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Research Article

Multivariate p-Adic Fermionic q-Integral on $\mathbb{Z}_p$ and Related Multiple Zeta-Type Functions

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In 2008, Jang et al. constructed generating functions of the multiple twisted Carlitz's type q-Bernoulli polynomials and obtained the distribution relation for them. They also raised the following problem: *"are there analytic multiple twisted Carlitz's type q-zeta functions which interpolate multiple twisted Carlitz's type q-Euler (Bernoulli) polynomials?"* The aim of this paper is to give a partial answer to this problem. Furthermore we derive some interesting identities related to twisted q-extension of Euler polynomials and multiple twisted Carlitz's type q-Euler polynomials.

1. Introduction, definitions, and notations

Let p be an odd prime. $\mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ will always denote, respectively, the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $v_p : \mathbb{C}_p \rightarrow \mathbb{Q} \cup \{\infty\}$ ($\mathbb{Q}$ is the field of rational numbers) denote the p-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. The absolute value on $\mathbb{C}_p$ will be denoted as $|\cdot|_p$, and $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{C}_p$. We let $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \mid 1/x \in \mathbb{Z}_p\}$. A p-adic integer in $\mathbb{Z}_p^\times$ is sometimes called a p-adic unit. For each integer $N \geq 0$, C_{p^N} will denote the multiplicative group of the primitive p^Nth roots of unity in $\mathbb{C}_p^\times = \mathbb{C}_p \setminus \{0\}$. Set

$$\mathbf{T}_p = \{\omega \in \mathbb{C}_p \mid \omega^{p^N} = 1 \text{ for some } N \geq 0\} = \bigcup_{N \geq 0} C_{p^N}. \tag{1.1}$$

The dual of $\mathbb{Z}_p$, in the sense of p-adic Pontrjagin duality, is $\mathbf{T}_p = C_{p^\infty}$, the direct limit (under inclusion) of cyclic groups C_{p^N} of order $p^N (N \geq 0)$, with the discrete topology.

When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $|1-q|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. If $q \in \mathbb{C}$, then we assume that $|q| < 1$.

Let

$$\mathbb{Z}_p = \lim_{\overleftarrow{N}} \left(\frac{\mathbb{Z}}{p^N \mathbb{Z}} \right), \qquad \mathbb{Z}_p^{\times} = \bigcup_{0<a<p} a + p\mathbb{Z}_p, \\ a + p^N \mathbb{Z}_p = \left\{ x \in \mathbb{Z}_p \mid x \equiv a\left(\bmod p^N \right) \right\}, \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < p^N$.

We use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}. \tag{1.3}$$

Hence

$$\lim_{q \to 1} [x]_q = x \tag{1.4}$$

for any x with $|x|_p \leq 1$ in the present p-adic case. The distribution $\mu_q(a + p^N \mathbb{Z}_p)$ is given as

$$\mu_q\left(a + p^N \mathbb{Z}_p\right) = \frac{q^a}{\left[p^N\right]_q} \tag{1.5}$$

(cf. [1–9]). For the ordinary p-adic distribution μ_0 defined by

$$\mu_0\left(a + p^N \mathbb{Z}_p\right) = \frac{1}{p^N}, \tag{1.6}$$

we see

$$\lim_{q \to 1} \mu_q = \mu_0. \tag{1.7}$$

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, we write $f \in \mathrm{UD}(\mathbb{Z}_p, \mathbb{C}_p)$ if the difference quotient

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y} \tag{1.8}$$

has a limit $f'(a)$ as $(x, y) \to (a, a)$. Also we use the following notation:

$$[x]_{-q} = \frac{1-(-q)^x}{1+q}, \tag{1.9}$$

(cf.[1–5]).

In [1–3], Kim gave a detailed proof of fermionic p-adic q-measures on $\mathbb{Z}_p$. He treated some interesting formulae-related q-extension of Euler numbers and polynomials; and he defined fermionic p-adic q-measures on $\mathbb{Z}_p$ as follows:

$$\mu_{-q}(a+p^N\mathbb{Z}_p) = \frac{(-q)^a}{[p^N]_{-q}}. \tag{1.10}$$

By using the fermionic p-adic q-measures, he defined the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x \tag{1.11}$$

for $f \in \mathrm{UD}(\mathbb{Z}_p, \mathbb{C}_p)$ (cf. [1–3]). Observe that

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{1.12}$$

From (1.12), we obtain

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.13}$$

where $f_1(x) = f(x+1)$. By substituting $f(x) = e^{tx}$ into (1.13), classical Euler numbers are defined by means of the following generating function:

$$\int_{\mathbb{Z}_p} e^{tx}d\mu_{-1}(x) = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}. \tag{1.14}$$

These numbers are interpolated by the Euler zeta function which is defined as follows:

$$\zeta_E(s) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}, \quad s \in \mathbb{C}, \tag{1.15}$$

(cf. [1–9]). From (1.12), we also obtain

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \tag{1.16}$$

where $f_1(x) = f(x+1)$. By substituting $f(x) = e^{tx}$ into (1.13), q-Euler numbers are defined by means of the following generating function:

$$\int_{\mathbb{Z}_p} e^{tx}d\mu_{-q}(x) = \frac{[2]_q}{qe^t+1} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}. \tag{1.17}$$

These numbers are interpolated by the Euler q-zeta function which is defined as follows:

$$\zeta_{q,E}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n^s}, \quad s \in \mathbb{C}, \tag{1.18}$$

(cf. [4]).

In [6], Ozden and Simsek defined generating function of q-Euler numbers by

$$\frac{2}{q+1} \int_{\mathbb{Z}_p} e^{tx} d\mu_{-q}(x) = \frac{2}{qe^t + 1}, \tag{1.19}$$

which are different from (1.17). But we observe that all these generating functions were obtained by the same fermionic p-adic q-measures on $\mathbb{Z}_p$ and the fermionic p-adic q-integrals on $\mathbb{Z}_p$.

In this paper, we define a multiple twisted Carlitz's type q-zeta functions, which interpolated multiple twisted Carlitz's type q-Euler polynomials at negative integers. This result gave us a partial answer of the problem proposed by Jang et al. [10], which is given by: "*Are there analytic multiple twisted Carlitz's type q-zeta functions which interpolate multiple twisted Carlitz's type q-Euler (Bernoulli) polynomials?*"

2. Preliminaries

In [10], Jang and Ryoo defined q-extension of Euler numbers and polynomials of higher order and studied multivariate q-Euler zeta functions. They also derived sums of products of q-Euler numbers and polynomials by using ferminonic p-adic q-integral.

In [5, 7], Ozden et al. defined multivariate Barnes-type Hurwitz q-Euler zeta functions and l-functions. They also gave relation between multivariate Barnes-type Hurwitz q-Euler zeta functions and multivariate q-Euler l-functions.

In this section, we consider twisted q-extension of Euler numbers and polynomials of higher order and study multivariate twisted Barnes-type Hurwitz q-Euler zeta functions and l-functions.

Let $\mathrm{UD}(\mathbb{Z}_p^h, \mathbb{C}_p)$ denote the space of all uniformly (or strictly) differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p^h = \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{h\text{-times}}$. For $f \in \mathrm{UD}(\mathbb{Z}_p^h, \mathbb{C}_p)$, the p-adic q-integral on $\mathbb{Z}_p^h$ is defined by

$$\begin{aligned} I_{-q}^{(h)}(f) &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} f(x_1, \ldots, x_h) d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}^h} \sum_{x_1=0}^{p^N-1} \cdots \sum_{x_h=0}^{p^N-1} f(x_1, \ldots, x_h)(-q)^{x_1+\cdots+x_h} \end{aligned} \tag{2.1}$$

(cf. [3]). If $q\to 1$, then

$$I_{-1}^{(h)}(f) = \lim_{q\to 1} I_{-q}^{(h)}(f) = \lim_{N\to\infty} \sum_{x_1=0}^{p^N-1} \cdots \sum_{x_h=0}^{p^N-1} f(x_1, \ldots, x_h)(-1)^{x_1+\cdots+x_h}. \tag{2.2}$$

For a fixed positive integer d with $(d,p)=1$, we set

$$X_p = \lim_{\overleftarrow{N}} \left(\frac{\mathbb{Z}}{dp^N\mathbb{Z}} \right). \tag{2.3}$$

For $f \in \mathrm{UD}(\mathbb{Z}_p^h, \mathbb{C}_p)$,

$$I_{-1}^{(h)}(f) = \underbrace{\int_{X_p} \cdots \int_{X_p}}_{h\text{-times}} f(x_1,\ldots,x_h)\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_h), \tag{2.4}$$

(cf. [2]).

We set $f(x_1,\ldots,x_h) = \omega^{x_1+\cdots+x_h} e^{(x+x_1+\cdots+x_h)t}$ in (2.2) and (2.4). Then we have

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} \omega^{x_1+\cdots+x_h} e^{(x+x_1+\cdots+x_h)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_h) = \underbrace{\left(\frac{2}{\omega e^t+1}\right)\cdots\left(\frac{2}{\omega e^t+1}\right)}_{h\text{-times}} = \sum_{n=0}^{\infty} E_{n,\omega}^{(h)}(x)\frac{t^n}{n!}, \tag{2.5}$$

where $E_{n,\omega}^{(h)}(x)$ are the twisted Euler polynomials of order h. From (2.5), we note that

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} \omega^{x_1+\cdots+x_h} (x+x_1+\cdots+x_h)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_h) = E_{n,\omega}^{(h)}(x). \tag{2.6}$$

We give an application of the multivariate q-deformed p-adic integral on $\mathbb{Z}_p^h$ in the fermionic sense related to [3]. Let

$$\int_{\mathbb{Z}_p^h} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}}. \tag{2.7}$$

By substituting

$$f(x_1,\ldots,x_h) = \omega^{x_1+\cdots+x_h} e^{(x+x_1+\cdots+x_h)t} \tag{2.8}$$

into (2.1), we define twisted q-extension of Euler numbers of higher order by means of the following generating function:

$$\int_{\mathbb{Z}_p^h} \omega^{x_1+\cdots+x_h} e^{(x_1+\cdots+x_h)t} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_h) = \underbrace{\left(\frac{[2]_q}{\omega q e^t+1}\right)\cdots\left(\frac{[2]_q}{\omega q e^t+1}\right)}_{h\text{-times}} = \sum_{n=0}^{\infty} E_{n,q,\omega}^{(h)}\frac{t^n}{n!}. \tag{2.9}$$

Then we have

$$\int_{\mathbb{Z}_p^h} \omega^{x_1+\cdots+x_h}(x_1+\cdots+x_h)^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_h) = E_{n,q,\omega}^{(h)}. \tag{2.10}$$

From (2.9), we obtain

$$\int_{\mathbb{Z}_p^h} \omega^{x_1+\cdots+x_h} e^{(x+x_1+\cdots+x_h)t} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_h) = \frac{[2]_q^h e^{xt}}{\underbrace{(\omega q e^t+1)\cdots(\omega q e^t+1)}_{h\text{-times}}} = \sum_{n=0}^{\infty} E_{n,q,\omega}^{(h)}(x)\frac{t^n}{n!}, \tag{2.11}$$

where $E_{n,q,\omega}^{(h)}(x)$ is called twisted q-extension of Euler polynomials of higher order (cf. [11]). We note that if $\omega = 1$, then $E_{n,q,\omega}^{(h)}(x) = E_{n,q}^{(h)}(x)$ and $E_{n,q,\omega}^{(h)} = E_{n,q}^{(h)}$ (cf. [6]). We also note that

$$E_{n,q,\omega}^{(h)}(x) = \sum_{k=0}^{n}\binom{n}{k} E_{k,q,\omega}^{(h)} x^{n-k}. \tag{2.12}$$

The twisted q-extension of Euler polynomials of higher order, $E_{n,q,\omega}^{(h)}(x)$, is defined by means of the following generating function:

$$\begin{aligned} G_{q,\omega}^{(h)}(x,t) &= \underbrace{\frac{[2]_q}{\omega q e^t+1}\cdots\frac{[2]_q}{\omega q e^t+1}}_{h\text{-times}} e^{xt} \\ &= [2]_q^h e^{tx}\sum_{l_1=0}^{\infty}(-\omega)^{l_1}q^{l_1}e^{l_1 t}\cdots\sum_{l_h=0}^{\infty}(-\omega)^{l_h}q^{l_h}e^{l_h t} \\ &= [2]_q^h \sum_{l_1,\ldots,l_h=0}^{\infty}(-\omega)^{l_1+\cdots+l_h}q^{l_1+\cdots+l_h}e^{(l_1+\cdots+l_h+x)t} \\ &= \sum_{n=0}^{\infty}E_{n,q,\omega}^{(h)}(x)\frac{t^n}{n!}, \end{aligned} \tag{2.13}$$

where $|t+\log(\omega q)| < \pi$. From these generating functions of twisted q-extension of Euler polynomials of higher order, we construct twisted multiple q-Euler zeta functions as follows.

For $s \in \mathbb{C}$ and $x \in \mathbb{R}$ with $0 < x \le 1$, we define

$$\zeta_{q,\omega,E}^{(h)}(s,x) = [2]_q^h \sum_{l_1,\ldots,l_h=0}^{\infty} \frac{(-\omega)^{l_1+\cdots+l_h}q^{l_1+\cdots+l_h}}{(l_1+\cdots+l_h+x)^s}. \tag{2.14}$$

By the mth differentiation on both sides of (2.13) at $t = 0$, we obtain the following

$$E_{m,q,\omega}^{(h)}(x) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m G_{q,\omega}^{(h)}(x,t)\bigg|_{t=0} = [2]_q^h \sum_{l_1,\ldots,l_h=0}^{\infty}(-\omega)^{l_1+\cdots+l_h}q^{l_1+\cdots+l_h}(x+l_1+\cdots+l_h)^m \tag{2.15}$$

for $m = 0, 1, \ldots$.

From (2.14) and (2.15), we arrive at the following

$$\zeta^{(h)}_{q,\omega,E}(-m,x) = E^{(h)}_{m,q,\omega}(x), \quad m = 0,1,\ldots. \tag{2.16}$$

We set

$$\int_{X_p^h} = \underbrace{\int_{X_p}\cdots\int_{X_p}}_{h\text{-times}}. \tag{2.17}$$

Let χ be Dirichlet's character with odd conductor d. We define twisted q-extension of generalized Euler polynomials of higher order by means of the following generating function (cf. [11]):

$$\int_{X_p^h}\chi(x_1+\cdots+x_h)\omega^{x_1+\cdots+x_h}e^{(x+x_1+\cdots+x_h)t}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_h) = \sum_{n=0}^{\infty}E^{(h)}_{n,q,\omega,\chi}(x)\frac{t^n}{n!}. \tag{2.18}$$

Note that

$$\begin{aligned}
&\sum_{n=0}^{\infty}E^{(h)}_{n,q,\omega,\chi}(x)\frac{t^n}{n!}\\
&\quad = e^{xt}\underbrace{\int_{X_p}\cdots\int_{X_p}}_{h\text{-times}}\chi(x_1+\cdots+x_h)\omega^{x_1+\cdots+x_h}e^{(x_1+\cdots+x_h)t}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_h)\\
&\quad = e^{xt}\frac{1}{[d]^h_{-q}}\lim_{N\to\infty}\frac{1}{[p^N]_{(-q)^d}}\sum_{a_1=0}^{d-1}\sum_{x_1=0}^{p^N-1}\cdots\sum_{a_h=0}^{d-1}\sum_{x_h=0}^{p^N-1}\chi(a_1+dx_1+\cdots+a_h+dx_h)\\
&\qquad\times\omega^{a_1+dx_1+\cdots+a_h+dx_h}e^{(a_1+dx_1+\cdots+a_h+dx_h)t}(-q)^{a_1+dx_1+\cdots+a_h+dx_h}\\
&\quad = e^{xt}\frac{1}{[d]^h_{-q}}\sum_{a_1,\ldots,a_h=0}^{d-1}\chi(a_1+\cdots+a_h)\omega^{a_1+\cdots+a_h}(-q)^{a_1+\cdots+a_h}e^{(a_1+\cdots+a_h)t}\\
&\qquad\times\underbrace{\lim_{N\to\infty}\frac{1+q^d}{1+q^{dp^N}}\frac{1+\omega^{dp^N}q^{dp^N}e^{dp^N}}{1+\omega^dq^de^{dt}}\cdots\lim_{N\to\infty}\frac{1+q^d}{1+q^{dp^N}}\frac{1+\omega^{dp^N}q^{dp^N}e^{dp^N}}{1+\omega^dq^de^{dt}}}_{h-\text{times}}\\
&\quad = e^{xt}\frac{1}{[d]^h_{-q}}\sum_{a_1,\ldots,a_h=0}^{d-1}\chi(a_1+\cdots+a_h)\omega^{a_1+\cdots+a_h}(-q)^{a_1+\cdots+a_h}e^{(a_1+\cdots+a_h)t}\\
&\qquad\times\underbrace{\frac{1+q^d}{1+\omega^dq^de^{dt}}\cdots\frac{1+q^d}{1+\omega^dq^de^{dt}}}_{h\text{-times}}
\end{aligned} \tag{2.19}$$

since

$$\lim_{N\to\infty}q^{p^N} = 1 \quad \text{for } |1-q|_p < 1. \tag{2.20}$$

This allows us to rewrite (2.18) as

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,q,\omega,\chi}^{(h)}(x)\frac{t^n}{n!}\\
&\quad= e^{xt}\frac{1}{[d]_{-q}^{h}}\sum_{a_1,\ldots,a_h=0}^{d-1}\chi(a_1+\cdots+a_h)\omega^{a_1+\cdots+a_h}(-q)^{a_1+\cdots+a_h}e^{(a_1+\cdots+a_h)t}\\
&\qquad\times \underbrace{\frac{1+q^d}{1+\omega^d q^d e^{dt}}\cdots\frac{1+q^d}{1+\omega^d q^d e^{dt}}}_{h\text{-times}}\\
&\quad= [2]_q^h e^{xt}\sum_{a_1,\ldots,a_h=0}^{d-1}\chi(a_1+\cdots+a_h)\omega^{a_1+\cdots+a_h}(-q)^{a_1+\cdots+a_h}e^{(a_1+\cdots+a_h)t}\\
&\qquad\times \underbrace{\sum_{x_1=0}^{\infty}\left(-\omega^d q^d e^{dt}\right)^{x_1}\cdots\sum_{x_h=0}^{\infty}\left(-\omega^d q^d e^{dt}\right)^{x_h}}_{h\text{-times}}\\
&\quad= [2]_q^h e^{xt}\sum_{x_1,\ldots,x_h=0}^{\infty}\sum_{a_1,\ldots,a_h=0}^{d-1}\chi(a_1+dx_1+\cdots+a_h+dx_h)\\
&\qquad\times \omega^{a_1+dx_1+\cdots+a_h+dx_h}(-q)^{a_1+dx_1+\cdots+a_h+dx_h}e^{(a_1+dx_1+\cdots+a_h+dx_h)t}\\
&\quad= [2]_q^h\sum_{l_1,\ldots,l_h=0}^{\infty}(-1)^{l_1+\cdots+l_h}\chi(l_1+\cdots+l_h)\omega^{l_1+\cdots+l_h}q^{l_1+\cdots+l_h}e^{(x+l_1+\cdots+l_h)t}.
\end{aligned}\tag{2.21}$$

By applying the mth derivative operator $(\mathrm{d}/\mathrm{d}t)^m|_{t=0}$ in the above equation, we have

$$E_{m,q,\omega,\chi}^{(h)}(x)= [2]_q^h\sum_{l_1,\ldots,l_h=0}^{\infty}\chi(l_1+\cdots+l_h)\prod_{i=1}^{h}(-1)^{l_i}\omega^{l_i}q^{l_i}(x+l_1+\cdots+l_h)^m \tag{2.22}$$

for $m=0,1,\ldots.$

From these generating functions of twisted q-extension of generalized Euler polynomials of higher order, we construct twisted multiple q-Euler l-functions as follows. For $s\in\mathbb{C}$ and $x\in\mathbb{R}$ with $0<x\le 1$, we define

$$l_{q,\omega,E}^{(h)}(s,x,\chi)=[2]_q^h\sum_{l_1,\ldots,l_h=0}^{\infty}\frac{\chi(l_1+\cdots+l_h)\prod_{i=1}^{h}(-1)^{l_i}\omega^{l_i}q^{l_i}}{(l_1+\cdots+l_h+x)^s}. \tag{2.23}$$

From (2.22) and (2.23), we arrive at the following

$$l_{q,\omega,E}^{(h)}(-m,x,\chi)=E_{m,q,\omega,\chi}^{(h)}(x),\quad m=0,1,\ldots. \tag{2.24}$$

Let $s \in \mathbb{C}$ and $a_i, F \in \mathbb{Z}$ with F is an odd integer and $0 < a_i < F$, where $i = 1, \ldots, h$. Then twisted partial multiple q-Euler ζ-functions are as follows:

$$H^{(h)}_{q,\omega,E}(s, a_1, \ldots, a_h, x \mid F) = [2]^h_q \sum_{\substack{l_1,\ldots,l_h=0 \\ l_i \equiv a_i (\operatorname{mod} F),\ i=1,\ldots,h}}^{\infty} \frac{(-1)^{l_1+\cdots+l_h} \omega^{l_1+\cdots+l_h} q^{l_1+\cdots+l_h}}{(l_1 + \cdots + l_h + x)^s}. \tag{2.25}$$

For $i = 1, \ldots, h$, substituting $l_i = a_i + n_i F$ with F is odd into (2.25), we have

$$\begin{aligned} H^{(h)}_{q,\omega,E}&(s, a_1, \ldots, a_h, x \mid F) \\ &= [2]^h_q \sum_{n_1,\ldots,n_h=0}^{\infty} \frac{(-1)^{a_1+n_1F+\cdots+a_h+n_hF} \omega^{a_1+n_1F+\cdots+a_h+n_hF} q^{a_1+n_1F+\cdots+a_h+n_hF}}{(a_1 + n_1F + \cdots + a_h + n_hF + x)^s} \\ &= \frac{[2]^h_q}{[2]^h_{q^F}} \frac{(-\omega q)^{a_1+\cdots+a_h}}{F^s} [2]^h_{q^F} \sum_{n_1,\ldots,n_h=0}^{\infty} \frac{(-1)^{n_1+\cdots+n_h} (\omega^F)^{n_1+\cdots+n_h} (q^F)^{n_1+\cdots+n_h}}{(n_1 + \cdots + n_h + (a_1 + \cdots + a_h + x)/F)^s} \\ &= \frac{[2]^h_q}{[2]^h_{q^F}} \frac{(-\omega q)^{a_1+\cdots+a_h}}{F^s} \zeta^{(h)}_{q^F,\omega^F,E}\left(s, \frac{a_1 + \cdots + a_h + x}{F}\right). \end{aligned} \tag{2.26}$$

Then we obtain

$$H^{(h)}_{q,\omega,E}(s, a_1, \ldots, a_h, x \mid F) = \frac{[2]^h_q}{[2]^h_{q^F}} \frac{(-\omega q)^{a_1+\cdots+a_h}}{F^s} \zeta^{(h)}_{q^F,\omega^F,E}\left(s, \frac{a_1 + \cdots + a_h + x}{F}\right). \tag{2.27}$$

By using (2.12) and (2.27) and by substituting $s = -m,\ m = 0, 1, \ldots$, we get

$$\begin{aligned} H^{(h)}_{q,\omega,E}(-m, a_1, \ldots, a_h, x \mid F) = {}& \frac{[2]^h_q}{[2]^h_{q^F}} (-\omega q)^{a_1+\cdots+a_h} (a_1 + \cdots + a_h + x)^m \\ & \times \sum_{k=0}^{m} \binom{m}{k} \left(\frac{F}{a_1 + \cdots + a_h + x}\right)^k E^{(h)}_{k,q^F,\omega^F}. \end{aligned} \tag{2.28}$$

Therefore, we modify twisted partial multiple q-Euler zeta functions as follows:

$$\begin{aligned} H^{(h)}_{q,\omega,E}(s, a_1, \ldots, a_h, x \mid F) = {}& \frac{[2]^h_q}{[2]^h_{q^F}} (-\omega q)^{a_1+\cdots+a_h} (a_1 + \cdots + a_h + x)^{-s} \\ & \times \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{F}{a_1 + \cdots + a_h + x}\right)^k E^{(h)}_{k,q^F,\omega^F}. \end{aligned} \tag{2.29}$$

Let χ be a Dirichlet character with conductors d and $d \mid F$. From (2.23) and (2.27), we have

$$\begin{aligned} l^{(h)}_{q,\omega,E}(s, x, \chi) = {}& \frac{[2]^h_q}{[2]^h_{q^F}} F^{-s} \sum_{a_1,\ldots,a_h=0}^{F-1} (-\omega q)^{a_1+\cdots+a_h} \\ & \times \chi(a_1 + \cdots + a_h) \zeta^{(h)}_{q^F,\omega^F,E}\left(s, \frac{a_1 + \cdots + a_h + x}{F}\right) \\ = {}& \sum_{a_1,\ldots,a_h=0}^{F-1} \chi(a_1 + \cdots + a_h) H^{(h)}_{q,\omega,E}(s, x, a_1, \ldots, a_h, x \mid F). \end{aligned} \tag{2.30}$$

3. The multiple twisted Carlitz's type q-Euler polynomials and q-zeta functions

Let us consider the multiple twisted Carlitz's type q-Euler polynomials as follows:

$$E_{n,q,\omega}^{(z,h)}(x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} [x_1 + \cdots + x_h + x]_q^n \omega^{x_1+\cdots+x_h} q^{x_1(z-1)+\cdots+x_h(z-h)} d\mu_q(x_1) \cdots d\mu_q(x_h) \tag{3.1}$$

(cf. [1, 3]). These can be written as

$$E_{n,q,\omega}^{(z,h)}(x) = \frac{[2]_q^h}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} q^{ix} (-1)^i \frac{1}{1+\omega q^{z+i}} \cdots \frac{1}{1+\omega q^{z+i-h+1}}. \tag{3.2}$$

We may now mention the following formulae which are easy to prove:

$$\omega q^z E_{n,q,\omega}^{(z,h)}(x+1) + E_{n,q,\omega}^{(z,h)}(x) = [2]_q E_{n,q,\omega}^{(z-1,h-1)}(x). \tag{3.3}$$

From (3.2), we can derive generating function for the multiple twisted Carlitz's type q-Euler polynomials as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,q,\omega}^{(z,h)}(x) \frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \frac{[2]_q^h}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} q^{ix} (-1)^i \frac{1}{1+\omega q^{z+i}} \cdots \frac{1}{1+\omega q^{z+i-h+1}} \frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \frac{[2]_q^h}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} q^{ix} (-1)^i \sum_{l_1=0}^{\infty} \left(-\omega q^{z+i}\right)^{l_1} \cdots \sum_{l_h=0}^{\infty} \left(-\omega q^{z+i-h+1}\right)^{l_h} \frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \frac{[2]_q^h}{(1-q)^n} \sum_{l_1,\ldots,l_h=0}^{\infty} (-1)^{l_1+\cdots+l_h} \sum_{i=0}^{n} \binom{n}{i} q^{(x+l_1+\cdots+l_h)i} (-1)^i \\
&\qquad \times \omega^{l_1+\cdots+l_h} q^{l_1 z + l_2(z-1)+\cdots+l_h(z-h+1)} \frac{t^n}{n!} \\
&\quad = [2]_q^h \sum_{l_1,\ldots,l_h=0}^{\infty} (-1)^{l_1+\cdots+l_h} \omega^{l_1+\cdots+l_h} q^{l_1 z + l_2(z-1)+\cdots+l_h(z-h+1)} \\
&\qquad \times \sum_{n=0}^{\infty} [x + l_1 + \cdots + l_h]_q^n \frac{t^n}{n!} \\
&\quad = [2]_q^h \sum_{l_1,\ldots,l_h=0}^{\infty} (-\omega)^{l_1+\cdots+l_h} q^{l_1 z + l_2(z-1)+\cdots+l_h(z-h+1)} e^{[x+l_1+\cdots+l_h]_q t}.
\end{aligned} \tag{3.4}$$

Also, an obvious generating function for the multiple twisted Carlitz's type q-Euler polynomials is obtained, from (3.2), by

$$[2]_q^h e^{t/(1-q)} \sum_{j=0}^{n} (-1)^j q^{jx} \left(\frac{1}{1-q}\right)^j \frac{1}{1+\omega q^{z+j}} \cdots \frac{1}{1+\omega q^{z+j-h+1}} = E_{n,q,\omega}^{(z,h)}(x). \tag{3.5}$$

From now on, we assume that $q \in \mathbb{C}$ with $|q| < 1$. From (3.2) and (3.4), we note that

$$\begin{aligned} G_{q,\omega}^{(z,h)}(x,t) &= [2]_q^h \sum_{l_1,\dots,l_h=0}^{\infty} (-\omega)^{l_1+\cdots+l_h} q^{l_1 z + l_2(z-1)+\cdots+l_h(z-h+1)} e^{[x+l_1+\cdots+l_h]_q t} \\ &= \sum_{n=0}^{\infty} E_{n,q,\omega}^{(z,h)}(x) \frac{t^n}{n!}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} E_{n,q,\omega}^{(z,h)}(x) &= \frac{[2]_q^h}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} q^{ix} (-1)^i \frac{1}{1+\omega q^{z+i}} \cdots \frac{1}{1+\omega q^{z+i-h+1}} \\ &= [2]_q^h \sum_{l_1,\dots,l_h=0}^{\infty} (-\omega)^{l_1+\cdots+l_h} q^{l_1 z + l_2(z-1)+\cdots+l_h(z-h+1)} \left[x + l_1 + \cdots + l_h\right]_q^n. \end{aligned} \tag{3.7}$$

Thus we can define the multiple twisted Carlitz's type q-zeta functions as follows:

$$\zeta_{q,\omega}^{(z,h)}(s,x) = [2]_q^h \sum_{l_1,\dots,l_h=0}^{\infty} \frac{(-\omega)^{l_1+\cdots+l_h} q^{l_1 z + l_2(z-1)+\cdots+l_h(z-h+1)}}{\left[x + l_1 + \cdots + l_h\right]_q^s}. \tag{3.8}$$

In [12, Proposition 3], Yamasaki showed that the series $\zeta_{q,\omega}^{(z,h)}(s,x)$ converges absolutely for Re $(z) > h-1$, and it can be analytically continued to the whole complex plane $\mathbb{C}$. Note that if $h = 1$, then

$$\zeta_{q,\omega}^{(z,h)}(s,x) \longrightarrow \zeta_{q,\omega}^{(z)}(s,x) = [2]_q \sum_{l=0}^{\infty} \frac{(-\omega)^l q^{lz}}{[x+l]_q^s}. \tag{3.9}$$

In [13], Wakayama and Yamasaki studied q-analogue of the Hurwitz zeta function

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \tag{3.10}$$

defined by the q-series with two complex variable $s, z \in \mathbb{C}$:

$$\zeta_q^{(z)}(s,x) = \sum_{n=0}^{\infty} \frac{q^{(n+x)z}}{[x+n]_q^s}, \quad \mathrm{Re}(z) > 0, \tag{3.11}$$

and special values at nonpositive integers of the q-analogue of the Hurwitz zeta function.

Therefore, by the mth differentiation on both sides of (3.6) at $t = 0$, we obtain the following:

$$\begin{aligned} E_{m,q,\omega}^{(z,h)}(x) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m G_{q,\omega}^{(z,h)}(x,t)\bigg|_{t=0} \\ &= [2]_q^h \sum_{l_1,\dots,l_h=0}^{\infty} (-\omega)^{l_1+\cdots+l_h} q^{l_1 z + l_2(z-1)+\cdots+l_h(z-h+1)} \left[x + l_1 + \cdots + l_h\right]_q^m \end{aligned} \tag{3.12}$$

for $m = 0, 1, \ldots$.

From (3.7), (3.8), and (3.12), we have (3.13) which shows that the multiple twisted Carlitz's type q-zeta functions interpolate the multiple twisted Carlitz's type q-Euler numbers and polynomials. For $m = 0, 1, \ldots,$ we have

$$\zeta_{q,\omega}^{(z,h)}(-m, x) = E_{m,q,\omega}^{(z,h)}(x), \tag{3.13}$$

where $x \in \mathbb{R}$ and $0 < x \leq 1$.

Thus, we derive the analytic multiple twisted Carlitz's type q-zeta functions which interpolate multiple twisted Carlitz's type q-Euler polynomials. This gives a part of the answer to the question proposed in [10].

4. Remarks

For nonnegative integers m and n, we define the q-binomial coefficient $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_q$ by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}, \tag{4.1}$$

where $(a;q)_m = \prod_{k=0}^{m-1}(1 - aq^k)$ for $m \geq 1$ and $(a;q)_0 = 1$. For $h \in \mathbb{N}$, it holds that

$$\sum_{\substack{l_1,\ldots,l_h \geq 0 \\ l_1+\cdots+l_h = l}} q^{-(l_1+2l_2+\cdots+hl_h)} = q^{-lh} \begin{bmatrix} l+h-1 \\ h-1 \end{bmatrix}_q \tag{4.2}$$

(cf. [12, Lamma 2.3]). From (3.8), it is easy to see that

$$\begin{aligned}
\zeta_{q,\omega}^{(z,h)}(s, x) &= [2]_q^h \sum_{l=0}^{\infty} \sum_{\substack{l_1,\ldots,l_h \geq 0 \\ l_1+\cdots+l_h = l}} \frac{(-\omega)^{l_1+\cdots+l_h} q^{(z+1)(l_1+\cdots+l_h)-(l_1+2l_2+\cdots+hl_h)}}{[x + l_1 + \cdots + l_h]_q^s} \\
&= [2]_q^h \sum_{l=0}^{\infty} \frac{(-\omega)^l q^{(z+1)l}}{[l+x]_q^s} \sum_{\substack{l_1,\ldots,l_h \geq 0 \\ l_1+\cdots+l_h = l}} q^{-(l_1+2l_2+\cdots+hl_h)} \\
&= [2]_q^h \sum_{l=0}^{\infty} \begin{bmatrix} l+h-1 \\ h-1 \end{bmatrix}_q \frac{(-\omega)^l q^{(z-h+1)l}}{[l+x]_q^s}.
\end{aligned} \tag{4.3}$$

We set $[m]_q! = [m]_q [m-1]_q \cdots [1]_q$ for $m \in \mathbb{N}$. The following identity has been studied in [12]:

$$\begin{bmatrix} l+h-1 \\ h-1 \end{bmatrix}_q = \frac{1}{[h-1]_q!} \prod_{j=1}^{h-1} \left([l+x]_q - q^{l+j} [x-j]_q \right) = \sum_{k=0}^{h-1} q^{l(h-1-k)} P_{q,h}^k(x) [l+x]_q^k, \tag{4.4}$$

where $P_{q,h}^k(x),\ 0 \leq k \leq h-1$, is a function of x defined by

$$P_{q,h}^k(x) = \frac{(-1)^{h-1-k}}{[h-1]_q!} \sum_{1 \leq m_1 < \cdots < m_{h-1-k} \leq h-1} q^{m_1+\cdots+m_{h-1-k}} [x - m_1]_q \cdots [x - m_{h-1-k}]_q \tag{4.5}$$

for $0 \leq k \leq h-2$ and $P_{q,h}^{h-1}(x) = 1/[h-1]_q!$. By using (3.9), (4.3), and (4.5), we have

$$\zeta_{q,\omega}^{(z,h)}(s,x) = [2]_q^h \sum_{k=0}^{h-1} P_{q,h}^k(x) \sum_{l=0}^{\infty} \frac{(-\omega)^l q^{(z-k)l}}{[l+x]_q^{s-k}} = [2]_q^{h-1} \sum_{k=0}^{h-1} P_{q,h}^k(x) \zeta_{q,\omega}^{(z-k)}(s-k,x), \tag{4.6}$$

and so

$$\zeta_{q,\omega}^{(z,h)}(-m,x) = [2]_q^{h-1} \sum_{k=0}^{h-1} P_{q,h}^k(x) \zeta_{q,\omega}^{(z-k)}(-m-k,x). \tag{4.7}$$

The values of $\zeta_{q,\omega}^{(z,h)}(-m,x)$ at $h = 2, 3$ are given explicitly as follows:

$$\begin{aligned}
\zeta_{q,\omega}^{(z,2)}(-m,x) &= (1+q)\Big(\zeta_{q,\omega}^{(z-1)}(-m-1,x) - q[x-1]_q \zeta_{q,\omega}^{(z)}(-m,x)\Big), \\
\zeta_{q,\omega}^{(z,3)}(-m,x) &= (1+q)\Big\{\zeta_{q,\omega}^{(z-2)}(-m-2,x) \\
&\qquad - \big(q[x-1]_q + q^2[x-2]_q\big)\zeta_{q,\omega}^{(z-1)}(-m-1,x) \\
&\qquad + q^3[x-1]_q[x-2]_q \zeta_{q,\omega}^{(z)}(-m,x)\Big\}.
\end{aligned} \tag{4.8}$$

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