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Research Article

A New q-Analogue of Bernoulli Polynomials Associated with p-Adic q-Integrals

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We will study a new q-analogue of Bernoulli polynomials associated with p-adic q-integrals. Furthermore, we examine the Hurwitz-type q-zeta functions, replacing p-adic rational integers x with a q-analogue $[x]_q$ for a p-adic number q with $|q-1|_p < 1$, which interpolate q-analogue of Bernoulli polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, represent the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$ and q is a p-adic number in $\mathbb{C}_p$ with $|q-1|_p < 1$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q} \tag{1.1}$$

(cf. [1–13]) for all $x \in \mathbb{Z}_p$. Hence, $\lim_{q\to 1}|x|_q = x$. For a fixed odd positive integer d with $(p,d)=1$, let

$$\begin{gathered} X = X_d = \lim_{\overleftarrow{n}} \frac{\mathbb{Z}}{dp^n\mathbb{Z}}, \qquad X_1 = \mathbb{Z}_p, \\ X^* = \underset{\substack{0<a<dp\\(a,p)=1}}{\longrightarrow} \cup \left(a+dp\mathbb{Z}_p\right), \\ a+dp^n\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \left(\operatorname{mod} dp^n\right)\right\}, \end{gathered} \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^n$. For any $n \in \mathbb{N}$,

$$\mu_q\left(a + dp^n \mathbb{Z}_p\right) = \frac{q^a}{\left[dp^n\right]_q} \tag{1.3}$$

is known to be a distribution on X (cf. [1–13]).

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y} \tag{1.4}$$

have a limit $l = f'(a)$ as $(x, y) \to (a, a)$ (cf. [2, 6, 7]). The p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$ was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{\left[p^N\right]_q} \sum_{x=0}^{p^N - 1} f(x) q^x. \tag{1.5}$$

By using p-adic q-integrals on $\mathbb{Z}_p$, it is well known that

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x + s)^n d\mu_1(s) \frac{t^n}{n!}, \tag{1.6}$$

where $\mu_1(x + p^n \mathbb{Z}_p) = 1/p^n$. Then we note that the Bernoulli polynomials $B_n(x)$ were defined as

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.7}$$

From (1.6) and (1.7), we have

$$B_n(x) = \int_{\mathbb{Z}_p} (x + s)^n d\mu_1(s) \tag{1.8}$$

for all $n \in \mathbb{N} \cup \{0\}$. We note that $[0]_q = (1 - q^0)/(1 - q) = 0$.

In Section 2, we study a q-analogue of Bernoulli polynomials associated with p-adic q-integrals—simply, we say q-Bernoulli polynomials. In Section 3, we examine Hurwitz-type q-zeta functions, replacing p-adic rational integers x with a q-analogue $[x]_q$ for a p-adic number q with $|q - 1|_p < 1$, which interpolate q-analogue of Bernoulli polynomials.

2. A new q-analogue of Bernoulli polynomials

In this section, from the view of (1.8), we can define a new q-analogue of Bernoulli polynomials as follows:

$$\beta_n^q(x)=\int_{\mathbb{Z}_p}\left([x]_q+[s]_q\right)^n d\mu_q(s). \tag{2.1}$$

We note that $\beta_n^q=\beta_n^q(0)$ are called the q-Bernoulli numbers. Then we find some properties of q-Bernoulli numbers and polynomials as follows.

Theorem 2.1. *For $n\in\mathbb{N}\cup\{0\}$, one has*

$$\beta_n^q=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{[l+1]_q}. \tag{2.2}$$

Proof. From (1.5) with $x=0$, we can find the following:

$$\begin{aligned}\beta_n^q&=\int_{\mathbb{Z}_p}[s]_q^n d\mu_q(s)\\&=\lim_{N\to\infty}\sum_{j=0}^{p^N-1}[j]_q^n\frac{q^j}{[p^N]_q}\\&=\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{(1-q)^{n-1}}\lim_{N\to\infty}\sum_{j=0}^{p^N-1}q^{j(l+1)}\frac{1}{1-q^{p^N}}\\&=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{[l+1]_q}.\end{aligned} \tag{2.3}$$

□

Theorem 2.2. *For $n\in\mathbb{N}\cup\{0\}$ and d being an odd positive integer with $(p,d)=1$, one has*

$$\beta_n^q(x)=[d]_q^{n-1}\sum_{l=0}^{n}\binom{n}{l}\beta_l^{q^d}\sum_{i=0}^{d-1}q^{i(l+1)}\left(\left[\frac{x}{d}\right]_{q^d}+\left[\frac{i}{d}\right]_{q^d}\right)^{n-l}. \tag{2.4}$$

Proof. From (1.5), we can derive (2.4) as follows:

$$\begin{aligned}\beta_n^q(x)&=\int_{\mathbb{Z}_p}\left([x]_q+[s]_q\right)^n d\mu_q(s)\\&=\lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{a=0}^{dp^N-1}\left([x]_q+[a]_q\right)^n q^a\\&=\lim_{N\to\infty}\frac{1-q}{1-q^{dp^N}}\sum_{i=0}^{d-1}\sum_{k=0}^{p^N-1}\left([x]_q+[i+dk]_q\right)^n q^{i+dk}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \frac{1}{[d]_q} \frac{1}{[p^N]_{q^d}} \sum_{i=0}^{d-1} q^i \sum_{k=0}^{p^N-1} [d]_q^n \left(\left[\frac{x}{d}\right]_{q^d} + \left[\frac{i}{d}\right]_{q^d} + q^i [k]_{q^d} \right)^n (q^d)^k \\
&= \lim_{N\to\infty} \frac{1}{[d]_q} \frac{1}{[p^N]_{q^d}} \sum_{i=0}^{d-1} q^i \sum_{k=0}^{p^N-1} \sum_{l=0}^{n} \binom{n}{l} \left(\left[\frac{x}{d}\right]_{q^d} + \left[\frac{i}{d}\right]_{q^d} \right)^{n-l} \left(q^i [k]_{q^d}\right)^l (q^d)^k \\
&= [d]_q^{n-1} \sum_{i=0}^{d-1} \sum_{l=0}^{n} \binom{n}{l} q^{i(l+1)} \left(\left[\frac{x}{d}\right]_{q^d} + \left[\frac{i}{d}\right]_{q^d} \right)^{n-l} \lim_{N\to\infty} \frac{1}{[p^N]_{q^d}} \sum_{k=0}^{p^N-1} [k]_{q^d}^l (q^d)^k \\
&= [d]_q^{n-1} \sum_{i=0}^{d-1} \sum_{l=0}^{n} \binom{n}{l} q^{i(l+1)} \left(\left[\frac{x}{d}\right]_{q^d} + \left[\frac{i}{d}\right]_{q^d} \right)^{n-l} \int_{\mathbb{Z}_p} [s]_{q^d}^l d\mu_{q^d}(s) \\
&= [d]_q^{n-1} \sum_{l=0}^{n} \binom{n}{l} \beta_l^{q^d} \sum_{i=0}^{d-1} q^{i(l+1)} \left(\left[\frac{x}{d}\right]_{q^d} + \left[\frac{i}{d}\right]_{q^d} \right)^{n-l},
\end{aligned}
\tag{2.5}
$$

since $a = i + dk$ and

$$
\left([x]_q + [i+dk]_q\right)^n = [d]_q^n \left(\left[\frac{x}{d}\right]_{q^d} + \left[\frac{i}{d}\right]_{q^d} + q^i [k]_{q^d} \right)^n \tag{2.6}
$$

for $a = 0, 1, \ldots, dp^N - 1$, $i = 0, 1, \ldots, d-1$, and $k = 0, 1, \ldots, p^N - 1$.

Let $G^q(x,t)$ be the generating function of q-Bernoulli polynomials as follows:

$$
G^q(x,t) = \sum_{n=0}^{\infty} \beta_n^q(x) \frac{t^n}{n!}. \tag{2.7}
$$

From (2.2) and (2.7), we can obtain the following theorem. □

Theorem 2.3. *Let $G^q(x,t)$ be as in the above generating function. Then, one has*

$$
G^q(x,t) = (1-q) \sum_{m=0}^{\infty} q^m e^{([x]_q + [m]_q)t}. \tag{2.8}
$$

Proof. By using (2.2) and (2.7), we can derive (2.8) as follows:

$$
\begin{aligned}
G^q(x,t) &= e^{([x]_q + \beta^q)t} = e^{[x]_q t} e^{\beta^q t} = e^{[x]_q t} \sum_{n=0}^{\infty} \beta_n^q \frac{t^n}{n!} \\
&= e^{[x]_q t} \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{[l+1]_q} \frac{t^n}{n!} \\
&= e^{[x]_q t} \sum_{n=0}^{\infty} \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1-q^{l+1}} \frac{t^n}{n!} \\
&= e^{[x]_q t} \sum_{n=0}^{\infty} \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{m=0}^{\infty} q^{(l+1)m} \frac{t^n}{n!}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{[x]_q t}(1-q)\sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty}\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lm}\frac{t^n}{n!}\\
&= e^{[x]_q t}(1-q)\sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty}[m]_q^n\frac{t^n}{n!}\\
&= (1-q)\sum_{m=0}^{\infty} q^m e^{([x]_q+[m]_q)t}.
\end{aligned}
\tag{2.9}
$$

□

3. A new formula for Hurwitz-type q-zeta functions

In this section, we consider the generating functions $F(t,x)$ which interpolate the q-Bernoulli polynomials $\beta_n^{*q}(x)$ as follows:

$$
F(t,x) = \sum_{m=0}^{\infty} q^m e^{([x]_q+[m]_q)t} = \sum_{m=0}^{\infty}\beta_n^{*q}(x)\frac{t^m}{m!}. \tag{3.1}
$$

From (3.1), we directly obtain the following theorem.

Theorem 3.1. *For each $k \in \mathbb{N}\cup\{0\}$, one has*

$$
\beta_k^{*q}(x) = \sum_{m=0}^{\infty} q^m\left([x]_q+[m]_q\right)^k. \tag{3.2}
$$

Proof. By the kth differentiation on both sides of (3.1), we can derive (3.2) as follows:

$$
\beta_k^{*q}(x) = \frac{d^k}{dt^k}F(x,t)|_{t=0} = \sum_{m=0}^{\infty} q^m\left([x]_q+[m]_q\right)^k. \tag{3.3}
$$

We remark that

$$
-\frac{\beta_k^{*q}(x)}{k} = \frac{1}{k}\sum_{m=0}^{\infty} q^m\left([x]_q+[m]_q\right)^k \tag{3.4}
$$

for $k \in \mathbb{N}$. From (3.2), we derive a q-extension of Hurwitz-type zeta function as follows: for $s \in \mathbb{C}$ with $\Re(s) > 1$ and $\Re(x) > 0$, we define

$$
\zeta^q(s,x) = \frac{1}{1-s}\sum_{m=0}^{\infty}\frac{q^m}{\left([x]_q+[m]_q\right)^s}. \tag{3.5}
$$

Note that the functions $\zeta^q(s,x)$ are analytic on $\Re(s) > 1$ and they have simple pole at $s = 1$. From (3.2), (3.4), and (3.5), we can see that Hurwitz-type q-zeta functions interpolate q-Bernoulli polynomials as follows. □

Theorem 3.2. *For each $k \in \mathbb{N}$, one has*

$$\zeta^q(1-k,x) = -\frac{\beta_k^{*q}(x)}{k}. \tag{3.6}$$

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