Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2007, Article ID 79406, 30 pages
doi:10.1155/2007/79406

Research Article
Navier-Stokes Equations with Potentials

Adriana-Ioana Lefter

Received 11 March 2007; Accepted 8 May 2007

Recommended by Viorel Barbu

We study Navier-Stokes equations perturbed with a maximal monotone operator, in a bounded domain, in 2D and 3D. Using the theory of nonlinear semigroups, we prove existence results for strong and weak solutions. Examples are also provided.

1. Introduction

Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$, $n = 2,3$, be an open and bounded domain, with a smooth boundary $\partial\Omega$ (of class C^2, e.g.). Consider the perturbed Navier-Stokes equations

$$\begin{gathered}
\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y + \Phi(y) + \nabla p \ni g, \quad \text{in } Q = \Omega\times(0,T), \\
\operatorname{div} y = 0, \quad \text{in } Q, \\
y = 0, \quad \text{on } \Sigma = \partial\Omega\times(0,T), \\
y(\cdot,0) = y_0, \quad \text{in } \Omega,
\end{gathered} \tag{1.1}$$

where $y = (y_1, y_2, \dots, y_n)$ is the velocity field, p is the scalar pressure. The density of external forces is $g = (g_1, g_2, \dots, g_n)$, the constant $\nu > 0$ is the kinematic viscosity coefficient, and the perturbation Φ is a maximal monotone operator. Such a nonlinear term Φ arises usually as a feedback nonlinear controller.

In this section, we describe the functional framework and we rewrite the Navier-Stokes equations in an abstract form. The main existence and uniqueness results for strong solutions are stated in Section 2. The first of these theorems is proved in Section 3 and the others in Section 4. Section 5 is concerned with weak solutions. The last section is devoted to examples.

We will use the standard spaces (see, e.g., [1–3])

$$H = \{y \in (L^2(\Omega))^n;\ \operatorname{div} y = 0 \text{ in } \Omega, y \cdot n_{\partial\Omega} = 0 \text{ on } \partial\Omega\},$$
$$V = \{y \in (H_0^1(\Omega))^n;\ \operatorname{div} y = 0 \text{ in } \Omega\}. \tag{1.2}$$

H is a real Hilbert space endowed with L^2-norm $|\cdot|$ and V is a real Hilbert space endowed with H_0^1-norm $\|\cdot\| = |\nabla\cdot|$. Moreover, denoting by V' the dual space of V and considering H identified with its own dual, we have $V \subset H \subset V'$ algebraically and topologically with compact injections.

Here, $(\cdot,\cdot)$ denotes the scalar product of H and the pairing between V and its dual V'. The norm of V' is denoted by $\|\cdot\|_{V'}$.

Let $A \in L(V,V')$ (the space of linear continuous operators from V in V'), $(Ay,z) = \sum_{i=1}^n \int_\Omega \nabla y_i \cdot \nabla z_i dx$, for all $y,z \in V$.

We have $(Ay,y) = \|y\|^2$, for all $y \in V$. We set $D(A) = \{y \in V;\ Ay \in H\}$ and denote again by A the restriction of A to H.

Let $b : V \times V \times V \to \mathbb{R}$ the trilinear continuous functional defined by

$$b(y,z,w) = \sum_{i,j=1}^n \int_\Omega y_i \frac{\partial z_j}{\partial x_i} w_j dx, \quad \forall y,z,w \in V. \tag{1.3}$$

The functional b satisfies (see, e.g., [1–3])

$$b(y,w,w) = 0, \quad b(y,z,w) = -b(y,w,z), \quad \forall y,z,w \in V, \tag{1.4}$$

$$|b(y,z,w)| \le C|y|^{1/2}\|y\|^{1/2}\|z\|^{1/2}|Az|^{1/2}|w|, \quad \forall y,w \in V,\ z \in D(A)\ (n=2), \tag{1.5}$$

$$|b(y,z,w)| \le C|y|^{1/2}\|y\|^{1/2}|z|^{1/2}\|z\|^{1/2}\|w\|, \quad \forall y,z,w \in V \text{ (for } n=2), \tag{1.6}$$

$$|b(y,z,w)| \le C\|y\|\|z\|^{1/2}|Az|^{1/2}|w|, \quad \forall y,w \in V,\ z \in D(A) \text{ (for } n=3), \tag{1.7}$$

$$|b(y,z,w)| \le C|y|^{1/2}\|y\|^{1/2}\|z\|\|w\|, \quad \forall y,z,w \in V \text{ (for } n=3), \tag{1.8}$$

$$|b(y,z,w)| \le C\|y\|\|z\|\|w\|, \quad \forall y,z,w \in V \text{ (for } n=2,3). \tag{1.9}$$

Let $B : V \to V'$ be defined by

$$(By,w) = b(y,y,w), \quad \forall y,w \in V. \tag{1.10}$$

In this setting, equations (1.1) may be rewritten

$$\frac{dy}{dt}(t) + \nu Ay(t) + By(t) + \Phi(y(t)) \ni f(t), \quad t \in (0,T)$$
$$y(0) = y_0, \tag{1.11}$$

where $f = Pg$, $P : (L^2(\Omega))^n \to H$ is the Leray projection.

Suppose Φ satisfies the following hypotheses:

(h_1) $\Phi = \partial\varphi$, where $\varphi : H \to \overline{\mathbb{R}}$ is a lower semicontinuous proper convex function (hence Φ is a maximal monotone operator in $H \times H$);

(h_2) $0 \in D(\Phi)$;

(h_3) there exist two constants $\gamma \geq 0$, $\alpha \in (0, (1/\nu))$ such that

$$(Au, \Phi_\lambda(u)) \geq -\gamma(1 + |u|^2) - \alpha \left|\Phi_\lambda(u)\right|^2, \quad \forall \lambda > 0,\ \forall u \in D(A), \tag{1.12}$$

where $\Phi_\lambda = (1/\lambda)(I - (I + \lambda\Phi)^{-1}) : H \to H$ is the Yosida approximation of Φ.

We consider the classical definition of the maximal monotone operator. We will denote $|\Phi(u)| = \inf\{|z|;\ z \in \Phi(u)\}$, where $u \in D(\Phi)$.

In the sequel, the symbol $\rightharpoonup$ will be used to denote convergence in the weak topology, while the strong convergence will be denoted by $\to$.

2. Main results for strong solutions

Theorem 2.1. *Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be an open and bounded domain, with a smooth boundary. Assume that $\Phi \subset H \times H$ satisfies the hypotheses (h_1)–(h_3). Let $y_0 \in D(A) \cap D(\Phi)$ and $f \in W^{1,1}(0, T; H)$.*

If $n = 2$, there exists a unique $y \in W^{1,\infty}(0, T; H) \cap L^\infty(0, T; D(A)) \cap C([0, T]; V)$ such that

$$\begin{gathered} \frac{dy}{dt}(t) + \nu Ay(t) + By(t) + \Phi(y(t)) \ni f(t), \quad \text{a.e. } t \in (0, T), \\ y(0) = y_0. \end{gathered} \tag{2.1}$$

Moreover, y is right differentiable, $(d^+/dt)y$ is right continuous, and

$$\frac{d^+}{dt} y(t) + \left(\nu Ay(t) + By(t) + \Phi(y(t)) - f(t)\right)^0 = 0, \quad \forall t \in [0, T). \tag{2.2}$$

If $n = 3$, the solution y exists on some interval $[0, T_0)$, where

$$T_0 = T_0\left(\|f\|_{L^2(0,T;H)}, \|y_0\|^2\right) \leq T. \tag{2.3}$$

We have denoted by $y \to (\nu Ay + By + \Phi(y) - f(t))^0$ the minimal section of the multivalued mapping $y \to (\nu Ay + By + \Phi(y) - f(t))$.

If we ask for lower regularity of the initial data, we obtain the following results.

Theorem 2.2 (case $n = 2$). *Let $T > 0$ and let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain, with a smooth boundary. Assume that $\Phi \subset H \times H$ satisfies the hypotheses (h_1)–(h_3). Let $y_0 \in V \cap D(\Phi)$, $f \in L^2(0, T; H)$. Then there exists a unique solution $y \in C([0, T]; H) \cap L^2(0, T; D(A)) \cap L^\infty(0, T; V)$ with $dy/dt \in L^2(0, T; H)$, $By \in L^2(0, T; H)$ for*

$$\begin{gathered} \frac{dy}{dt}(t) + \nu Ay(t) + By(t) + \Phi(y(t)) \ni f(t), \quad \text{a.e. } t \in (0, T), \\ y(0) = y_0. \end{gathered} \tag{2.4}$$

Theorem 2.3 (case $n = 3$). *Let $T > 0$ and let $\Omega \subset \mathbb{R}^3$ be an open and bounded domain, with a smooth boundary. Assume that $\Phi \subset H \times H$ satisfies the hypotheses (h_1)–(h_3). Let $y_0 \in V \cap D(\Phi)$, $f \in L^2(0,T;H)$.*

Then there exists $T_0 = T_0(\|f\|_{L^2(0,T;H)}, \|y_0\|^2) \le T$ such that the problem

$$\frac{dy}{dt}(t) + \nu Ay(t) + By(t) + \Phi(y(t)) \ni f(t), \quad \text{a.e. } t \in (0,T_0)$$
$$y(0) = y_0 \tag{2.5}$$

has a unique solution

$$y \in C([0,T_0];H) \cap L^2(0,T_0;D(A)) \cap L^\infty(0,T_0;V),$$
$$\frac{dy}{dt} \in L^2(0,T_0;H), \qquad By \in L^2(0,T_0;H). \tag{2.6}$$

Remark 2.4. We obtain the same results if Φ satisfies the following hypotheses:

(H_1) Φ is a single-valued maximal monotone operator in $H \times H$;

(H_2) there exist three constants $\gamma_1, \gamma_2 \ge 0$, and $\alpha \in (0,\nu)$ such that

$$|\Phi(u)| \le \alpha|Au| + \gamma_1\|u\| + \gamma_2, \quad \forall u \in D(A). \tag{2.7}$$

In the sequel, we use the same symbol C for various positive constants.

3. Proof of Theorem 2.1

The proof uses the theory of nonlinear differential equations of accretive type in Banach spaces. In order to obtain a quasi-m-accretive operator in the left-hand side of the Navier-Stokes equation (Proposition 3.1), we have to substitute the nonlinearity B with a truncation B_N, $N \in \mathbb{N}^*$. We may then state existence and uniqueness results for the approximate equations (3.2), (3.34) involving B_N, Φ, and B_N, Φ_λ, $\lambda > 0$ instead of B, Φ (Propositions 3.2, 3.3).

We intend to prove that for N large enough, the solution of the truncated problem involving B_N, Φ coincides with the solution of the initial problem. To this aim, we need to obtain estimates on the solution y_N of problem (3.2). In order to do this, we are obliged to deduce the convenient estimates first on problem (3.34) (the one involving Φ_λ) because relation (1.12) does not extend in a suitable way to arbitrary elements of $\Phi(y_N(t))$. Passing to the limit with $\lambda \to 0$ in (3.34), we return to the problem in B_N, Φ and conclude the proof.

3.1. Approximate problems: existence and uniqueness. For $N \in \mathbb{N}^*$, define the modified nonlinearity $B_N : V \to V'$,

$$B_N y = \begin{cases} By & \text{if } \|y\| \le N, \\ \left(\dfrac{N}{\|y\|}\right)^2 By & \text{if } \|y\| > N, \end{cases} \tag{3.1}$$

and consider the approximate equation

$$\frac{dy_N}{dt}(t) + \nu A y_N(t) + B_N y_N(t) + \Phi(y_N(t)) \ni f(t), \quad t \in (0,T) \\ y_N(0) = y_0. \tag{3.2}$$

Proposition 3.1 is one of the main ingredients of the proof.

Proposition 3.1. *Let $N \in \mathbb{N}^*$ be fixed. Let $\Phi \subset H \times H$ be a maximal monotone operator with $0 \in D(\Phi)$. Assume that there exist two constants $\gamma \geq 0$, $\alpha \in (0, 1/\nu)$ such that relation (1.12) is verified.*

Define the operator $\Lambda_N : D(\Lambda_N) \to H$, $\Lambda_N = \nu A + B_N + \Phi + \alpha_N I$, $\alpha_N > 0$, where $D(\Lambda_N) = \{u \in H;\ \varnothing \neq \Lambda_N(u) \subset H\}$. Then $D(\Lambda_N) = D(A) \cap D(\Phi)$ and Λ_N is a maximal monotone in $H \times H$ for α_N large enough.

Moreover, there exists a constant $C_N > 0$ such that

$$|Aw| \leq C_N \left(1 + |w|^2 + \left|\nu A w + B_N w + \Phi_\lambda(w)\right|^2\right)^{3/2}, \quad \forall w \in D(A),\ \forall \lambda > 0, \tag{3.3}$$

$$|Aw| \leq C_N \left(1 + |w|^2 + \left|\nu A w + B_N w + \eta\right|^2\right)^{3/2}, \quad \forall w \in D(A) \cap D(\Phi),\ \forall \eta \in \Phi(w). \tag{3.4}$$

Proof. It has been proved in [4] (see Lemma 5.1, page 292) that $\nu A + B_N$ applies $D(A)$ into H and that for α_N large enough, the operator $\Gamma_N = \nu A + B_N + \alpha_N I$ with $D(\Gamma_N) = D(A)$ is (maximal) monotone in $H \times H$. Then $D(A) \cap D(\Phi) \subset D(\Lambda_N)$ and $\Lambda_N = \Gamma_N + \Phi$ is the sum of two monotone operators, and by consequence it is a monotone. In order to obtain the maximal monotony of Λ_N, it is sufficient to prove that $R(I + \Lambda_N) = H$.

Let $f \in H$ and $\lambda > 0$ a fixed. We approximate the equation

$$u + \nu A u + B_N u + \Phi(u) + \alpha_N u \ni f \tag{3.5}$$

by the equation

$$u_\lambda + \nu A u_\lambda + B_N u_\lambda + \Phi_\lambda(u_\lambda) + \alpha_N u_\lambda = f, \quad \lambda > 0, \tag{3.6}$$

that is

$$u_\lambda + \Gamma_N u_\lambda + \Phi_\lambda(u_\lambda) = f, \tag{3.7}$$

where Φ_λ is the Yosida approximation of Φ. By the properties of the Yosida approximation, Φ_λ is demicontinuous monotone and its sum with the maximal monotone operator Γ_N is maximal monotone, which implies the existence of a solution $u_\lambda \in D(A)$ for (3.6). The uniqueness follows by monotony arguments.

Let $\mu_N = \alpha_N + 1$; then (3.6) reads

$$\nu A u_\lambda + B_N u_\lambda + \Phi_\lambda(u_\lambda) + \mu_N u_\lambda = f, \quad \lambda > 0. \tag{3.8}$$

We first multiply (3.8) by u_λ and infer that

$$\nu \|u_\lambda\|^2 + (B_N u_\lambda, u_\lambda) + (\Phi_\lambda(u_\lambda), u_\lambda) + \mu_N |u_\lambda|^2 = (f, u_\lambda). \tag{3.9}$$

But $b(u_\lambda,u_\lambda,u_\lambda)=0$. Also, the operator Φ_λ is monotone, with $0\in D(\Phi_\lambda)=H$, which implies that $(\Phi_\lambda(u_\lambda),u_\lambda)\geq(\Phi_\lambda(0),u_\lambda)$, and we have

$$-(\Phi_\lambda(0),u_\lambda)\leq\frac{1}{\mu_N}|\Phi(0)|^2+\frac{\mu_N}{4}|u_\lambda|^2. \tag{3.10}$$

Equation (3.9) yields

$$\nu\|u_\lambda\|^2+\frac{\mu_N}{2}|u_\lambda|^2\leq\frac{1}{\mu_N}|f|^2+\frac{1}{\mu_N}|\Phi(0)|^2. \tag{3.11}$$

Consequently,

$$|u_\lambda|^2,\|u_\lambda\|^2\leq C(1+|f|^2),\quad \forall\lambda>0, \tag{3.12}$$

where the constant $C>0$ does not depend on λ.

Next, (3.8) is multiplied by Au_λ, which gives

$$\nu|Au_\lambda|^2+(B_Nu_\lambda,Au_\lambda)+(\Phi_\lambda(u_\lambda),Au_\lambda)+\mu_N\|u_\lambda\|^2=(f,Au_\lambda). \tag{3.13}$$

But

$$\begin{aligned}|(B_Nu_\lambda,Au_\lambda)|&\leq|b(u_\lambda,u_\lambda,Au_\lambda)|\\&\leq\begin{cases}C|u_\lambda|^{1/2}\|u_\lambda\||Au_\lambda|^{3/2}\leq\dfrac{\nu}{4}|Au_\lambda|^2+C|u_\lambda|^2\|u_\lambda\|^4, & n=2,\\ C\|u_\lambda\|^{3/2}|Au_\lambda|^{3/2}\leq\dfrac{\nu}{4}|Au_\lambda|^2+C\|u_\lambda\|^6, & n=3,\end{cases}\\&\leq\frac{\nu}{4}|Au_\lambda|^2+C(1+|f|^2)^3,\end{aligned} \tag{3.14}$$

where $C>0$ denotes several positive constants (not depending on λ). We used estimates (1.5) in the case $n=2$, (1.7) in the case $n=3$, then Young inequality and (3.12).

Recalling also hypothesis (1.12), (3.13) implies that

$$\begin{aligned}&\nu|Au_\lambda|^2-\frac{\nu}{4}|Au_\lambda|^2-C(1+|f|^2)^3-\gamma(1+|u_\lambda|^2)-\alpha|\Phi_\lambda(u_\lambda)|^2+\mu_N\|u_\lambda\|^2\\&\quad\leq\frac{1}{\nu}|f|^2+\frac{\nu}{4}|Au_\lambda|^2.\end{aligned} \tag{3.15}$$

Ignoring the term $\mu_N\|u_\lambda\|^2\geq 0$, by (3.12) the above relation reads

$$\frac{\nu}{2}|Au_\lambda|^2\leq\alpha|\Phi_\lambda(u_\lambda)|^2+C(1+|f|^2)^3,\quad\forall\lambda>0, \tag{3.16}$$

where $C>0$ denotes several positive constants (not depending on λ).

Finally, we multiply (3.8) by $\Phi_\lambda(u_\lambda)$ and obtain

$$\nu(Au_\lambda,\Phi_\lambda(u_\lambda))+(B_Nu_\lambda,\Phi_\lambda(u_\lambda))+|\Phi_\lambda(u_\lambda)|^2+\mu_N(u_\lambda,\Phi_\lambda(u_\lambda))=(f,\Phi_\lambda(u_\lambda)). \tag{3.17}$$

As shown before,

$$\mu_N(u_\lambda,\Phi_\lambda(u_\lambda)) \geq -\frac{\mu_N}{2}(|\Phi(0)|^2+|u_\lambda|^2). \tag{3.18}$$

Also,

$$\begin{aligned} |(B_N u_\lambda,\Phi_\lambda(u_\lambda))| &\leq |b(u_\lambda,u_\lambda,\Phi_\lambda(u_\lambda))| \\ &\leq \begin{cases} C|u_\lambda|^{1/2}\|u_\lambda\||Au_\lambda|^{1/2}|\Phi_\lambda(u_\lambda)|, & n=2, \\ C\|u_\lambda\|^{3/2}|Au_\lambda|^{1/2}|\Phi_\lambda(u_\lambda)|, & n=3, \end{cases} \\ &\leq \frac{1-\nu\alpha}{4}|\Phi_\lambda(u_\lambda)|^2+C(1+|f|^2)^{3/2}|Au_\lambda|. \end{aligned} \tag{3.19}$$

We used again (1.5) in the case $n=2$, (1.7) in the case $n=3$, and (3.12). The constants C do not depend on λ. Together with (1.12), (3.17) implies that

$$\begin{aligned} &-\nu\gamma(1+|u_\lambda|^2)-\nu\alpha|\Phi_\lambda(u_\lambda)|^2-\frac{1-\nu\alpha}{4}|\Phi_\lambda(u_\lambda)|^2-C(1+|f|^2)^{3/2}|Au_\lambda| \\ &\quad+|\Phi_\lambda(u_\lambda)|^2-\frac{\mu_N}{2}(|\Phi(0)|^2+|u_\lambda|^2) \leq \frac{1}{1-\nu\alpha}|f|^2+\frac{1-\nu\alpha}{4}|\Phi_\lambda(u_\lambda)|^2, \end{aligned} \tag{3.20}$$

and by (3.12),

$$|\Phi_\lambda(u_\lambda)|^2 \leq C(1+|f|^2)^{3/2}|Au_\lambda|+C(1+|f|^2), \quad \forall\lambda>0. \tag{3.21}$$

Substituting (3.21) into (3.16), we obtain

$$\frac{\nu}{2}|Au_\lambda|^2 \leq C(1+|f|^2)^{3/2}|Au_\lambda|+C(1+|f|^2)^3, \tag{3.22}$$

which implies that

$$\begin{aligned} |Au_\lambda| &\leq C(1+|f|^2)^{3/2}, \\ |\Phi_\lambda(u_\lambda)|^2 &\leq C(1+|f|^2)^3. \end{aligned} \tag{3.23}$$

The constants C do not depend on λ or $|f|$.

From the boundedness in H of the sequences $(u_\lambda)_{\lambda>0}$, $(\Phi_\lambda(u_\lambda))_{\lambda>0}$, $(f_\lambda)_{\lambda>0}$, where $f_\lambda = f-u_\lambda-\Phi_\lambda(u_\lambda)=\Gamma_N u_\lambda$, it follows that on a sequence $\lambda_j\to 0$, we have the weak convergences in H:

$$u_{\lambda_j} \rightharpoonup u, \quad \Phi_{\lambda_j}(u_{\lambda_j}) \rightharpoonup f_1, \quad f_{\lambda_j} = \Gamma_N u_{\lambda_j} \rightharpoonup f_2. \tag{3.24}$$

Because (u_{λ_j}) is bounded in V by (3.12), we get that $u_{\lambda_j}\to u$.

Passing to the weak limit in the equality $f-u_{\lambda_j}-\Phi_{\lambda_j}(u_{\lambda_j})=f_{\lambda_j}$, we obtain $f=u+f_1+f_2$. If we prove that $f_2=\Gamma_N u$, $f_1\in\Phi(u)$, it will follow that $f\in u+\Gamma_N u+\Phi(u)$, as claimed.

We multiply by $u_\lambda - u_\mu$ the difference of (3.7) written for $\lambda > 0$ and the same equation written for $\mu > 0$. We find

$$(\Phi_\lambda(u_\lambda) - \Phi_\mu(u_\mu), u_\lambda - u_\mu) + ((\Gamma_N + I)u_\lambda - (\Gamma_N + I)u_\mu, u_\lambda - u_\mu) = 0. \tag{3.25}$$

Since $\Gamma_N + I$ is the sum of two monotone operators and by consequence monotone, we get $(\Phi_\lambda(u_\lambda) - \Phi_\mu(u_\mu), u_\lambda - u_\mu) \le 0$, for all $\lambda, \mu > 0$. Then $(u, f_1) \in \Phi$ and

$$\lim_{\lambda,\mu\to 0} (\Phi_\lambda(u_\lambda) - \Phi_\mu(u_\mu), u_\lambda - u_\mu) = 0 \tag{3.26}$$

(see [5, Proposition 1.3(iv), page 49]).

Relation (3.25) implies that

$$\lim_{\lambda,\mu\to 0} ((\Gamma_N + I)u_\lambda - (\Gamma_N + I)u_\mu, u_\lambda - u_\mu) = 0. \tag{3.27}$$

Using also $u_{\lambda_j} \to u$, $\Gamma_N u_{\lambda_j} \rightharpoonup f_2$ and the fact that $\Gamma_N + I$ is maximal monotone (Γ_N maximal monotone), it follows that $(u, u + f_2) \in \Gamma_N + I$, and thus $\Gamma_N u = f_2$ (see [5, Lemma 1.3, page 49]).

From $(u, f_1) \in \Phi$ and $\Gamma_N u = f_2$, we also get $u \in D(\Gamma_N) \cap D(\Phi) = D(A) \cap D(\Phi)$. Consequently, $D(\Lambda_N) = D(A) \cap D(\Phi)$.

Let us prove now (3.3) and (3.4).

For the first one, we consider $\lambda > 0$ fixed, $w \in D(A)$, and let $g_\lambda = \nu Aw + B_N w + \Phi_\lambda(w) + \mu_N w$. In the same way as we deduced (3.23), we may obtain $|Aw| \le C(1 + |g_\lambda|^2)^{3/2}$, hence

$$\begin{aligned} |Aw|^{2/3} &\le C(1 + |g_\lambda|^2) = C(|\nu Aw + B_N w + \Phi_\lambda(w) + \mu_N w|^2 + 1) \\ &\le C(1 + 2\mu_N^2 |w|^2 + 2|\nu Aw + B_N w + \Phi_\lambda(w)|^2), \end{aligned} \tag{3.28}$$

where the constant $C > 0$ does not depend on λ. Thus (3.3) is proved.

In order to prove the second relation, we take $w \in D(A) \cap D(\Phi)$ and $\eta \in \Phi(w)$. Let $g = \nu Aw + B_N w + \eta + \mu_N w$. For this g, we may construct as in the first part of the proof a sequence $(w_\lambda)_{\lambda > 0} \subset H$ such that

$$\nu A w_\lambda + B_N w_\lambda + \Phi_\lambda(w_\lambda) + \mu_N w_\lambda = g, \quad \forall \lambda > 0. \tag{3.29}$$

Moreover, $w_\lambda \to w$, $Aw_\lambda \rightharpoonup Aw$ because Λ_N is maximal monotone.

Passing to the limit with $\lambda \to 0$ in (3.23) written for (w_λ), we obtain $|Aw| \le C(1 + |g|^2)^{3/2}$, hence

$$\begin{aligned} |Aw|^{2/3} &\le C(1 + |g|^2) = C(1 + |\nu Aw + B_N w + \eta + \mu_N w|^2) \\ &\le C(1 + 2\mu_N^2 |w|^2 + 2|\nu Aw + B_N w + \eta|^2), \end{aligned} \tag{3.30}$$

which proves relation (3.4). This concludes the proof of Proposition 3.1. □

Proposition 3.2. *Let $\Phi \subset H \times H$ verifying the hypotheses in Proposition 3.1. Let $f \in W^{1,1}(0,T;H)$ and $y_0 \in D(A) \cap D(\Phi)$. Then there exists a unique strong solution*

$$y_N \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;D(A)) \cap C([0,T];V) \tag{3.31}$$

to problem (3.2). Moreover, y_N is right differentiable, $(d^+/dt)y_N$ is right continuous, and

$$\frac{d^+}{dt} y_N(t) + \big(\nu A y_N(t) + B_N y_N(t) + \Phi(y_N(t)) - f(t)\big)^0 = 0, \quad \forall t \in [0,T). \tag{3.32}$$

Proof. From Proposition 3.1 and [5, Theorems 1.4, 1.6, pages 214–216], it follows that problem (3.2) has a unique solution $y_N \in W^{1,\infty}(0,T;H)$ verifying relation (3.32). In order to prove that $y_N \in L^\infty(0,T;D(A)) \cap C([0,T];V)$, let $\zeta_N(t) \in \Phi(y_N(t))$ such that

$$\frac{dy_N}{dt}(t) + \nu A y_N(t) + B_N y_N(t) + \zeta_N(t) = f(t). \tag{3.33}$$

We know $f - (dy_N/dt) \in L^\infty(0,T;H)$. Consequently, $\nu A y_N + B_N y_N + \zeta_N \in L^\infty(0,T;H)$. Applying (3.4) for $y_N(t)$ and $\zeta_N(t) \in \Phi(y_N(t))$, we get $A y_N \in L^\infty(0,T;H)$, which implies that $y_N \in L^\infty(0,T;D(A))$. Together with $(dy_N/dt) \in L^\infty(0,T;H)$, we infer that $y_N \in C([0,T];V)$. □

A similar result takes place if we use the Yosida approximation Φ_λ instead of Φ.

Proposition 3.3. *Let $\Phi \subset H \times H$ verifying the hypotheses in Proposition 3.1. Let $f \in W^{1,1}(0,T;H)$ and $y_0 \in D(A) \cap D(\Phi)$. Then for all $\lambda > 0$, there exists a unique strong solution $y_N^\lambda \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;D(A)) \cap C([0,T];V)$ for problem*

$$\begin{gathered} \frac{dy_N^\lambda}{dt}(t) + \nu A y_N^\lambda(t) + B_N y_N^\lambda(t) + \Phi_\lambda(y_N^\lambda(t)) = f(t), \quad \text{a.e. } t \in (0,T), \\ y_N^\lambda(0) = y_0. \end{gathered} \tag{3.34}$$

Moreover, y_N^λ is right differentiable, $(d^+/dt)y_N^\lambda$ is right continuous, and

$$\frac{d^+}{dt} y_N^\lambda(t) + \nu A y_N^\lambda(t) + B_N y_N^\lambda(t) + \Phi_\lambda(y_N^\lambda(t)) = f(t), \quad \forall t \in [0,T). \tag{3.35}$$

Proof. $\Gamma_N = \nu A + B_N + \alpha_N I$ is maximal monotone (for α_N large enough), Φ_λ is demicontinuous monotone, which implies that $\nu A + B_N + \Phi_\lambda + \alpha_N I$ is maximal monotone in $H \times H$. Then, problem (3.34) has a unique solution $y_N^\lambda \in W^{1,\infty}(0,T;H)$ verifying relation (3.35). Moreover, we infer that $\nu A y_N^\lambda + B_N y_N^\lambda + \Phi_\lambda(y_N^\lambda) = f - (dy_N^\lambda/dt) \in L^\infty(0,T;H)$. Applying (3.3) for $y_N^\lambda(t) \in D(A)$, we get $A y_N^\lambda \in L^\infty(0,T;H)$, which implies that $y_N^\lambda \in L^\infty(0,T;D(A))$. Together with $(dy_N^\lambda/dt) \in L^\infty(0,T;H)$, we obtain $y_N^\lambda \in C([0,T];V)$. □

3.2. Estimates for the solution of problem (3.34). By Proposition 3.2, problem (3.2) has a unique strong solution

$$y_N \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;D(A)) \cap C([0,T];V). \tag{3.36}$$

However, in order to get better estimates, we will further approximate problem (3.2) by problem (3.34), which also has a unique strong solution, by Proposition 3.3.

First, we multiply (3.34) by $y_N^\lambda(t)$ and integrate on $(0,t)$,

$$\begin{aligned}\int_0^t \frac{d}{ds}\left(\frac{1}{2}|y_N^\lambda(s)|^2\right)ds+\nu\int_0^t (Ay_N^\lambda(s),y_N^\lambda(s))\,ds+\int_0^t (B_N y_N^\lambda(s),y_N^\lambda(s))\,ds \\ +\int_0^t (\Phi_\lambda(y_N^\lambda(s)),y_N^\lambda(s))\,ds=\int_0^t (f(s),y_N^\lambda(s))\,ds.\end{aligned}\tag{3.37}$$

But $(B_N y_N^\lambda(s),y_N^\lambda(s))=0$ and $(\Phi_\lambda(y_N^\lambda(s)),y_N^\lambda(s))\geq(\Phi_\lambda(0),y_N^\lambda(s))$ because Φ_λ is monotone,

$$\begin{aligned}&\frac{1}{2}|y_N^\lambda(t)|^2+\nu\int_0^t\|y_N^\lambda(s)\|^2ds \\ &\quad\leq-\int_0^t(\Phi_\lambda(0),y_N^\lambda(s))\,ds+\frac{1}{2}|y_0|^2+\int_0^t(f(s),y_N^\lambda(s))\,ds\leq\frac{1}{2}|y_0|^2 \\ &\qquad+\frac{1}{2}\int_0^t|y_N^\lambda(s)|^2ds+\frac{1}{2}\int_0^t 2(|\Phi(0)|^2+|f(s)|^2)\,ds.\end{aligned}\tag{3.38}$$

In particular, it follows that

$$|y_N^\lambda(t)|^2\leq|y_0|^2+2\int_0^T(|\Phi(0)|^2+|f(s)|^2)\,ds+\int_0^t|y_N^\lambda(s)|^2ds,\tag{3.39}$$

and by Gronwall's inequality,

$$|y_N^\lambda(t)|^2\leq\left(|y_0|^2+2\int_0^T(|\Phi(0)|^2+|f(s)|^2)\,ds\right)e^t.\tag{3.40}$$

Finally, we infer that

$$\frac{1}{2}|y_N^\lambda(t)|^2+\nu\int_0^t\|y_N^\lambda(s)\|^2ds\leq\left(\frac{1}{2}|y_0|^2+\int_0^T(|\Phi(0)|^2+|f(s)|^2)\,ds\right)e^t,\tag{3.41}$$

and thus

$$\frac{1}{2}|y_N^\lambda(t)|^2+\nu\int_0^t\|y_N^\lambda(s)\|^2ds\leq C_1(\|f\|_{L^2(0,T;H)},|y_0|^2),\tag{3.42}$$

where C_1 is a positive bounded function depending on $\|f\|_{L^2(0,T;H)}$, $|y_0|^2$, but independent of N,λ.

Next we multiply (3.34) with $Ay_N^\lambda(t)$ and integrate on $(0,t)$:

$$\begin{aligned}\int_0^t\frac{d}{ds}\left(\frac{1}{2}\|y_N^\lambda(s)\|^2\right)ds+\nu\int_0^t|Ay_N^\lambda(s)|^2ds+\int_0^t(B_N y_N^\lambda(s),Ay_N^\lambda(s))\,ds \\ +\int_0^t(\Phi_\lambda(y_N^\lambda(s)),Ay_N^\lambda(s))\,ds=\int_0^t(f(s),Ay_N^\lambda(s))\,ds.\end{aligned}\tag{3.43}$$

Recalling (1.12), this yields

$$\frac{1}{2}\|y_N^\lambda(t)\|^2+\nu\int_0^t|Ay_N^\lambda(s)|^2ds-\gamma\int_0^t\left(1+|y_N^\lambda(s)|^2\right)ds-\alpha\int_0^t|\Phi_\lambda(y_N^\lambda(s))|^2ds$$
$$\leq\frac{1}{2}\|y_0\|^2+\int_0^t\left(f(s),Ay_N^\lambda(s)\right)ds-\int_0^t\left(B_Ny_N^\lambda(s),Ay_N^\lambda(s)\right)ds. \tag{3.44}$$

But if $n=2$,

$$\begin{aligned}\left|\int_0^t\left(B_Ny_N^\lambda(s),Ay_N^\lambda(s)\right)ds\right| &\leq\int_0^t C|y_N^\lambda(s)|^{1/2}\|y_N^\lambda(s)\||Ay_N^\lambda(s)|^{3/2}ds\\ &\leq\frac{\nu}{4}\int_0^t|Ay_N^\lambda(s)|^2ds+\widetilde{C}\int_0^t\|y_N^\lambda(s)\|^4ds,\end{aligned} \tag{3.45}$$

($|y_N^\lambda(s)|$ being bounded from (3.42)) and if $n=3$,

$$\begin{aligned}\left|\int_0^t\left(B_Ny_N^\lambda(s),Ay_N^\lambda(s)\right)ds\right| &\leq\int_0^t C\|y_N^\lambda(s)\|^{3/2}|Ay_N^\lambda(s)|^{3/2}ds\\ &\leq\frac{\nu}{4}\int_0^t|Ay_N^\lambda(s)|^2ds+\widetilde{C}\int_0^t\|y_N^\lambda(s)\|^6ds.\end{aligned} \tag{3.46}$$

In both cases, the constant $\widetilde{C}$ is independent of N, λ.

We get

$$\begin{aligned}&\frac{1}{2}\|y_N^\lambda(t)\|^2+\nu\int_0^t|Ay_N^\lambda(s)|^2ds\\ &\quad\leq\frac{1}{2}\|y_0\|^2+\frac{1}{\nu}\int_0^t|f(s)|^2ds\\ &\qquad+\frac{\nu}{4}\int_0^t|Ay_N^\lambda(s)|^2ds+\frac{\nu}{4}\int_0^t|Ay_N^\lambda(s)|^2ds+\widetilde{C}\int_0^t\|y_N^\lambda(s)\|^dds\\ &\qquad+\gamma\left(t+\int_0^t|y_N^\lambda(s)|^2ds\right)+\alpha\int_0^t|\Phi_\lambda(y_N^\lambda(s))|^2ds,\end{aligned} \tag{3.47}$$

where $d=4$ for $n=2$ and $d=6$ for $n=3$. Using also (3.42), it follows that

$$\begin{aligned}&\frac{1}{2}\|y_N^\lambda(t)\|^2+\frac{\nu}{2}\int_0^t|Ay_N^\lambda(s)|^2ds\\ &\quad\leq\alpha\int_0^t|\Phi_\lambda(y_N^\lambda(s))|^2ds+\begin{cases}\widetilde{C}_2\int_0^t\|y_N^\lambda(s)\|^4ds+C_2, & n=2,\\ \widetilde{C}_2\int_0^t\|y_N^\lambda(s)\|^6ds+C_2, & n=3,\end{cases}\end{aligned} \tag{3.48}$$

where $\widetilde{C}_2$, C_2 are positive bounded functions of $\|f\|_{L^2(0,T;H)}$, $|y_0|^2$, $\|y_0\|^2$, but do not depend on N,λ.

Finally, we multiply (3.34) with $\Phi_\lambda(y_N^\lambda(t))$ and integrate on $(0,t)$. We recall that $\Phi = \partial\varphi$ and that the Yosida approximation $\Phi_\lambda = \nabla\varphi_\lambda$, the Gâteaux differential of φ_λ, where

$$\varphi_\lambda(u) = \inf\left\{\frac{|u-v|^2}{2\lambda} + \varphi(v);\ v \in H\right\}, \quad \forall u \in H, \tag{3.49}$$

is the regularization of φ. So,

$$\begin{aligned} &\int_0^t \left(\frac{d}{ds}y_N^\lambda(s), (\nabla\varphi_\lambda)(y_N^\lambda(s))\right)ds \\ &\quad + \nu\int_0^t (Ay_N^\lambda(s), \Phi_\lambda(y_N^\lambda(s)))ds + \int_0^t (B_N y_N^\lambda(s), \Phi_\lambda(y_N^\lambda(s)))ds \\ &\quad + \int_0^t |\Phi_\lambda(y_N^\lambda(s))|^2 ds = \int_0^t (f(s), \Phi_\lambda(y_N^\lambda(s)))ds. \end{aligned} \tag{3.50}$$

Using $((d/ds)y_N^\lambda(s), (\nabla\varphi_\lambda)(y_N^\lambda(s))) = (d/ds)[\varphi_\lambda(y_N^\lambda(s))]$ and (1.12), we get

$$\begin{aligned} &(1-\nu\alpha)\int_0^t |\Phi_\lambda(y_N^\lambda(s))|^2 ds \\ &\quad \le \varphi_\lambda(y_0) - \varphi_\lambda(y_N^\lambda(t)) + \nu\gamma\left(t + \int_0^t |y_N^\lambda(s)|^2 ds\right) \\ &\quad + \int_0^t (f(s), \Phi_\lambda(y_N^\lambda(s)))ds - \int_0^t (B_N y_N^\lambda(s), \Phi_\lambda(y_N^\lambda(s)))ds. \end{aligned} \tag{3.51}$$

Any proper lower semicontinuous convex function is bounded from below by an affine function, consequently there are $h \in H$ and $p \in \mathbb{R}$ such that

$$\varphi(x) \ge (x,h) + p, \quad \forall x \in H. \tag{3.52}$$

Also, $J_\lambda = (\lambda\Phi + I)^{-1}$ is bounded on bounded subsets of H and $\varphi(J_\lambda(x)) \le \varphi_\lambda(x) \le \varphi(x)$, for all $\lambda > 0$, for all $x \in H$. Using again (3.42), we infer that

$$\begin{aligned} -\varphi_\lambda(y_N^\lambda(t)) &\le -\varphi(J_\lambda(y_N^\lambda(t))) \le -(J_\lambda(y_N^\lambda(t)), h) - p \\ &\le |J_\lambda(y_N^\lambda(t))||h| + |p| \le c, \quad c \text{ constant not depending of } N,\ \lambda,\ t. \end{aligned} \tag{3.53}$$

On the other hand, $\varphi_\lambda(y_0) \le \varphi(y_0)$.

If $n = 2$, using (1.5) and Young's inequality,

$$\begin{aligned} &\left|\int_0^t (B_N y_N^\lambda(s), \Phi_\lambda(y_N^\lambda(s)))ds\right| \\ &\quad \le \int_0^t C|y_N^\lambda(s)|^{1/2}\|y_N^\lambda(s)\||Ay_N^\lambda(s)|^{1/2}|\Phi_\lambda(y_N^\lambda(s))|ds \\ &\quad \le \frac{C_1}{2\varepsilon^4}\int_0^t \|y_N^\lambda(s)\|^4 ds + \frac{C^4\varepsilon^4\beta^4}{4}\int_0^t |Ay_N^\lambda(s)|^2 ds + \frac{1}{2\beta^2}\int_0^t |\Phi_\lambda(y_N^\lambda(s))|^2 ds \end{aligned} \tag{3.54}$$

($|y_N^\lambda(s)|^2$ being bounded by $2C_1$ from (3.42)).

If $n = 3$, using (1.7) and Young's inequality,

$$\begin{aligned}
&\left| \int_0^t (B_N y_N^\lambda(s), \Phi_\lambda(y_N^\lambda(s)))\,ds \right| \\
&\quad \le \int_0^t C \|y_N^\lambda(s)\|^{3/2} |Ay_N^\lambda(s)|^{1/2} |\Phi_\lambda(y_N^\lambda(s))|\,ds \\
&\quad \le \frac{1}{4\varepsilon^4}\int_0^t \|y_N^\lambda(s)\|^6 ds + \frac{C^4\varepsilon^4\beta^4}{4}\int_0^t |Ay_N^\lambda(s)|^2 ds + \frac{1}{2\beta^2}\int_0^t |\Phi_\lambda(y_N^\lambda(s))|^2 ds.
\end{aligned} \tag{3.55}$$

The constants $C, \varepsilon, \beta > 0$ do not depend on N, λ. While C (occurring in (1.5), (1.7), resp.) is fixed, ε and β are at our choice and will be precised later.

Then (3.51) becomes

$$\begin{aligned}
&(1-\nu\alpha)\int_0^t |\Phi_\lambda(y_N^\lambda(s))|^2 ds \\
&\quad \le \varphi(y_0) + c + \nu\gamma T(1+2C_1) + \frac{1}{1-\nu\alpha}\int_0^t |f(s)|^2 ds + \frac{1-\nu\alpha}{4}\int_0^t |\Phi_\lambda(y_N^\lambda(s))|^2 ds \\
&\qquad + \frac{C^4\varepsilon^4\beta^4}{4}\int_0^t |Ay_N^\lambda(s)|^2 ds + \frac{1}{2\beta^2}\int_0^t |\Phi_\lambda(y_N^\lambda(s))|^2 ds \\
&\qquad + \begin{cases} \dfrac{C_1}{2\varepsilon^4}\displaystyle\int_0^t \|y_N^\lambda(s)\|^4 ds, & n = 2, \\ \dfrac{1}{4\varepsilon^4}\displaystyle\int_0^t \|y_N^\lambda(s)\|^6 ds, & n = 3. \end{cases}
\end{aligned} \tag{3.56}$$

In order to absorb $\int_0^t |\Phi_\lambda(y_N^\lambda(s))|^2 ds$ in the left-hand side, we choose $1/2\beta^2 = (1-\nu\alpha)/4$, that is, $\beta^2 = 2/(1-\nu\alpha)$.

Relation (3.56) is then rewritten as

$$\begin{aligned}
&\int_0^t |\Phi_\lambda(y_N^\lambda(s))|^2 ds \\
&\quad \le \frac{2C^4\varepsilon^4}{(1-\nu\alpha)^3}\int_0^t |Ay_N^\lambda(s)|^2 ds + C_3\left(\|f\|_{L^2(0,T;H)}, |y_0|^2, \varphi(y_0)\right) \\
&\qquad + \begin{cases} \dfrac{C_1}{\varepsilon^4(1-\nu\alpha)}\displaystyle\int_0^t \|y_N^\lambda(s)\|^4 ds, & n = 2, \\ \dfrac{1}{2\varepsilon^4(1-\nu\alpha)}\displaystyle\int_0^t \|y_N^\lambda(s)\|^6 ds, & n = 3. \end{cases}
\end{aligned} \tag{3.57}$$

Case 1 ($n = 2$: global boundedness results). Now we substitute relation (3.57) into (3.48),

$$\begin{aligned}\frac{1}{2}\|y_N^\lambda(t)\|^2 + \frac{\nu}{2}\int_0^t |Ay_N^\lambda(s)|^2 ds \\ \leq \frac{2\alpha C^4\varepsilon^4}{(1-\nu\alpha)^3}\int_0^t |Ay_N^\lambda(s)|^2 ds + C_4\int_0^t \|y_N^\lambda(s)\|^4 ds + C_5, \\ \text{where } C_4 = \widetilde{C}_2 + \frac{\alpha C_1}{\varepsilon^4(1-\nu\alpha)},\ C_5 = C_2 + \alpha C_3.\end{aligned} \tag{3.58}$$

We would like that

$$k = \frac{\nu}{2} - \frac{2\alpha C^4\varepsilon^4}{(1-\nu\alpha)^3} > 0, \quad \text{that is, } 0 < \varepsilon < \left[\frac{\nu(1-\nu\alpha)^3}{4\alpha C^4}\right]^{1/4}. \tag{3.59}$$

This yields

$$\frac{1}{2}\|y_N^\lambda(t)\|^2 + k\int_0^t |Ay_N^\lambda(s)|^2 ds \leq C_4\int_0^t \|y_N^\lambda(s)\|^4 ds + C_5. \tag{3.60}$$

In particular,we have

$$\|y_N^\lambda(t)\|^2 \leq 2\left(C_4\int_0^t \|y_N^\lambda(s)\|^4 ds + C_5\right). \tag{3.61}$$

By Gronwall's lemma, we infer that

$$\|y_N^\lambda(t)\|^2 \leq 2C_5 e^{2C_4\int_0^t \|y_N^\lambda(s)\|^2 ds} \tag{3.62}$$

and recalling that $\int_0^t \|y_N^\lambda(s)\|^2 ds$ is bounded by C_1/ν from (3.42),

$$\|y_N^\lambda(t)\|^2 \leq 2C_5 e^{2C_4(C_1/\nu)} = C_6, \quad \text{a.e. } t \in [0,T]. \tag{3.63}$$

Substituting (3.63) into (3.60), we get

$$\begin{aligned}\frac{1}{2}\|y_N^\lambda(t)\|^2 + k\int_0^t |Ay_N^\lambda(s)|^2 ds \\ \leq C_7(\|f\|_{L^2(0,T;H)}, \|y_0\|^2, \varphi(y_0)), \quad \text{a.e. } t \in [0,T].\end{aligned} \tag{3.64}$$

Relation (3.64) implies that $\int_0^t |Ay_N^\lambda(s)|^2 ds \leq C_7/k$ and, together with (3.63), transforms (3.57) into

$$\int_0^t |\Phi_\lambda(y_N^\lambda(s))|^2 ds \leq C_8(\|f\|_{L^2(0,T;H)}, \|y_0\|^2, \varphi(y_0)), \quad \text{a.e. } t \in [0,T]. \tag{3.65}$$

Case 2 ($n = 3$: local boundedness results). Proceeding in the same way, we get

$$\begin{aligned}
&\frac{1}{2}\|y_N^\lambda(t)\|^2 + \frac{\nu}{2}\int_0^t |Ay_N^\lambda(s)|^2 \\
&\quad \le \frac{2\alpha C^4\varepsilon^4}{(1-\nu\alpha)^3}\int_0^t |Ay_N^\lambda(s)|^2 ds \\
&\qquad + C_4\int_0^t \|y_N^\lambda(s)\|^6 ds + C_5, \quad \text{where } C_4 = \widetilde{C}_2 + \frac{\alpha}{2\varepsilon^4(1-\nu\alpha)},\ C_5 = C_2 + \alpha C_3.
\end{aligned} \tag{3.66}$$

Choosing again $0 < \varepsilon < [\nu(1-\nu\alpha)^3/4\alpha C^4]^{1/4}$, we obtain

$$\frac{1}{2}\|y_N^\lambda(t)\|^2 + k\int_0^t |Ay_N^\lambda(s)|^2 ds \le C_4\int_0^t \|y_N^\lambda(s)\|^6 ds + C_5. \tag{3.67}$$

In particular, we have

$$\|y_N^\lambda(t)\|^2 \le 2\left(C_4\int_0^t \|y_N^\lambda(s)\|^6 ds + C_5\right). \tag{3.68}$$

Using a comparison result, we infer that

$$\begin{gathered}
\|y_N^\lambda(t)\|^2 \le W(t), \\
\text{where}\begin{cases} W'(t) = 2C_4W^3(t) \\ W(0) = 2C_5, \end{cases} \quad \text{that is } W(t) = \frac{2C_5}{\sqrt{1-16C_4C_5^2t}}.
\end{gathered} \tag{3.69}$$

The solution W exists on a maximal interval $[0, \overline{T^*})$, $\overline{T^*} = 1/16C_4C_5^2$. Let $T^* = \min\{T, \overline{T^*}\}$.

We get $\|y_N^\lambda(t)\|^2 \le 2C_5/(1-16C_4C_5^2t)^{1/2}$, $t \in [0, T^*)$. Substituting into (3.67), we obtain

$$\begin{aligned}
&\frac{1}{2}\|y_N^\lambda(t)\|^2 + k\int_0^t |Ay_N^\lambda(s)|^2 ds \\
&\quad \le C_5 + C_4\int_0^t \frac{8C_5^3}{(1-16C_4C_5^2s)^{3/2}} ds, \quad t \in [0, T^*).
\end{aligned} \tag{3.70}$$

The term $\int_0^t (8C_5^3/(1-16C_4C_5^2s)^{3/2}) ds$ is of the order of $\int_0^t (1/(\overline{T^*}-s)^{3/2}) ds = 2/\sqrt{\overline{T^*}-t} - 2/\sqrt{\overline{T^*}}$, which explodes in $t = \overline{T^*}$, and consequently is bounded on intervals of the type $[0, T^*-\delta]$, $\delta \in (0, T^*)$.

Finally, for $n = 3$, we have

$$\begin{aligned}
&\frac{1}{2}\|y_N^\lambda(t)\|^2 + k\int_0^t |Ay_N^\lambda(s)|^2 ds \\
&\quad \le C_7(\|f\|_{L^2(0,T;H)}, \|y_0\|^2, \varphi(y_0), \delta), \quad \text{a.e. } t \in [0, T^*-\delta].
\end{aligned} \tag{3.71}$$

Using the local boundedness of $\|y_N^\lambda(t)\|^2$ and $\int_0^t |Ay_N^\lambda(s)|^2 ds$, estimate (3.57) becomes

$$\int_0^t \left|\Phi_\lambda(y_N^\lambda(s))\right|^2 ds \le C_8\left(\|f\|_{L^2(0,T;H)}, \|y_0\|^2, \varphi(y_0), \delta\right), \quad \text{a.e. } t \in [0, T^* - \delta]. \tag{3.72}$$

If $\overline{T}^* > T$, estimates (3.71) and (3.72) take place a.e $t \in [0, T]$.

Estimates (3.42) and (3.64), (3.65) in the case $n = 2$, respectively, (3.71), (3.72) in the case $n = 3$, will allow us to pass to the limit for $\lambda \to 0$ (maintaining N fixed). The positive bounded functions C_1, C_7, C_8 are independent of λ.

3.3. Passing to the limit for $\lambda \to 0$. We recall that Proposition 3.3 implies that

$$y_N^\lambda \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;D(A)) \cap C([0,T];V). \tag{3.73}$$

Let $T_0 = T$ for $n = 2$ and $T_0 < T^*$ for $n = 3$ (we may take $T_0 = T$ if $\overline{T}^* > T$). We have

$$(y_N^\lambda)_\lambda \quad \text{is bounded in } C([0,T_0];V) \cap L^2(0,T_0;D(A)), \tag{3.74}$$

$$(Ay_N^\lambda)_\lambda, (\Phi_\lambda(y_N^\lambda))_\lambda \quad \text{are bounded in } L^2(0,T_0;H). \tag{3.75}$$

From (1.5), (1.7), (3.42), (3.64), and (3.71) we infer that

$$\left|B_N y_N^\lambda(s)\right| \le C\left|y_N^\lambda(s)\right|^{1/2} \|y_N^\lambda(s)\| \left|Ay_N^\lambda(s)\right|^{1/2} \le C\left|Ay_N^\lambda(s)\right|^{1/2} \quad (n = 2), \tag{3.76}$$

respectively,

$$\left|B_N y_N^\lambda(s)\right| \le C\|y_N^\lambda(s)\|^{3/2} \left|Ay_N^\lambda(s)\right|^{1/2} \le C\left|Ay_N^\lambda(s)\right|^{1/2} \quad (n = 3). \tag{3.77}$$

The constant C does not depend on N, λ. Together with (3.75), we get

$$(B_N y_N^\lambda)_\lambda \quad \text{is bounded in } L^2(0,T_0;H). \tag{3.78}$$

From (3.75), (3.78), and (3.34) we also have

$$\left(\frac{dy_N^\lambda}{dt}\right)_\lambda \quad \text{is bounded in } L^2(0,T_0;H). \tag{3.79}$$

From (3.74), (3.79), and [6, Theorem A2.2], we infer that

$$(y_N^\lambda)_\lambda \quad \text{is relatively compact in } C([0,T_0];H). \tag{3.80}$$

These yield that, on a subsequence again denoted by $(y_N^\lambda)_\lambda$, we have for $\lambda \to 0$,

$$\begin{aligned} y_N^\lambda &\longrightarrow y_N \quad \text{in } C([0,T_0];H), \\ \frac{dy_N^\lambda}{dt} &\rightharpoonup \frac{dy_N}{dt} \quad \text{in } L^2(0,T_0;H), \end{aligned} \tag{3.81}$$

$$\begin{aligned} Ay_N^\lambda &\rightharpoonup Ay_N \quad \text{in } L^2(0,T_0;H), \\ B_N y_N^\lambda &\rightharpoonup \beta_N \quad \text{in } L^2(0,T_0;H), \\ \Phi_\lambda(y_N^\lambda) &\rightharpoonup \eta_N \quad \text{in } L^2(0,T_0;H). \end{aligned} \tag{3.82}$$

Moreover, by Aubin's compactness theorem,

$$y_N^\lambda \longrightarrow y_N \quad \text{in } L^2(0,T_0;V). \tag{3.83}$$

But $\Phi_\lambda(y_N^\lambda) = \Phi(I+\lambda\Phi)^{-1}(y_N^\lambda)$ and $(I+\lambda\Phi)^{-1}(y_N^\lambda) \to y_N$ in $L^2(0,T_0;H)$. Φ being maximal monotone, it follows that $\eta_N \in \Phi(y_N)$ a.e. t.

We prove now that $\beta_N = B_N y_N$ a.e. t.

Case 3 ($n=2$). Using (1.5), we obtain for any $\psi \in V$ that

$$\begin{aligned} &|(B_N y_N^\lambda(s) - B_N y_N(s), \psi)| \\ &\quad \le |b(y_N^\lambda(s) - y_N(s), y_N^\lambda(s), \psi)| + |b(y_N(s), y_N^\lambda(s) - y_N(s), \psi)| \\ &\quad \le C\big(|y_N^\lambda(s) - y_N(s)|^{1/2} \|y_N^\lambda(s) - y_N(s)\|^{1/2} \cdot \|y_N^\lambda(s)\|^{1/2} |Ay_N^\lambda(s)|^{1/2} |\psi| \\ &\qquad + |y_N(s)|^{1/2} \|y_N(s)\|^{1/2} \|y_N^\lambda(s) - y_N(s)\|^{1/2} \cdot |A(y_N^\lambda(s) - y_N(s))|^{1/2} |\psi|\big). \end{aligned} \tag{3.84}$$

Using (3.81) and (3.64), we get

$$\begin{aligned} &|B_N y_N^\lambda(s) - B_N y_N(s)| \\ &\quad \le C\|y_N^\lambda(s) - y_N(s)\|^{1/2} \big(|Ay_N^\lambda(s)|^{1/2} + |A(y_N^\lambda(s) - y_N(s))|^{1/2}\big), \end{aligned} \tag{3.85}$$

that implies

$$\begin{aligned} &\int_0^T |B_N y_N^\lambda(s) - B_N y_N(s)|^2 ds \\ &\quad \le C\left(\int_0^T \|y_N^\lambda(s) - y_N(s)\|^2 ds\right)^{1/2} \cdot \left[\int_0^T \big(|Ay_N^\lambda(s)|^2 + |A(y_N^\lambda(s) - y_N(s))|^2\big) ds\right]^{1/2}. \end{aligned} \tag{3.86}$$

Recalling that $(A(y_N^\lambda(s)))_\lambda$ is bounded by (3.64), relation (3.83) yields

$$B_N y_N^\lambda \longrightarrow B_N y_N \quad \text{in } L^2(0,T;H). \tag{3.87}$$

Case 4 ($n=3$).

$$\begin{aligned}
&|(B_N y_N^\lambda(s) - B_N y_N(s), \psi)| \\
&\quad \le |b(y_N^\lambda(s) - y_N(s), y_N^\lambda(s), \psi)| + |b(y_N(s), y_N^\lambda(s) - y_N(s), \psi)| \\
&\quad \le C(\|y_N^\lambda(s) - y_N(s)\|^{1/2} |A(y_N^\lambda(s) - y_N(s))|^{1/2} \cdot \|y_N^\lambda(s)\| |\psi| \\
&\qquad\quad + \|y_N(s)\|^{1/2} |A(y_N(s))|^{1/2} \|y_N^\lambda(s) - y_N(s)\| |\psi|), \quad \forall \psi \in V,
\end{aligned} \tag{3.88}$$

hence, using also (3.71),

$$\begin{aligned}
&|B_N y_N^\lambda(s) - B_N y_N(s)| \\
&\quad \le C\|y_N^\lambda(s) - y_N(s)\|^{1/2} (|A(y_N^\lambda(s) - y_N(s))|^{1/2} + \|y_N^\lambda(s) - y_N(s)\|^{1/2}).
\end{aligned} \tag{3.89}$$

Then,

$$\begin{aligned}
&\int_0^{T_0} |B_N y_N^\lambda(s) - B_N y_N(s)|^2 ds \\
&\quad \le C\left(\int_0^{T_0} \|y_N^\lambda(s) - y_N(s)\|^2 ds\right)^{1/2} \\
&\qquad \cdot \left[\int_0^{T_0} (|A(y_N^\lambda(s) - y_N(s))|^2 + \|y_N^\lambda(s) - y_N(s)\|^2) ds\right]^{1/2}.
\end{aligned} \tag{3.90}$$

Using (3.71) and (3.83), we get

$$B_N y_N^\lambda \longrightarrow B_N y_N \quad \text{in } L^2(0,T_0;H). \tag{3.91}$$

Letting λ tend to zero in (3.34), we obtain that y_N satisfies problem (3.2).

3.4. The uniqueness of the solution for problem (3.2). We will prove that the solution obtained by passing to the limit with $\lambda \to 0$ is unique. In order to prove the uniqueness of the solution, we assume that $y_N^1, y_N^2 \in C([0,T_0];H) \cap L^2(0,T_0;D(A)) \cap L^\infty(0,T_0;V)$ are two solutions for (3.2). Then $(y_N^1 - y_N^2)(0) = 0$ and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|y_N^1(t) - y_N^2(t)|^2 + \nu\|y_N^1(t) - y_N^2(t)\|^2 \\
&\quad + (B_N y_N^1(t) - B_N y_N^2(t), y_N^1(t) - y_N^2(t)) \\
&\quad + (\eta_N^1(t) - \eta_N^2(t), y_N^1(t) - y_N^2(t)) = 0, \quad \eta_N^j(t) \in \Phi(y_N^j(t)), \quad j = 1,2.
\end{aligned} \tag{3.92}$$

We use the monotony of Φ and the estimate

$$\begin{aligned}
&|(B_N y_N^1(t) - B_N y_N^2(t), y_N^1(t) - y_N^2(t))| \\
&\quad \le |b(y_N^1(t) - y_N^2(t), y_N^1(t), y_N^1(t) - y_N^2(t))| \text{ (by (1.7))} \\
&\quad \le C\|y_N^1(t) - y_N^2(t)\| \|y_N^1(t)\|^{1/2} |Ay_N^1(t)|^{1/2} |y_N^1(t) - y_N^2(t)| \\
&\quad \le \frac{\nu}{2}\|y_N^1(t) - y_N^2(t)\|^2 + C|Ay_N^1(t)| |y_N^1(t) - y_N^2(t)|^2
\end{aligned} \tag{3.93}$$

(the solution $y_N^1 \in L^\infty(0,T_0;V)$) and we get

$$\begin{aligned}
&\frac{d}{dt}|y_N^1(t) - y_N^2(t)|^2 \\
&\qquad \le 2C|Ay_N^1(t)| |y_N^1(t) - y_N^2(t)|^2, \quad (y_N^1 - y_N^2)(0) = 0.
\end{aligned} \tag{3.94}$$

Then $|y_N^1(t) - y_N^2(t)|^2 \le 2C\int_0^t |Ay_N^1(s)||y_N^1(s) - y_N^2(s)|^2 ds$ and by Gronwall's inequality,

$$|y_N^1(t) - y_N^2(t)|^2 \le 0 \cdot e^{(2C\int_0^t |Ay_N^1(s)|ds)} = 0, \quad t \in [0,T_0] \tag{3.95}$$

($[0,T_0]$ is bounded and $Ay_N^1 \in L^2(0,T_0;H) \subset L^1(0,T_0;H)$).

We infer that $y_N^1(t) = y_N^2(t)$, $t \in [0,T_0]$.

3.5. Proof of Theorem 2.1(the final part). We know that problem (3.2)

(i) has a unique solution in $C([0,T_0];H) \cap L^2(0,T_0;D(A)) \cap L^\infty(0,T_0;V)$, obtained by letting $\lambda \to 0$ in problem (3.34);

(ii) has a solution in $C([0,T];V) \cap L^\infty(0,T;D(A)) \cap W^{1,\infty}(0,T;H)$ (unique), given by Proposition 3.2.

Thus the two solutions must coincide and the resulting function has the regularity properties given by Proposition 3.2. Moreover, the solution of problem (3.2) satisfies a.e. on $[0,T_0]$ the estimates

$$\frac{1}{2}|y_N(t)|^2 + \nu\int_0^t \|y_N(s)\|^2 ds \le C_1, \tag{3.96}$$

$$\frac{1}{2}\|y_N(t)\|^2 + k\int_0^t |Ay_N(s)|^2 ds \le C_7, \tag{3.97}$$

$$\int_0^t |\eta_N(s)|^2 ds \le C_8, \tag{3.98}$$

where $\eta_N(t) = f(t) - ((dy_N/dt)(t) + \nu Ay_N(t) + B_N y_N(t)) \in \Phi(y_N(t))$,

$$\int_0^{T_0} \left(\left| \frac{dy_N}{dt}(t) \right|^2 + |B_N y_N(t)|^2 \right) dt \le C_9(\|f\|_{L^2(0,T;H)}, \|y_0\|, \varphi(y_0)). \tag{3.99}$$

The positive bounded functions C_1, C_7, C_8, C_9 do not depend on N.

From (3.97), we infer that

$$\|y_N(t)\|^2 \le 2C_7, \quad t \in [0, T_0]. \tag{3.100}$$

It yields that for N large enough, $\|y_N(t)\| \le N$, $t \in [0, T_0]$, and by consequence $B_N y_N = B y_N$ in $[0, T_0]$ and $y_N = y$ is a solution (defined on $[0, T_0]$) of the initial problem (2.1), conserving on $[0, T_0]$ all regularity properties of y_N. The uniqueness comes from the uniqueness of the solution of problem (3.2). □

Remark 3.4. If Φ is single valued, it is no longer necessary to use approximate problem (3.34) because hypothesis (1.12) implies that

$$(Au, \Phi(u)) \ge -\gamma(1 + |u|^2) - \alpha|\Phi(u)|^2, \quad \forall u \in D(A) \cap D(\Phi). \tag{3.101}$$

4. Proof of Theorems 2.2 and 2.3

The idea of the proof is to approximate the initial data with sequences of functions satisfying the hypotheses of Theorem 2.1 and then to pass to the limit.

Let $(y_0^j)_{j\in\mathbf{N}} \subset D(A) \cap D(\Phi)$ and $(f_j)_{j\in\mathbf{N}} \subset W^{1,1}(0, T; H)$ such that

$$y_0^j \longrightarrow y_0 \quad \text{in } V, \quad f_j \longrightarrow f \quad \text{in } L^2(0, T; H). \tag{4.1}$$

According to Theorem 2.1, problem

$$\begin{gathered} \frac{dy_j(t)}{dt} + \nu A y_j(t) + B y_j(t) + \Phi(y_j(t)) \ni f_j(t), \quad \text{a.e. } t \in (0, T), \\ y_j(0) = y_0^j \end{gathered} \tag{4.2}$$

has a unique solution $y_j \in W^{1,\infty}(0, T_0; H) \cap L^\infty(0, T_0; D(A)) \cap C([0, T_0]; V)$, where $T_0 = T$ if $n = 2$ and $T_0 \le T$ in $n = 3$. Moreover, y_j satisfy the estimates

$$\begin{gathered} \frac{1}{2}|y_j(t)|^2 + \nu \int_0^t \|y_j(s)\|^2 ds \le C, \quad t \in [0, T_0], \\ \frac{1}{2}\|y_j(t)\|^2 + k \int_0^t |Ay_j(s)|^2 ds \le C, \quad t \in [0, T_0], \\ \int_0^t |\eta_j(s)|^2 ds \le C, \quad t \in [0, T_0], \end{gathered} \tag{4.3}$$

where $\eta_j(t) = f_j(t) - ((dy_j/dt)(t) + \nu A y_j(t) + B y_j(t)) \in \Phi(y_j(t))$,

$$\int_0^{T_0} \left(\left| \frac{dy_j}{dt}(t) \right|^2 + |By_j(t)|^2 \right) dt \le C. \tag{4.4}$$

The constants are independent of j.

Consequently,

$$\begin{aligned} &(y_j)_j \quad \text{is bounded in } C([0,T_0];V) \cap L^2(0,T_0;D(A)),\\ &(Ay_j)_j,(By_j)_j,(\eta_j)_j \quad \text{are bounded in } L^2(0,T_0;H),\\ &\left(\frac{dy_j}{dt}\right)_j \quad \text{is bounded in } L^2(0,T_0;H). \end{aligned} \tag{4.5}$$

Equation (4.5) imply that

$$(y_j)_j \quad \text{is relatively compact in } C([0,T_0];H). \tag{4.6}$$

Then, on a subsequence again denoted by $(y_j)_j$, we have for $j \to \infty$,

$$\begin{aligned} &y_j \longrightarrow y \quad \text{in } C([0,T_0];H),\\ &\frac{dy_j}{dt} \rightharpoonup \frac{dy}{dt} \quad \text{in } L^2(0,T_0;H),\\ &Ay_j \rightharpoonup Ay \quad \text{in } L^2(0,T_0;H),\\ &By_j \rightharpoonup \beta \quad \text{in } L^2(0,T_0;H),\\ &\eta_j \rightharpoonup \eta \quad \text{in } L^2(0,T_0;H). \end{aligned} \tag{4.7}$$

Moreover, by Aubin's compactness theorem,

$$y_j \longrightarrow y \quad \text{in } L^2(0,T_0;V). \tag{4.8}$$

Using (4.7), (4.8), and Φ being maximal monotone, we get $\eta \in \Phi(y)$ a.e. t. Proceeding in the same way as we did in Theorem 2.1 to prove that $\beta_N = B_N y_N$ a.e. t, we deduce also that $\beta = By$ a.e. t.

Passing to the limit with $j \to \infty$, we prove the existence of the strong solution.

In order to prove the uniqueness of the solution, we proceed as in the proof of Theorem 2.1 in Section 3.4. □

5. Weak solutions

Consider the operator $\Phi : V \to V'$ monotone and demicontinuous. From the definition of demicontinuity, we infer that Φ is also single valued and its domain is V. Moreover, Φ is maximal monotone in $V \times V'$.

Let $D(\Phi) = \{v \in V; \Phi(v) \in H\}$.

We will denote by the same symbol Φ the operator $\Phi : V \to V'$ and its restriction from $D(\Phi)$ to H. The operator $\Phi : D(\Phi) \subset V \to H$ is maximal monotone in $H \times H$.

Assume in addition that

(h_1') $0 \in D(\Phi)$;

(h_2') there exist two constants $\gamma \geq 0$, $\alpha \in (0, 1/\nu)$ such that

$$(Au, \Phi_\lambda(u)) \geq -\gamma(1 + |u|^2) - \alpha \left| \Phi_\lambda(u) \right|^2, \quad \forall \lambda > 0,\ \forall u \in D(A), \tag{5.1}$$

where Φ_λ is the Yosida approximation of $\Phi \subset H \times H$;

(h_3') there exist $p \geq 2$, $\omega_1, \omega_2 > 0$, $\mu \geq 0$ constants such that

$$(\Phi(u), u) \geq \omega_1 \|u\|^p - \mu, \quad \forall u \in V, \tag{5.2}$$

$$\left\|\Phi(u)\right\|_{V'} \leq \omega_2 \|u\|^{p-1}, \quad \forall u \in V. \tag{5.3}$$

The following result on weak solutions takes place.

Theorem 5.1. *Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be an open and bounded domain, with a smooth boundary. Let $y_0 \in H$ and $f \in L^2(0, T; V')$. Assume that $\Phi : V \to V'$ satisfies the above hypotheses. Then problem*

$$\begin{gathered} \frac{dy(t)}{dt} + \nu A y(t) + B y(t) + \Phi(y(t)) = f(t), \quad t \in (0, T), \\ y(0) = y_0 \end{gathered} \tag{5.4}$$

admits at least one weak solution $y \in L^p(0, T; V) \cap C_w([0, T]; H)$. Moreover, denoting by p' the conjugate of p (i.e., p' satisfies $1/p + 1/p' = 1$), one has

$$\begin{gathered} \frac{dy}{dt} \in L^{p'}(0, T; V') \quad \textit{for } n = 2, \\ \frac{dy}{dt} \in L^r(0, T; V') \quad \textit{for } n = 3, \textit{ where } r = \min\left\{\frac{2p}{3}, p'\right\}. \end{gathered} \tag{5.5}$$

The weak solution is unique if $n = 2$.

Proof. First we will fix $N \in \mathbb{N}^*$ and we will prove that problem

$$\begin{gathered} \frac{dy_N(t)}{dt} + \nu A y_N(t) + B_N y_N(t) + \Phi(y_N(t)) = f(t), \quad \text{a.e. } t \in (0, T), \\ y_N(0) = y_0 \end{gathered} \tag{5.6}$$

has a unique solution

$$y_N \in L^p(0, T; V) \cap C([0, T]; H), \qquad \frac{dy_N}{dt} \in \begin{cases} L^{p'}(0, T; V') & \text{if } n = 2, \\ L^r(0, T; V') & \text{if } n = 3. \end{cases} \tag{5.7}$$

Then we will pass to the limit with $N \to \infty$.

Let $(y_0^j)_{j \in \mathbf{N}} \subset D(A) \cap D(\Phi)$ and $(f_j)_{j \in \mathbf{N}} \subset W^{1,1}(0, T; H)$ such that

$$y_0^j \longrightarrow y_0 \quad \text{in } H, \qquad f_j \longrightarrow f \quad \text{in } L^2(0, T; V'). \tag{5.8}$$

Φ satisfies the hypotheses of Proposition 3.2, and supposing $N \in \mathbb{N}^*$ fixed, we infer that problem:

$$\frac{dy_N^j(t)}{dt} + \nu A y_N^j(t) + B_N y_N^j(t) + \Phi(y_N^j(t)) = f_j(t), \quad \text{a.e. } t \in (0,T),$$
$$y_N^j(0) = y_0^j \tag{5.9}$$

has a unique solution $y_N^j \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;D(A)) \cap C([0,T];V)$.

Equation (5.9) implies that

$$\int_0^t \frac{d}{ds}\left(\frac{1}{2}|y_N^j(s)|^2\right)ds + \nu\int_0^t (Ay_N^j(s), y_N^j(s))\,ds + \int_0^t (B_N y_N^j(s), y_N^j(s))\,ds$$
$$+\int_0^t (\Phi(y_N^j(s)), y_N^j(s))\,ds = \int_0^t (f_j(s), y_N^j(s))\,ds. \tag{5.10}$$

But $(B_N y_N^j(s), y_N^j(s)) = 0$ and from the monotony of Φ,

$(\Phi(y_N^j(s)), y_N^j(s)) \geq (\Phi(0), y_N^j(s))$,

which yields $\frac{1}{2}|y_N^j(t)|^2 + \nu\int_0^t ||y_N^j(s)||^2 ds$

$$\leq \frac{1}{2}|y_0^j|^2 + \frac{\nu}{2}\int_0^t ||y_N^j(s)||^2 ds + \frac{1}{2\nu}\int_0^t 2(||\Phi(0)||_{V'}^2 + ||f_j(s)||_{V'}^2)\,ds. \tag{5.11}$$

Finally, we infer that

$$|y_N^j(t)|^2 + \nu\int_0^t ||y_N^j(s)||^2 ds \leq C\left(\int_0^T ||f(s)||_{V'}^2 ds, |y_0|^2\right), \tag{5.12}$$

where C is a positive bounded function independent of N, j.

On the other side, from (5.10) and (5.2), we get

$$\frac{1}{2}|y_N^j(t)|^2 + \nu\int_0^t ||y_N^j(s)||^2 ds + \omega_1\int_0^t ||y_N^j(s)||^p ds - \mu t$$
$$\leq \frac{1}{2}|y_0^j|^2 + \frac{\nu}{2}\int_0^t ||y_N^j(s)||^2 ds + \frac{1}{2\nu}\int_0^t ||f_j(s)||_{V'}^2 ds, \tag{5.13}$$

which implies that

$$|y_N^j(t)|^2 + \nu\int_0^t ||y_N^j(s)||^2 ds + 2\omega_1\int_0^t ||y_N^j(s)||^p ds \leq |y_0^j|^2 + \frac{1}{\nu}\int_0^T ||f_j(t)||_{V'}^2 dt + 2\mu T. \tag{5.14}$$

In particular,

$$2\omega_1 \int_0^t ||y_N^j(s)||^p ds \le C\left(\int_0^T ||f(s)||_{V'}^2 ds, |y_0|^2 \right), \tag{5.15}$$

where C is a positive bounded function independent of N, j.

For $n = 2$, (1.6), (5.12) and (5.15) give

$$\begin{aligned} ||B_N y_N^j(s)||_{V'} &\le C|y_N^j(s)|\,||y_N^j(s)|| \\ &\le C||y_N^j(s)||, \quad \text{which yield} \int_0^T ||B_N y_N^j(s)||_{V'}^p ds \le C, \text{ if } n = 2. \end{aligned} \tag{5.16}$$

If $n = 3$, from (1.8), (5.12), and (5.15) we get

$$\begin{aligned} ||B_N y_N^j(s)||_{V'} &\le C|y_N^j(s)|^{1/2}||y_N^j(s)||^{3/2} \\ &\le C||y_N^j(s)||^{3/2}, \quad \text{which yield} \int_0^T ||B_N y_N^j(s)||_{V'}^{2p/3} ds \le C, \text{ if } n = 3. \end{aligned} \tag{5.17}$$

From (5.3) and (5.15), we obtain

$$\int_0^T ||\Phi(y_N^j(t))||_{V'}^{p'} ds \le \omega_2 \int_0^T ||y_N^j(t)||^p dt \le C, \tag{5.18}$$

where $p' \le 2$ is the conjugate of $p \ge 2$.

From (5.16)-(5.18) and from the continuity of the operator $A : V \to V'$, we infer that

$$\int_0^T \left\| \frac{dy_N^j}{dt} \right\|_{V'}^{p'} ds \le C \quad \text{for } n = 2, \tag{5.19}$$

$$\int_0^T \left\| \frac{dy_N^j}{dt} \right\|_{V'}^{r} ds \le C \quad \text{for } n = 3, \tag{5.20}$$

where $r = \min\{2p/3, p'\}$.

On a subsequence again denoted by $(y_N^j)_j$, we have for $j \to \infty$,

$$y_N^j \rightharpoonup y_N \quad \text{in } L^p(0,T;V),$$

$$Ay_N^j \rightharpoonup Ay_N \quad \text{in } L^p(0,T;V'),$$

$$\frac{dy_N^j}{dt} \rightharpoonup \frac{dy_N}{dt} \quad \text{in} \begin{cases} L^{p'}(0,T;V') & \text{if } n = 2, \\ L^r(0,T;V') & \text{if } n = 3, \end{cases}$$

$$B_N y_N^j \rightharpoonup \beta_N \quad \text{in } \begin{cases} L^p(0,T;V') & \text{if } n=2, \\ L^{2p/3}(0,T;V') & \text{if } n=3, \end{cases}$$
$$\Phi(y_N^j) \rightharpoonup \eta_N \quad \text{in } L^{p'}(0,T;V'). \tag{5.21}$$

In order to prove that $\beta_N = B_N y_N$ and $\eta_N = \Phi y_N$, we use that for μ_N large enough, B_N verifies

$$|(B_N u - B_N v, u - v)| \leq \frac{\nu}{2}\|u-v\|^2 + \mu_N |u-v|^2, \quad \forall u,v \in V, \tag{5.22}$$

(see [4, relation (5.4), page 292]) and we observe that

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\left| y_N^j(s) - y_N^k(s)\right|^2 + \frac{\nu}{2}\left\| y_N^j(s) - y_N^k(s)\right\|^2 \\ &\quad \leq \mu_N \left| y_N^j(s) - y_N^k(s)\right|^2 + \left\| f_j(s) - f_k(s)\right\|_{V'}\left\| y_N^j(s) - y_N^k(s)\right\|. \end{aligned} \tag{5.23}$$

Integrating from 0 to t and applying the Gronwall lemma, it follows that

$$\left| y_N^j(t) - y_N^k(t)\right|^2 + \frac{\nu}{2}\int_0^t \left\| y_N^j(s) - y_N^k(s)\right\|^2 ds \leq C_N\left(\left| y_0^j - y_0^k\right|^2 + \int_0^T \left\| f_j(s) - f_k(s)\right\|_{V'}^2 ds\right). \tag{5.24}$$

Consequently,

$$y_N^j \longrightarrow y_N \quad \text{in } L^2(0,T;V) \cap C([0,T];H). \tag{5.25}$$

But $\|B_N y_N^j(s) - B_N y_N(s)\|_{V'} \leq C\|y_N^j(s) - y_N(s)\|(\|y_N^j(s)\| + \|y_N(s)\|)$ and (5.25) implies that

$$B_N y_N^j \longrightarrow B_N y_N \quad \text{in } L^1(0,T;V'). \tag{5.26}$$

On the other side, from (5), (5.25), and the properties of Φ, we get $\eta_N(t) = \Phi(y_N(t))$, $t \in (0,T)$.

Thus problem (5.6) has a unique solution

$$y_N \in L^p(0,T;V) \cap C([0,T];H), \qquad \frac{dy_N}{dt} \in \begin{cases} L^{p'}(0,T;V') & \text{if } n=2, \\ L^r(0,T;V') & \text{if } n=3, \end{cases}$$
$$\Phi(y_N) \in L^{p'}(0,T;V'), \qquad B_N y_N \in \begin{cases} L^p(0,T;V') & \text{if } n=2, \\ L^{2p/3}(0,T;V') & \text{if } n=3. \end{cases} \tag{5.27}$$

Moreover, y_N satisfies the estimates (5.12), (5.15)–(5.20). Then, on a subsequence again denoted by $(y_N)_N$, we have for $N \to \infty$,

$$\begin{gathered} y_N \rightharpoonup y \quad \text{in } L^p(0,T;V), \qquad y_N \rightharpoonup y \quad \text{weakly} * \text{ in } L^\infty(0,T;H), \\ Ay_N \rightharpoonup Ay \quad \text{in } L^p(0,T;V'), \end{gathered} \tag{5.28}$$

$$\begin{gathered} \frac{dy_N}{dt} \rightharpoonup \frac{dy}{dt} \quad \text{in } \begin{cases} L^{p'}(0,T;V') & \text{if } n=2, \\ L^r(0,T;V') & \text{if } n=3, \end{cases} \\ B_N y_N \rightharpoonup \beta \quad \text{in } \begin{cases} L^p(0,T;V') & \text{if } n=2, \\ L^{2p/3}(0,T;V') & \text{if } n=3, \end{cases} \\ \Phi y_N \rightharpoonup \eta \quad \text{in } L^{p'}(0,T;V'). \end{gathered} \tag{5.29}$$

From (5.28), (5.29), and Aubin's compactness theorem, it follows that

$$y_N \longrightarrow y \quad \text{in } L^p(0,T;H). \tag{5.30}$$

From the properties of Φ, we have $\eta = \Phi y$. It remains to prove that $\beta(t) = By(t)$ a.e. $t \in (0,T)$.

Let $E_N = \{t \in [0,T];\ \|y_N(t)\| \le N\}$. Then $By_N = B_N y_N$ in E_N. We note that the Lebesgue measure

$$m([0,T] - E_N) = m(\{t \in [0,T];\ \|y_N(t)\| > N\}) \le \frac{C}{N^2}. \tag{5.31}$$

Let $\psi \in L^\infty(0,T;\mathcal{V})$, where $\mathcal{V} = \{\rho \in C_0^\infty(\Omega);\ \operatorname{div}\rho = 0\}$. We have

$$\begin{aligned} &\int_0^T |(B_N y_N - By, \psi)|\,dt \\ &\quad \le \int_{E_N} |(By_N - By, \psi)|\,dt + C\int_{[0,T]-E_N} \|\psi\|(|y_N|^{1/2}\|y_N\|^{3/2} + |y|^{1/2}\|y\|^{3/2})\,dt. \end{aligned} \tag{5.32}$$

Hence

$$\begin{aligned} &\int_0^T |(B_N y_N - By, \psi)|\,dt \\ &\quad \le \int_0^T (|b(y_N - y, y_N, \psi)| + |b(y, y_N - y, \psi)|)\,dt + CN^{-2}\|\psi\|_{L^\infty(0,T;V)}. \end{aligned} \tag{5.33}$$

Recalling (5.28), (5.30), which imply that $y_N \to y$ in $L^2(0,T;H)$ and $y_N \rightharpoonup y$ in $L^2(0,T;V)$, we get

$$\lim_{N\to\infty} \int_0^T (B_N y_N - By, \psi)\,dt = 0, \quad \forall \psi \in L^\infty(0,T;\mathcal{V}). \tag{5.34}$$

Then $\eta = By$, which concludes the existence part of the theorem.

If $n = 2$, the solution is unique. Indeed, taking two solutions y_1, y_2, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left| y_1(t) - y_2(t) \right|^2 &+ \nu\|y_1(t) - y_2(t)\|^2 \\ &+ b(y_1(t) - y_2(t), y_1(t), y_1(t) - y_2(t)) \\ &+ (\Phi y_1(t) - \Phi y_2(t), y_1(t) - y_2(t)) = 0.\end{aligned} \tag{5.35}$$

From the monotony of Φ and the properties of b, we infer that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left| y_1(t) - y_2(t) \right|^2 + \nu\|y_1(t) - y_2(t)\|^2 \\ &\leq C\|y_1(t) - y_2(t)\|\|y_1(t)\| \cdot |y_1(t) - y_2(t)| \\ &\leq \frac{\nu}{2}\|y_1(t) - y_2(t)\|^2 + C\|y_1(t)\|^2 \left| y_1(t) - y_2(t) \right|^2,\end{aligned} \tag{5.36}$$

and by Gronwall's lemma, $y_1 \equiv y_2$. □

6. Examples

Example 6.1 (a time optimal problem in 2D). Consider the controlled Navier-Stokes equations ($n = 2$)

$$\begin{gathered}\frac{dy}{dt}(t) + \nu A y(t) + B(y(t)) = u(t), \quad t > 0, \\ y(0) = y_0.\end{gathered} \tag{6.1}$$

Given $\rho > 0$, $y_0 \in V$, $y_1 \in D(A)$, one searches a $u \in L^\infty(\mathbb{R}_+; H)$, $|u(t)| \leq \rho$ a.e. $t > 0$, which steers y_0 to y_1 in minimum time.

In order to prove the existence of admissible controllers, in [7] $u(t) \in -\rho\,\mathrm{sgn}(y(t) - y_1)$, $t > 0$ is chosen. By sgn we mean the multivalued operator:

$$\mathrm{sgn} : H \longrightarrow H, \quad \mathrm{sgn}\, x = \begin{cases} \dfrac{x}{|x|} & \text{if } |x| > 0, \\ \{p \in H;\ |p| \leq 1\} & \text{if } x = 0, \end{cases} \tag{6.2}$$

where $|\cdot|$ denotes the norm of H.

So, we need to show that the problem

$$\begin{gathered}\frac{dy}{dt}(t) + \nu A y(t) + B y(t) + \rho\,\mathrm{sgn}\left(y(t) - y_1\right) \ni 0, \quad \text{a.e. } t > 0, \\ y(0) = y_0\end{gathered} \tag{6.3}$$

has a unique strong solution on $(0, T)$, for all $T > 0$.

The operator $\rho\,\mathrm{sgn}(\cdot - y_1)$ is the subdifferential of $\rho|\cdot - y_1|$, hence maximal monotone in $H \times H$. A standard calculation gives

$$(Ay, (\rho\,\mathrm{sgn}(\cdot - y_1))_\lambda(y)) \geq 0, \quad \forall y \in D(A),\ \forall \lambda > 0. \tag{6.4}$$

Thus we may apply the existence and uniqueness theorems for strong solutions in Section 2.

Then (see [7]), problem

$$\frac{dz}{dt}+\nu Az+B(z+y_1)-By_1+\rho\,\mathrm{sgn}\,z\ni-(\nu Ay_1+By_1),\qquad z(0)=y_0-y_1, \tag{6.5}$$

where $z(t)=y(t)-y_1$, has finite time extinction property, that is, $z(T)=0$ for ρ large enough.

Example 6.2 (invariance preserving (see [4])). Consider the controlled Navier-Stokes equation

$$\frac{dy}{dt}(t)+\nu Ay(t)+By(t)=f(t)+u(t),\quad t\in(0,T),\qquad y(0)=y_0. \tag{6.6}$$

Let $K\subset H$ be a closed and convex set such that $0\in K$ and

$$(I+\lambda A)^{-1}K\subset K,\quad \forall\lambda>0. \tag{6.7}$$

We search a feedback controller $u(t)$ such that

$$y_0\in K\quad \text{implies } y(t)\in K,\ t\in[0,T]. \tag{6.8}$$

This is done by solving the problem

$$\frac{dy}{dt}(t)+\nu Ay(t)+By(t)-f(t)+N_K(y(t))\ni 0,\quad t\in(0,T)\qquad y(0)=y_0, \tag{6.9}$$

where $N_K(y)=\{z\in H;\ (z,y-x)\geq 0,\text{ for all } x\in K\}$ is the Clarke normal cone to K at y.

The operator $\Phi(y)=N_K(y)$ coincides with the subdifferential $(\partial I_K)(y)$ of the indicator function

$$I_K(x)=\begin{cases}0 & \text{for } x\in K,\\ +\infty & \text{for } x\notin K,\end{cases} \tag{6.10}$$

(consequently, Φ is maximal monotone). Also, $0\in D(\Phi)=K$.

The resolvent $(I+\lambda\partial I_K)^{-1}=P_K$, where $P_K:(L^2(\Omega))^n\to K$ is the projection operator on K. It follows that the Yosida approximation of ∂I_K is

$$(\partial I_K)_\lambda(y)=\frac{1}{\lambda}\left(y-(I+\lambda\partial I_K)^{-1}(y)\right)=\frac{1}{\lambda}(y-P_K(y)),\quad \forall y\in H. \tag{6.11}$$

In the mean time, $(\partial I_K)_\lambda$ is the subdifferential of $(I_K)_\lambda(y)=(1/(2\lambda))|y-P_K(y)|^2$.

Using (6.7) and applying [8, Proposition 4.5, page 131] (or [9, Theorem 1.7, page 183]) for the single-valued maximal monotone operator $A \subset H \times H$, with $\overline{D(A)} = H$, we get

$$(Ay, (\partial I_K)_\lambda(y)) \geq 0, \quad \forall y \in D(A), \forall \lambda > 0. \tag{6.12}$$

Then hypotheses (h$_1$)–(h$_3$), Section 1 on Φ, are verified by $\partial I_K = N_K$ and we may apply the results in Section 2.

Example 6.3 (stabilizing feedback controllers). Consider the controlled system

$$\begin{gathered} \frac{dy}{dt}(t) + \nu A y(t) + B y(t) = f_e + u(t), \quad t > 0, \\ y(0) = y_0. \end{gathered} \tag{6.13}$$

Let $y_e \in D(A)$ be a steady-state solution for (6.13), that is, y_e verifies

$$\nu A y_e + B y_e = f_e. \tag{6.14}$$

Given $K \subset H$ a closed and convex set, with $0 \in K$, we look for a feedback controller u such that $y(t) - y_e \in K$, for all $t \geq 0$ and $\lim_{t\to\infty} |y(t) - y_e| = 0$.

We set $z = y - y_e$ and take $u(t) \in -\lambda z(t) - (\partial I_K)(z(t))$, with $\lambda > 0$ large enough. Then (6.13) becomes

$$\begin{gathered} \frac{dz}{dt}(t) + \nu A z(t) + B z(t) + A_0 z(t) + (\partial I_K)(z(t)) + \lambda z(t) \ni 0, \quad t > 0, \\ z(0) = y_0 - y_e, \end{gathered} \tag{6.15}$$

where

$$A_0 \in L(V, V'), \quad (A_0 z, w) = b(z, y_e, w) + b(y_e, z, w), \quad \forall z, w \in V. \tag{6.16}$$

Assume that

$$(I + \varepsilon A)^{-1} K \subset K, \quad \forall \varepsilon > 0. \tag{6.17}$$

Let $\Phi(w) = \lambda w + (\partial I_K)(w) = \partial((\lambda/2)|\cdot|^2 + I_K)(w)$, for all $w \in H$. Then Φ is a maximal monotone operator in $H \times H$. From $0 \in K = D(\partial I_K)$, we get $0 \in D(\Phi) = K$. Moreover, relation (6.17) yields

$$(\Phi_\varepsilon(w), Aw) \geq 0, \quad \forall w \in D(A), \ \forall \varepsilon > 0 \tag{6.18}$$

(see [8, Theorem 4.4, (i)$\Leftrightarrow$(iv) page 130]). As a result, Φ satisfies hypotheses (h$_1$)–(h$_3$), Section 1.

In the proofs of Theorems 2.1, 2.2, 2.3, the operator B will be replaced by $B + A_0$. This fact does not change the estimates in Section 3. Of course, the positive constants in the right-hand side of the estimates will depend on $\|y_e\|$, $|Ay_e|$. Although, unlike B, $(A_0 w, w) = b(w, y_e, w) \neq 0$ in V, the resulting terms are absorbed by other terms.

By consequence, we have existence and uniqueness results for the solution z of system (6.15), with the invariance property $z(t) \in K$, $t > 0$. Stability will be considered in a further paper.

References

[1] P. Constantin and C. Foiaş, *Navier-Stokes Equations*, The University of Chicago Press, Chicago, Ill, USA, 1989.

[2] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, vol. 2 of *Studies in Mathematics and Its Applications*, North-Holland, Amsterdam, The Netherlands, 1979.

[3] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, vol. 66 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 2nd edition, 1995.

[4] V. Barbu and S. S. Sritharan, "Flow invariance preserving feedback controllers for the Navier-Stokes equation," *Journal of Mathematical Analysis and Applications*, vol. 255, no. 1, pp. 281–307, 2001.

[5] V. Barbu, *Analysis and Control of Nonlinear Infinite-Dimensional Systems*, vol. 190 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1993.

[6] I. I. Vrabie, *C_0-Semigroups and Applications*, vol. 191 of *North-Holland Mathematics Studies*, North-Holland, Amsterdam, The Netherlands, 2003.

[7] V. Barbu, "The time optimal control of Navier-Stokes equations," *Systems & Control Letters*, vol. 30, no. 2-3, pp. 93–100, 1997.

[8] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland Mathematics Studies, no. 5. Notas de Matemática (50), North-Holland, Amsterdam, The Netherlands, 1973.

[9] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei Republicii Socialiste România, Bucharest, Romania, 1976.

Adriana-Ioana Lefter: Department of Applied Mathematics, Faculty of Mathematics, University "Al. I. Cuza", 11 Bd. Carol I, 700506 Iaşi, Romania; Institute of Mathematics "Octav Mayer", Romanian Academy, Iaşi Branch, 8 Bd. Carol I, 700506 Iaşi, Romania
Email address: ilefter@uaic.ro