{1}{8}z^6 + \left(\frac{1}{4} - \frac{1}{4}\omega\right)z^7. \tag{3.15}$$

With the choice of $n = 3$ and $L = 1$, and by (3.14)

$$b^{[3,1]}(z) = \frac{1}{4}\alpha^{(3)}(z) = \frac{z^3}{1 + z + z^2 + z^3}\alpha^{(2)}(z) = \xi_3(z). \tag{3.16}$$

For C^2 continuity, it is necessary that Laurent polynomial $\alpha^{(2)}(z)$ satisfy (2.5), which is incidentally true, and also for first integer value of $L > 0$ for which $\|((1/4)S_3)^L\|_\infty < 1$.

$$\left\|\frac{1}{4}S_3\right\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}}\left|b^{[3,1]}_{i+4j}\right| : i = 0,1,2,3\right\}. \tag{3.17}$$

This implies for $-1.5 < \omega < 2.5$

$$\left\|\frac{1}{4}S_3\right\|_\infty = \max\{\aleph_6, \aleph_7\} < 1, \tag{3.18}$$

where

$$\aleph_6 = \left|\frac{1}{4} - \frac{1}{4}\omega\right| + \left|\frac{1}{4}\omega\right|, \qquad \aleph_7 = \frac{1}{4}. \tag{3.19}$$

Therefore $(1/4)S_3$ converges uniformly. Hence, by Corollary 2.3, $S^\infty f^0 \in C^2$. Now from (3.16) Laurent polynomial of scheme S_3 can be written as

$$\alpha^{(3)}(z) = z^{-3}\left(1 + z + z^2 + z^3\right)\xi_4(z), \tag{3.20}$$

where

$$\xi_4(z) = (1-\omega)z^4 + \left(-\frac{1}{2} + \omega\right)z^5 + \left(-\frac{1}{2} + \omega\right)z^6 + (1-\omega)z^7. \tag{3.21}$$

With the choice of $n = 4$ and $L = 1$, we have following by (3.20)

$$b^{[4,1]}(z) = \frac{1}{4}\alpha^{(4)}(z) = \frac{z^3}{1 + z + z^2 + z^3}\alpha^{(3)}(z) = \xi_4(z). \tag{3.22}$$

For C^3 continuity, it is necessary that Laurent polynomial $\alpha^{(3)}(z)$ satisfy (2.5), which is incidentally true, and also for first integer value of $L > 0$ for which $\|((1/4)S_4)^L\|_\infty < 1$:

$$\left\|\frac{1}{4}S_4\right\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}}\left|b^{[4,1]}_{i+4j}\right| : i = 0,1,2,3\right\}. \tag{3.23}$$

This implies for $0 < \omega < 1.5$

$$\left\|\frac{1}{4}S_4\right\|_\infty = \max\{\aleph_8, \aleph_9\} < 1, \tag{3.24}$$

where

$$\aleph_8 = |1-\omega|, \qquad \aleph_9 = \left|-\frac{1}{2}+\omega\right|. \tag{3.25}$$

Therefore $(1/4)S_4$ converges uniformly. Hence, by Corollary 2.3, $S^{\infty}f^0 \in C^3$. □

3.1. Hölder Exponent

From the above discussion, we conclude that our scheme is C^3. In the following paragraph we generalize its smoothness based on Rioul's method [11] and Hassan et al. [8] (in generalize sense). We conclude that scheme has Hölder regularity $R_H = 3 + \vartheta^{\eta}$ for all $\eta \geq 1$, where ϑ^{η} is defined by

$$4^{-\eta\vartheta^{\eta}} = \left\|\left(\frac{1}{4}S_4\right)^{\eta}\right\|_{\infty}. \tag{3.26}$$

For the convenience of computation, we set $\eta = 1$. Since by (3.24)

$$\left\|\frac{1}{4}S_4\right\|_{\infty} = \begin{cases} 1-\omega, & 0 < \omega \leq 0.75, \\ \omega - \dfrac{1}{2}, & 0.75 \leq \omega < 1.5, \end{cases} \tag{3.27}$$

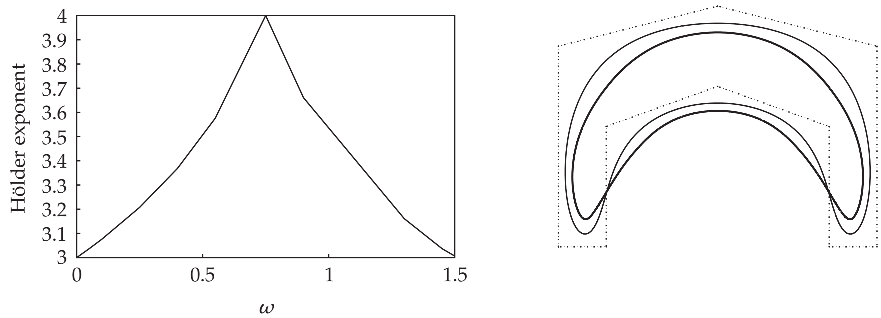
we obtain that Hölder regularity against ω of the 4-point quaternary scheme is

$$R(\omega) = \begin{cases} 3 - \log_4(1-\omega), & 0 < \omega \leq 0.75, \\ 3 - \log_4\left(\omega - \dfrac{1}{2}\right), & 0.75 \leq \omega < 1.5. \end{cases} \tag{3.28}$$

Figure 1(a) shows a graph of the Hölder exponent against ω. Notice that the highest smoothness of the 4-point quaternary scheme is achieved at $\omega = 0.75$, and its Hölder exponent is $R_H = R(0.75) = 4$. Figure 1(b) shows the result of proposed scheme (1.1) after four subdivision levels. In this figure the control polygon is drawn by dotted lines. The thin solid line is produced by setting $\omega = 0.75$, and the bold solid line is produced by setting $\omega = 0.02$.

3.2. Subdivision Rule for Open Polygon

When dealing with open initial polygon $f^0 = \{f_i^0 : i = 0, \ldots, N\}$, it is not possible to refine the first and last edges by rule (1.1). However, the extension of this strategy to deal with open polygon requires a well-defined neighborhood of end points. Since the first and last edges can be treated analogously, it will be sufficient to derive the rules only for one side of the open polygon. To this aim, we see that if we define just one auxiliary point $f_{-1}^0 = 2f_0^0 - f_1^0$



(a)

(b)

Figure 1: (a) Graph against Höder exponent and parameter $0 < \omega < 1.5$ for a 4-point quaternary scheme. (b) The dotted lines are control polygon; whereas the bold and thin solid lines are produced by setting $\omega = 0.02, 0.75$, respectively.

as extrapolatory rule in the initial polygon f^0. Then the first edge $\overline{f_0^k f_1^k}$ of the nonrefined polygon $\{f_i^k : i = 0, \ldots, 4^k N\}$ can be refined by the following rules:

$$\begin{aligned}
f_0^{k+1} &= (2\alpha_1 + \alpha_2) f_0^k + (\alpha_3 - \alpha_1) f_1^k + \alpha_4 f_2^k, \\
f_1^{k+1} &= (2\alpha_5 + \alpha_6) f_0^k + (\alpha_7 - \alpha_5) f_1^k + \alpha_8 f_2^k, \\
f_2^{k+1} &= (2\alpha_8 + \alpha_7) f_0^k + (\alpha_6 - \alpha_8) f_1^k + \alpha_5 f_2^k, \\
f_3^{k+1} &= (2\alpha_4 + \alpha_3) f_0^k + (\alpha_2 - \alpha_4) f_1^k + \alpha_1 f_2^k.
\end{aligned} \tag{3.29}$$

Remark 3.2. Subdivision rule (3.29) for last edges does not affect the convergence of the proposed scheme to a continuously differentiable limit. It is sufficient to show that, taken $f_{-2}^0 = 2f_0^0 - f_2^0$ and $f_{-1}^0 = 2f_0^0 - f_1^0$, and refining the polygon f^0 by (1.1), after k steps of subdivision the expression of the point f_{-1}^k turns out to coincide with $f_{-1}^k = 2f_0^k - f_1^k$.

4. Basic Properties of the Scheme

In this section, we discuss approximation order and support of basic limit function of 4-point quaternary approximating scheme.

4.1. Approximation Order

Here we show that the approximation order of proposed scheme is five. The following lemma based on the technique of Sabin [12] is needed to follow up the claim.

Lemma 4.1. *The proposed 4-point quaternary subdivision scheme reproduces all the cubic polynomials for $\omega \in (0, 1.5)$ and quartic at $\omega = 0.75$.*

Proof. We carry out this result by taking our origin the middle of an original span with ordinate $\ldots, (-5)^n, (-3)^n, (-1)^n, 1^n, 3^n, 5^n, \ldots$. If $y = x^n$, then we have,

$$\begin{aligned}
[y] = \ldots, &\alpha_1(-5)^n + \alpha_2(-3)^n + \alpha_3(-1)^n + \alpha_4(1)^n, \\
&\alpha_5(-5)^n + \alpha_6(-3)^n + \alpha_7(-1)^n + \alpha_8(1)^n, \\
&\alpha_8(-5)^n + \alpha_7(-3)^n + \alpha_6(-1)^n + \alpha_5(1)^n, \\
&\alpha_4(-5)^n + \alpha_3(-3)^n + \alpha_2(-1)^n + \alpha_1(1)^n, \\
&\alpha_1(-3)^n + \alpha_2(-1)^n + \alpha_3(1)^n + \alpha_4(3)^n, \\
&\vdots \\
&\alpha_8(-1)^n + \alpha_7(1)^n + \alpha_6(3)^n + \alpha_5(5)^n, \\
&\alpha_4(-1)^n + \alpha_3(1)^n + \alpha_2(3)^n + \alpha_1(5)^n, \ldots,
\end{aligned} \tag{4.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_8$ are defined by (1.1).

If $y = x^1$, then

$$\begin{aligned}
[y] &= \ldots, -2.75, -2.25, -1.75, -1.25, -0.75, \\
&\quad -0.25, 0.25, 0.75, 1.25, 1.75, 2.25, 2.75, \ldots \\
[\delta y] &= \ldots, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, \ldots \\
\left[\delta^2 y\right] &= 0,
\end{aligned} \tag{4.2}$$

where δ represents the differences of the vertices.

If $y = x^2$, then

$$\begin{aligned}
[y] = \ldots, &9.875 - \omega, 7.375 - \omega, 5.375 - \omega, 3.875 - \omega, 2.875 - \omega, 2.375 - \omega, \\
&2.375 - \omega, 2.875 - \omega, 3.875 - \omega, 5.375 - \omega, 7.375 - \omega, 9.875 - \omega, \ldots.
\end{aligned} \tag{4.3}$$

Taking further differences, we get $[\delta^3 y] = 0$.

If $y = x^3$, then

$$\begin{aligned}
[y] = \ldots, &-39.875 + 8.25\omega, -27 + 6.75\omega, -17.5 + 5.25\omega, -10.625 + 3.75\omega, \\
&-5.625 + 2.25\omega, -1.75 + 0.75\omega, 1.75 - 0.75\omega, 5.625 - 2.25\omega, \\
&10.625 - 3.75\omega, 17.5 - 5.25\omega, 27 - 6.75\omega, 39.875 - 8.25\omega, \ldots.
\end{aligned} \tag{4.4}$$

This implies that $[\delta^4 y] = 0$.

If $y = x^4$, then

$$\begin{aligned}[y] = {} & \dots, 173.75 - 52\omega, 109.75 - 40\omega, 65.75 - 28\omega, 35.75 - 16\omega, \\ & 19.75 - 10\omega, 14.75 - 10\omega, 14.75 - 10\omega, 19.75 - 10\omega, \\ & 35.75 - 16\omega, 65.75 - 28\omega, 109.75 - 40\omega, 173.75 - 52\omega, \dots .\end{aligned} \tag{4.5}$$

By taking differences, we have

$$\begin{aligned}\left[\delta^4 y\right] &= \dots, 6 - 6\omega, -3 + 6\omega, -3 + 6\omega, 6 - 6\omega, 6 - 6\omega, \\ &\quad - 3 + 6\omega, -3 + 6\omega, 6 - 6\omega, \dots \\ \left[\delta^5 y\right] &= 0, \quad \text{at } \omega = 0.75.\end{aligned} \tag{4.6}$$

Thus the proposed scheme has cubic in all $\omega \in (0, 1.5)$ and quartic precision at $\omega = 0.75 \in (0, 1.5)$. □

The theorem is an easy consequences of Lemma 4.1 and Theorem 2.4.

Theorem 4.2. *A 4-point quaternary approximating subdivision scheme has approximation order 5.*

4.2. Support of Basic Limit Function

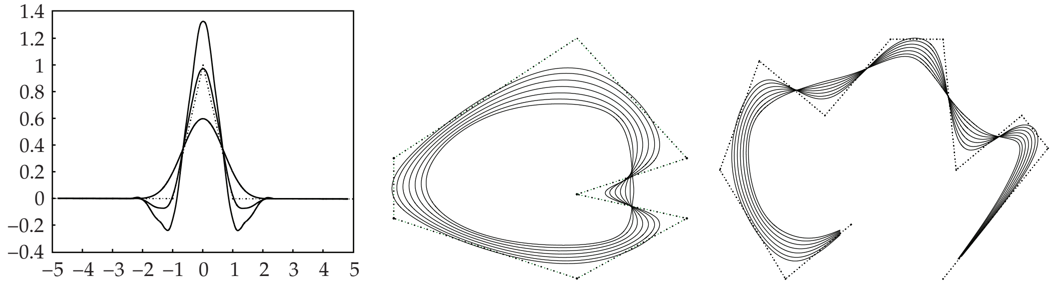
The basic function of a subdivision scheme is the limit function of proposed scheme for the following data:

$$f_i^0 = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases} \tag{4.7}$$

Figure 2(a) shows the basic limit function $\psi = S^\infty f_i^0$ of proposed scheme. The following theorem is related to the support of limit function.

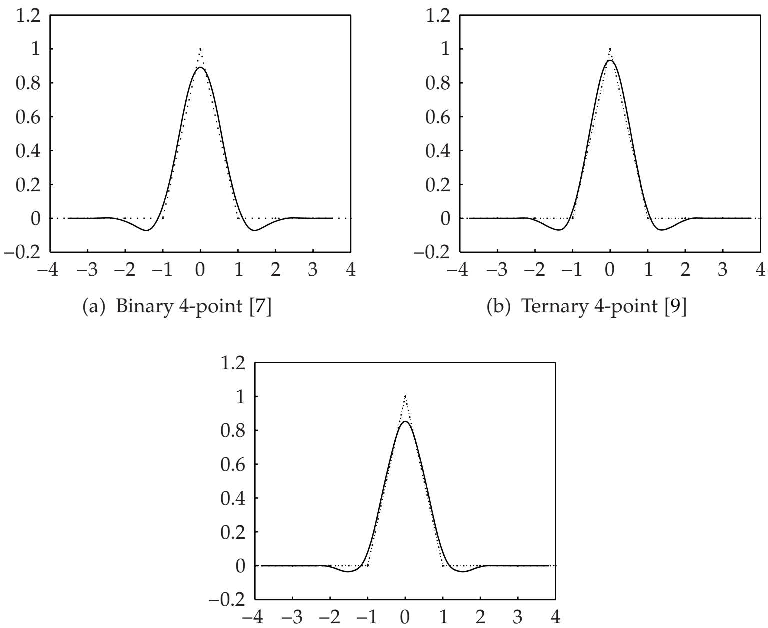
Theorem 4.3. *The basic limit function ψ of proposed 4-point scheme has support width $s = 5$, which implies that it vanishes outside the interval $[-5/2, 5/2]$.*

Proof. Since the basic function is the limit function of the scheme for the data (4.7), its support width s can be determine by computing how for the effect of the nonzero vertex f_0^0 will propagate along by. As the mask of the scheme is a 16-long sequence by centering it on that vertex, the distances to the last of its left and right nonzero coefficients are equal to 8 and 7, respectively. At the first subdivision step we see that the vertices on the left and right sides of f_0^1 at $8/4$ & $7/4$ are the furthest nonzero new vertices. At each refinement, the distance on both sides is reduced by the factor $1/4$. At the next step of the scheme this will propagate along by $8/4 \times 1/4$ on the left and $7/4 \times 1/4$ on the right. Hence after k subdivision steps the furthest nonzero vertex on the left will be at $8(1/4 + 1/4^2 + \cdots + 1/4^k) = (8/4)(\sum_{j=0}^{k-1} 1/4^j)$ and



(a) Basic limit functions (b) Close curves (c) Open curves

Figure 2: The effect of parameter on the shape of the basic limit function/limit curve of the proposed scheme. Doted lines show control polygons; whereas solid lines indicate basic limit functions/curves. (a) Here, $\omega = 0.75, 2.40$, and 3.75 from origin to the top. (b) and (c) $\omega = 1.45, 1.25, 1.00, 0.75, 0.50, 0.25$, and 0.05 from left to right.



(a) Binary 4-point [7] (b) Ternary 4-point [9]

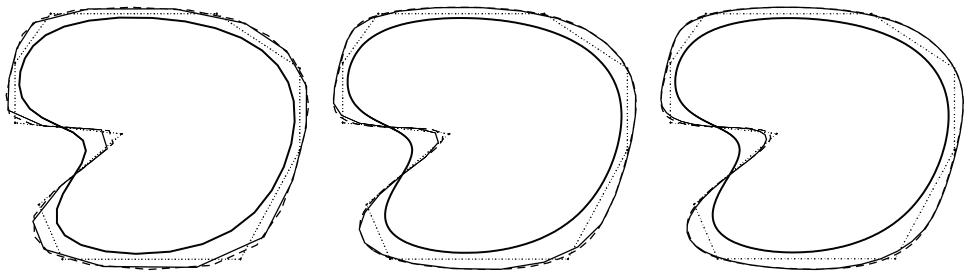
(c) Proposed

Figure 3: Support width of for binary, ternary, and proposed 4-point approximating schemes has been shown in (a), (b), and (c), respectively.

on the right will be at $7(1/4+1/4^2+\cdots+1/4^k) = (7/4)(\sum_{j=0}^{k-1} 1/4^j)$. So the total support width is $(8/4)(\sum_{j=0}^{\infty} 1/4^j) + (7/4)(\sum_{j=0}^{\infty} 1/4^j) = (15/4)(1/(1-1/4)) = 5$. □

5. Comparison and Application

In Table 1, we compare some properties of proposed 4-point subdivision scheme with those of other 4-point schemes having smaller arity.



(a) Level 1 (b) Level 2 (c) Level 3

Figure 4: It shows the comparison at 1st, 2nd, and 3rd level of binary, ternary, and quaternary 4-point approximating schemes. Dotted lines indicate initial control polygons, whereas dashed, thin solid, and bold solid continuous curves are generated by binary, ternary, and quaternary schemes, respectively.

Table 1: Comparison of proposed 4-point scheme with other 4-point schemes.

Scheme	Type	Approximation order	Support	C^n
Binary 4-point [5]	Interpolating	4	6	1
Binary 4-point [6]	Interpolating	4	6	1
Binary 4-point [7]	Approximating	4	7	2
Ternary 4-point [4]	Interpolating	3	5	2
Ternary 4-point [8]	Interpolating	3	5	2
Ternary 4-point [9]	Approximating	4	5.5	2
Proposed scheme	Approximating	5	5	3

In Figures 2(b) and 2(c), we illustrate performance of our scheme by setting the shape parameter to various values, which illustrate how this parameter affect the shape of limit curve. Moreover, Figures 3 and 4 show the comparison of support width and approximation order of proposed scheme with other existing 4-point approximating schemes.

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