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Weighted weak type $(1,1)$ estimates for singular integrals with non-isotropic homogeneity

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Abstract. We prove sharp weighted weak type $(1,1)$ estimates for rough singular integral operators on homogeneous groups. Similar results are shown for singular integrals on $\mathbb{R}^2$ with the generalized homogeneity.

1. Introduction

We consider singular integrals defined by kernels homogeneous with respect to non-isotropic dilations, which generalize homogeneous singular integrals studied in Calderón-Zygmund [3]. In this note we deal with weighted weak type boundedness, for rough singular integrals on $\mathbb{R}^2$ with generalized homogeneity and for rough singular integrals on homogeneous groups; we shall prove analogues of a result of Vargas [30] concerning weighted weak type $(1,1)$ estimates for homogeneous singular integrals on $\mathbb{R}^2$.

Let P be an $n\times n$ real matrix whose eigenvalues have positive real parts. A dilation group $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ is defined by $A_t=\exp((\log t)P)$. We assume $n\geq 2$. Then, there is a norm function r on $\mathbb{R}^n$ associated with $\{A_t\}_{t>0}$, which is non-negative, continuous, even on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n\setminus\{0\}$; furthermore it satisfies

(1) $r(A_t x)=tr(x)$ for all $t>0$ and $x\in\mathbb{R}^n$;

(2) $r(x+y)\leq N_1(r(x)+r(y))$ for a positive constant N_1;

(3) if $\Sigma=\{x\in\mathbb{R}^n : r(x)=1\}$, then $\Sigma=\{\theta\in\mathbb{R}^n : \langle B\theta,\theta\rangle=1\}$ for a positive symmetric matrix B, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$.

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We have a polar coordinates expression for the Lebesgue measure:

$$\int_{\mathbb{R}^n} f(x)\,dx=\int_0^\infty\int_\Sigma f(A_t\theta)t^{\gamma-1}\,dS(\theta)\,dt$$

with γ=trace P and $dS=\omega\,dS_0$, where ω is a strictly positive C^∞ function on Σ and dS_0 is the Lebesgue surface measure on Σ. Also, there are positive constants $c_1, c_2, d_1, d_2, \alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$c_1|x|^{\alpha_1}\le r(x)\le c_2|x|^{\alpha_2}\quad\text{or}\quad d_1|x|^{\beta_1}\le r(x)\le d_2|x|^{\beta_2}$$

according as $r(x)>1$ or $r(x)\le 1$, where $|\cdot|$ denotes the Euclidean norm. Let $S^{n-1}=\{x\in\mathbb{R}^n:|x|=1\}$ be the unit sphere with the Euclidean norm. The Lebesgue surface measure on S^{n-1} will be denoted also by $d\sigma$. See [2], [21] and [28] for more details.

We denote by $L\log L(\Sigma)$ the Zygmund class on Σ with the norm defined as

$$\|\Omega\|_{L\log L}=\inf\left\{\lambda>0:\int_\Sigma\left|\Omega(\theta)/\lambda\right|\log\left(2+\left|\Omega(\theta)/\lambda\right|\right)dS(\theta)\le 1\right\}.$$

Also, we consider the $L^q(\Sigma)$ spaces. We write $\|\Omega\|_q=(\int_\Sigma|\Omega(\theta)|^q\,dS(\theta))^{1/q}$, $0<q<\infty$, $\|\Omega\|_\infty=\sup_{\theta\in\Sigma}|\Omega(\theta)|$.

Let Ω be locally integrable in $\mathbb{R}^n\setminus\{0\}$ and homogeneous of degree 0 with respect to the dilation group $\{A_t\}$. Thus $\Omega(A_tx)=\Omega(x)$ for $x\ne 0$ and $t>0$. We assume the cancellation property

$$\int_\Sigma\Omega(\theta)\,dS(\theta)=0. \tag{1.1}$$

Let $K(x)=r(x)^{-\gamma}\Omega(x')$, where $x'=A_{r(x)^{-1}}(x)$ for $x\ne 0$, and define the singular integral

$$Tf(x)=\text{p.v.}\int_{\mathbb{R}^n}f(x-y)K(y)\,dy. \tag{1.2}$$

Then it is known that T is bounded on $L^p(\mathbb{R}^n)$ for all $p\in(1,\infty)$ if $\Omega\in L\log L(\Sigma)$. A proof of this based on [10] can be found in [22], where a wider class of singular integrals including the one in (4.1) below is treated. Also, the following result is known (see [24]).

Theorem A. *Let $n=2$ and $\Omega\in L\log L(\Sigma)$. Then T is of weak type $(1,1)$ on $\mathbb{R}^2$.*

For $\Omega \in L^1(S^{n-1})$, let

$$M_\Omega(f)(x) = \sup_{t>0} t^{-n} \int_{|y|<t} |f(x-y)| |\Omega(y')| \, dy, \quad y' = y/|y|. \tag{1.3}$$

Put $M_{\Omega,s}(f) = [M_\Omega(|f|^s)]^{1/s}$, $s>0$. We then recall a result of Vargas [30] on $\mathbb{R}^2$ with the usual isotropic dilation and the Euclidean norm.

Theorem B. *Let $\Omega \in L^q(S^1)$, $q>1$, $\int_{S^1} \Omega \, d\sigma = 0$, and let T be defined as in (1.2) with $n=2$ and $K(x) = \Omega(x')/|x|^2$ ($x' = x/|x|$). For a weight w define*

$$W(x) = \|\Omega\|_q^{1/\beta'} M_\beta M_{\tilde{\Omega},\beta} M_\beta(w)(x) + \|\Omega\|_q M_\beta(w)(x),$$

where $\beta \in (1,\infty)$, $1/\beta + 1/\beta' = 1$, $\widetilde{\Omega}(\theta) = \Omega(-\theta)$, $M_\beta(f) = [M(|f|^\beta)]^{1/\beta}$ with M denoting the Hardy-Littlewood maximal operator on $\mathbb{R}^2$. Then

$$\sup_{\lambda>0} \lambda w\big(\{x \in \mathbb{R}^2 : |Tf(x)| > \lambda\}\big) \le C \int_{\mathbb{R}^2} |f(x)| W(x) \, dx$$

for a positive constant C independent of Ω, where $w(E) = \int_E w(x) \, dx$.

Theorem B is generalized to higher dimensions by [12] on the basis of [26]. In this note we shall extend Theorem B to the cases of singular integrals on $\mathbb{R}^2$ with generalized homogeneity and singular integrals on homogeneous groups.

We regard $\mathbb{R}^n$ as the underlying manifold of a homogeneous group. The multiplication is given by a polynomial mapping and there is a dilation family $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ such that each A_t is an automorphism of the group structure with the form

$$A_t x = \big(t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_n} x_n\big), \quad x = (x_1, \dots, x_n),$$

for some real numbers $a_1, \dots, a_n$ satisfying $0 < a_1 \le a_2 \le \dots \le a_n$. We denote by $\mathbb{H}$ the homogeneous group. Thus $\mathbb{H}$ is equipped with a homogeneous nilpotent Lie group structure, where Lebesgue measure is a bi-invariant Haar measure, the identity is the origin 0, $x^{-1} = -x$ and multiplication xy, $x, y \in \mathbb{H}$, satisfies

(4) $A_t(xy) = (A_t x)(A_t y)$, $x, y \in \mathbb{H}$, $t>0$;

(5) if $z = xy$, then $z_1 = x_1 + y_1$ and $z_k = x_k + y_k + R_k(x,y)$ for $k \ge 2$ with a polynomial $R_k(x,y)$ depending only on x_i, y_j, $1 \le i, j \le k-1$;

(6) $r(xy) \le N_2(r(x) + r(y))$ for a positive constant N_2.

We may assume that $\Sigma = S^{n-1}$, where Σ is as in (3). The space $\mathbb{H}$ with a left invariant quasi-metric $d(x,y) = r(x^{-1}y)$ can be regarded as a space of homogeneous type. We refer to [5], [8], [13], [20] and [29] for more details.

If we define the multiplication

$$(x,y,u)(x',y',u') = (x+x', y+y', u+u'+(xy'-yx')/2),$$

then $\mathbb{R}^3$ with this group law is the Heisenberg group $\mathbb{H}_1$, which is an example of homogeneous groups. Dilations $A_t(x,y,u)=(tx,ty,t^2u)$ (2-step) and $A'_t(x,y,u)=(tx,t^2y,t^3u)$ (3-step) are automorphisms on $\mathbb{H}_1$. The Euclid space $\mathbb{R}^n$ with the usual addition is also a homogeneous group; for the associated dilation $A_t=\exp((\log t)P)$, we may choose any diagonal matrix P with entries in ascending order.

The convolution on $\mathbb{H}$ is defined by

$$f*g(x) = \int_{\mathbb{H}} f(y)g(y^{-1}x)\,dy.$$

Let $K(x)=r(x)^{-\gamma}\Omega(x')$ be the homogeneous kernel associated with $\{A_t\}$ as above and define

$$Tf(x) = \text{p.v.} f*K(x) = \text{p.v.} \int_{\mathbb{H}} f(y)K(y^{-1}x)\,dy. \tag{1.4}$$

Then, in [29], the following two results are proved.

Theorem C. *Suppose that $\Omega\in L\log L(\Sigma)$. Then, T is bounded on $L^p(\mathbb{H})$ for all $p\in(1,\infty)$.*

Theorem D. *Let $\Omega\in L\log L(\Sigma)$. Then, T is of weak type $(1,1)$ on $\mathbb{H}$.*

Theorem A follows from Theorem D when the matrix P is diagonal.

A result similar to Theorem C is proved in [25] for the maximal singular integrals T_* defined by

$$T_*f(x) = \sup_{N,\varepsilon>0} \left| \int_{\varepsilon<r(y)<N} f(xy^{-1})K(y)\,dy \right|.$$

We shall prove weighted versions of Theorems A and D, generalizing Theorem B. Let w be a measurable, almost everywhere positive function on $\mathbb{R}^n$, which we call a weight function. We denote by $L^p(w)$ $(p>0)$ the space of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \right)^{1/p} < \infty$$

and by $L^{1,\infty}(w)$ the weak $L^1(w)$ space of all those functions f which satisfy

$$\|f\|_{L^{1,\infty}(w)} = \sup_{\lambda>0} \lambda w(\{x\in\mathbb{R}^n : |f(x)|>\lambda\}) < \infty,$$

where we recall that $w(E)=\int_E w(x)\,dx$.

Let $B(a,s)=\{x\in\mathbb{H}:d(a,x)<s\}$ be a ball in $\mathbb{H}$ with center a and radius s. Note that $|B(a,s)|=cs^\gamma$, where $|S|$ denotes the Lebesgue measure of a set S and $c=|B(0,1)|$. If $s=2^k$ for some $k\in\mathbb{Z}$ (the set of all integers), then $S=B(a,2^k)$ is called a dyadic ball. Also, we write $a=x_S$, $k=k(S)$. We define $\tau B(a,s)=B(a,\tau s)$ for $\tau>0$.

Let A_p, $1<p<\infty$, be the weight class of Muckenhoupt on $\mathbb{H}$ defined to be the collection of all weight functions w on $\mathbb{H}$ satisfying

$$\sup_B\left(|B|^{-1}\int_B w(x)\,dx\right)\left(|B|^{-1}\int_B w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty,$$

where the supremum is taken over all balls B in $\mathbb{H}$. Let M be the Hardy-Littlewood maximal operator defined as usual by

$$Mf(x)=\sup_B|B|^{-1}\int_B\left|f(y)\right|dy,$$

where the supremum is taken over all balls B in $\mathbb{H}$ containing x. We then recall the class A_1 is defined to be the family of all weight functions w on $\mathbb{H}$ satisfying $Mw\le Cw$ a.e. (See [1], [8] and [14].)

For functions Ω on Σ and f on $\mathbb{H}$, we define a maximal function

$$M_\Omega(f)(x)=\sup_{t>0}t^{-\gamma}\int_{r(y)<t}\left|f\left(xy^{-1}\right)\right|\left|\Omega\left(y'\right)\right|dy,$$

generalizing (1.3). Some weighted estimates for M_Ω, T, T_* analogous to those in the Euclidean case of [9] and [31] are shown in [25].

Put $M_s(f)=[M(|f|^s)]^{1/s}$, $s>0$, and $M_{\Omega,s}(f)=[M_\Omega(|f|^s)]^{1/s}$. Then

$$M_{\Omega,s}(f)\le\left(\|\Omega\|_1/\gamma\right)^{1/s-1/t}M_{\Omega,t}(f),\quad\text{if } s<t;\tag{1.5}$$

$$M_{\Omega,s}(f)\le C\|\Omega\|_q^{1/s}M_{sq'}(f)\quad\text{if } q>1.\tag{1.6}$$

We have the following result.

Theorem 1.1. *Let $w\in A_2$ and $\beta\in(1,\infty)$. Suppose that T is as in (1.4) with $\Omega\in L^q(\Sigma)$ for some q, $1<q\le\infty$. Then, there exists a positive constant C independent of Ω such that*

$$\|Tf\|_{L^{1,\infty}(w)}\le C\int_{\mathbb{H}}\left|f(x)\right|\left(\|\Omega\|_q^{1/\beta'}M_\beta M_{\tilde\Omega,\beta}(w)(x)+\|\Omega\|_qM_\beta(w)(x)\right)dx.$$

Here $\tilde g(x)=g(x^{-1})$. See [6], [7], [12], [15], [16] and [26] for relevant results. By Theorem 1.1 and (1.5) we can easily prove the weighted weak type estimates for T analogous to Theorem B (see Remark 2 of [12]).

Corollary 1.2. *Suppose that* $\Omega \in L^q(\Sigma)$ *for some* $1<q\le\infty$ *and* $w^{q'}\in A_1$. *Then* T *is bounded from* $L^1(w)$ *to* $L^{1,\infty}(w)$.

This follows from Theorem 1.1 with β sufficiently close to 1 and (1.6).

To prove Theorem 1.1, we use the following weighted L^2-estimates.

Theorem 1.3. *Let* Ω, q, T, w, β *be as in Theorem* 1.1. *Then, there exists a constant* C *independent of* q *and* Ω *such that*

$$\|Tf\|_{L^2(w)} \le Cq'\|\Omega\|_q^{1-1/(2\beta)} \left(\int_{\mathbb{H}} \bigl|f(x)\bigr|^2 M_\beta M_{\tilde{\Omega},\beta}(w)(x)\,dx\right)^{1/2}.$$

To state results with $\Omega \in L\log L(\Sigma)$, we consider the maximal function

$$M^*(f)(x) = \sup_F M_F(f)(x),$$

where the supremum is taken over all the functions $F\in L^1(\Sigma)$ with $\|F\|_1=1$. Put $M_s^*(f)=[M^*(|f|^s)]^{1/s}$. Then we have the following.

Theorem 1.4. *Let* T *be as in* (1.4) *with* $\Omega\in L\log L(\Sigma)$. *Suppose that* $w\in A_2$ *and* $\beta\in(1,\infty)$. *Then*

$$\|Tf\|_{L^{1,\infty}(w)} \le C\|\Omega\|_{L\log L}\|f\|_{L^1(M_\beta M_\beta^*(w))}$$

for a constant C *independent of* Ω.

Let $\mathcal{A}_1(M^*)$ be the collection of all the weight functions w such that $M^*w\le C_w w$ a.e. for some constant C_w. Then, if w satisfies that $w^\tau\in\mathcal{A}_1(M^*)$ for some $\tau>1$ and $\Omega\in L\log L(\Sigma)$, by Theorem 1.4 it follows that T is bounded from $L^1(w)$ to $L^{1,\infty}(w)$ on $\mathbb{H}$.

To prove Theorem 1.4 we apply the following.

Theorem 1.5. *Let* w, β, Ω *and* T *be as in Theorem* 1.4. *Then*

$$\|Tf\|_{L^2(w)} \le C\|\Omega\|_{L\log L}\|f\|_{L^2(M_\beta M_\beta^*(w))}$$

with a constant C *independent of* Ω.

Remark 1.6. Let $M_\Omega(f)$ be as in (1.3). Let $w(x)=|x|^\alpha$, $\Omega\in L^q$, $q\ge1$. Then, $M_\Omega(w)(x)\le C\|\Omega\|_q w(x)$ if $-n+(n-1)/q<\alpha\le0$ (see [19]). In the case of the Euclidean structure, this observation and Theorem 1.1 will be used to get a better result when $w(x)=|x|^\alpha$ than the one Corollary 1.2 can provide (see [27] for relevant results).

Proofs of Theorems 1.3 and 1.5 will be given in Section 2. Theorem 1.3 will be shown by applying two parameter Littlewood-Paley type decomposition of T depending on $q>1$ in the theorem (see (2.4)), which is introduced in [25], and using the decay estimates (Lemma 2.3) which can be proved through orthogonality via convolution. Such two parameter decomposition is needed at the present stage of the research, since in general homogeneous groups Fourier transform estimates cannot be applied as effectively as in the Euclidean situation of [10] (see the proof of Theorem 4.3 in Section 4) and the group convolution may be noncommutative. Theorem 1.5 will be proved by extrapolation using Theorem 1.3.

In Section 3 we shall prove Theorems 1.1 and 1.4. We apply the Koranyi-Vagi version of the Calderón-Zygmund decomposition $f=g+b$. The evaluation of Tg can be accomplished by the weighted L^2 estimates of Theorems 1.3 and 1.5 as usual. To treat Tb we apply a result of Tao [29] (Proposition 3.4). We interpolate with change of measures, between unweighted estimates shown from the result of Tao (Lemma 3.2) and weighted estimates which can be obtained by a straightforward computation (Lemma 3.3), to prove some key estimates. The interpolation is a variant of the methods of Vargas [30]. Since Vargas's interpolation arguments cannot be applied directly to get necessary estimates for the proofs of the theorems, we need to further develop the idea of the methods and suitably modify the arguments to be adapted for the present situation (see also [11] and [12]).

We shall consider singular integrals on $\mathbb{R}^2$ with generalized homogeneity defined by (1.2) in Section 4. We are able to prove results analogous to those stated above for singular integrals on $\mathbb{H}$ (Theorems 4.1–4.4). To prove analogues for Theorems 1.1 and 1.4 (Theorems 4.1, 4.2), we apply Proposition 2.1 of [24], which will play a role similar to the one Proposition 3.4 performs in the proofs of Theorems 1.1 and 1.4.

In Section 5, we focus on the case of $\mathbb{R}^n$ with the Euclidean structure and give an application of Theorem 1.4. We shall show a sharp weighted weak type $(1,1)$ estimate conjectured in [9].

2. Proofs of Theorems 1.3 and 1.5

Let ϕ be a non-negative, smooth function on $\mathbb{H}$ with support in $B(0,1)\setminus B(0,1/2)$ satisfying $\int \phi=1$, $\phi=\tilde{\phi}$. For $\rho\geq 2$, define

$$\Delta_k=\delta_{\rho^{k-1}}\phi-\delta_{\rho^k}\phi, \quad k\in\mathbb{Z},$$

where $\delta_t\phi(x)=t^{-\gamma}\phi(A_t^{-1}x)$. Note that $\operatorname{supp}(\Delta_k)\subset B(0,\rho^k)\setminus B(0,\rho^{k-2})$, $\Delta_k=\tilde{\Delta}_k$ and $\sum_k \Delta_k=\delta$, where δ is the delta function.

For any $\rho \geq 2$ we can find a sequence $\{\psi_j\}_{j\in\mathbb{Z}}$ of non-negative functions in $C_0^\infty(\mathbb{R})$ such that

$$\operatorname{supp}(\psi_j)\subset\big\{t\in\mathbb{R}:\rho^j\leq t\leq\rho^{j+2}\big\},$$
$$\sum_{j\in\mathbb{Z}}\psi_j(t)=1 \quad \text{for } t>0,$$
$$\big|(d/dt)^m\psi_j(t)\big|\leq c_m|t|^{-m} \quad \text{for } m=0,1,2,\dots,$$

where c_m is a constant independent of ρ, which is possible as $\rho\geq 2$.

Define the operator S_j by

$$S_jF(x)=(\log 2)^{-1}\int_0^\infty \psi_j(t)\delta_tF(x)\,dt/t. \tag{2.1}$$

Then $S_jK_0(x)=r(x)^{-\gamma}\Omega(x')\Psi_j(r(x))$, where

$$K_0(x)=K(x)\chi_{D_0}(x), \quad D_0=\big\{x\in\mathbb{R}^n:1\leq r(x)\leq 2\big\},$$
$$\Psi_j(s)=(\log 2)^{-1}\int_{1/2}^1\psi_j(ts)\,dt/t.$$

Here χ_E denotes the characteristic function of a set E. It follows that $\sum_{j\in\mathbb{Z}}S_jK_0=K$. Thus

$$Tf=\sum_j f*S_jK_0.$$

We have the following L^2 estimates.

Lemma 2.1. *Suppose that $1<q\leq\infty$, $\rho=2^{q'}$ and $\Omega\in L^q(\Sigma)$. Then, for $j,k\in\mathbb{Z}$ we have*

$$\|f*S_jK_0*\Delta_k\|_2\leq Cq'2^{-\varepsilon|j-k|}\|\Omega\|_q\|f\|_2$$

for some positive constants C, ε independent of q and Ω.

This is proved in [29] when $q=\infty$. The result for the whole range of q is shown in [25] by further developing the methods of [29].

We use the weighted Littlewood-Paley inequalities given in [25].

Lemma 2.2. *Let $w \in A_p$, $1<p<\infty$. Then*

$$\left\|\sum_k f_k * \Delta_k\right\|_{L^p(w)} \le C_{p,w}\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_{L^p(w)}, \tag{2.2}$$

$$\left\|\left(\sum_k |f*\Delta_k|^2\right)^{1/2}\right\|_{L^p(w)} \le C_{p,w}\|f\|_{L^p(w)}, \tag{2.3}$$

where $C_{p,w}$ is independent of $\rho \ge 2$.

To prove Theorem 1.3, we apply a Littlewood-Paley decomposition depending on ρ and obtain a decomposition of T analogous to the one used in [10] in the case of the Euclidean structure. We write

$$Tf = \sum_{k_1,k_2} U_{k_1,k_2} f, \tag{2.4}$$

where

$$U_{k_1,k_2} f = \sum_j f * \Delta_{k_1+j} * S_j K_0 * \Delta_{k_2+j}, \quad k_1, k_2 \in \mathbb{Z}.$$

This two parameter decomposition is to be compared with (4.2) based on the Fourier transform. Lemma 2.1 enables us to get the following estimates.

Lemma 2.3. *Let $\Omega \in L^q(\Sigma)$, $1<q\le\infty$, and $\rho = 2^{q'}$. Then, for any integers k_1, k_2 we have*

$$\|U_{k_1,k_2} f\|_2 \le C q' 2^{-\varepsilon|k_1|} 2^{-\varepsilon|k_2|} \|\Omega\|_q \|f\|_2$$

with some positive constants C and ε independent of q and Ω.

See [25, Section 4]. We give a proof for completeness.

Proof of Lemma 2.3. Let $L_j = S_j K_0$. We may assume that all the functions under discussion are real valued. We note that Lemma 2.1 and duality imply

$$\|f * \Delta_k * L_j\|_2 \le C q' 2^{-\varepsilon|j-k|} \|\Omega\|_q \|f\|_2.$$

If we apply this estimate and Lemma 2.1 for L_j and $\widetilde{L_j}$ along with Young's inequality and the evaluations

$$\|\Delta_{k_2+j} * \Delta_{k_2+j'}\|_1 \le C \min\left(1, \rho^{-\varepsilon(|j-j'|-c)}\right), \quad \|\Delta_k\|_1 \le C,$$

which hold for some positive constants C, c independent of q and can be proved easily, then we obtain

$$\big\|f*(\Delta_{k_1+j}*L_j)*(\Delta_{k_2+j}*\Delta_{k_2+j'})*(\tilde{L}_{j'}*\Delta_{k_1+j'})\big\|_2 \\ \le C\big(q'\|\Omega\|_q\big)^2 2^{-2\varepsilon|k_1|}\min\big(1,\rho^{-\varepsilon(|j-j'|-c)}\big)\|f\|_2.$$

Similarly, by the associative law of convolution the same quantity can be estimated as

$$\big\|f*\Delta_{k_1+j}*(L_j*\Delta_{k_2+j})*(\Delta_{k_2+j'}*\tilde{L}_{j'})*\Delta_{k_1+j'}\big\|_2 \le C\big(q'\|\Omega\|_q\big)^2 2^{-2\varepsilon|k_2|}\|f\|_2.$$

Thus, taking the geometric mean we have

$$\|f*\Delta_{k_1+j}*L_j*\Delta_{k_2+j}*\Delta_{k_2+j'}*\tilde{L}_{j'}*\Delta_{k_1+j'}\|_2 \\ \le C\big(q'\|\Omega\|_q\big)^2 2^{-\varepsilon|k_1|}2^{-\varepsilon|k_2|}\min\big(1,\rho^{-\varepsilon(|j-j'|-c)/2}\big)\|f\|_2.$$

Obviously, a similar estimate is valid for

$$\|f*\Delta_{k_2+j'}*\tilde{L}_{j'}*\Delta_{k_1+j'}*\Delta_{k_1+j}*L_j*\Delta_{k_2+j}\|_2.$$

Thus, applying the Cotlar-Knapp-Stein lemma we can reach the conclusion. □

On the other hand, since $w\in A_2$, applying (2.2) of Lemma 2.2 with $p=2$ we have

$$\|U_{k_1,k_2}f\|_{L^2(w)} \le C\bigg(\sum_j \|f*\Delta_{k_1+j}*S_jK_0\|^2_{L^2(w)}\bigg)^{1/2}. \tag{2.5}$$

We note that

$$\begin{aligned}\|g*S_jK_0\|_{L^1(w)} &\le \int_{\mathbb{H}}\bigg(\int_{\rho^j\le r(y^{-1}x)\le\rho^{j+3}}\bigg|g(y)\frac{\Omega((y^{-1}x)')}{r(y^{-1}x)^\gamma}\bigg|\,dy\bigg)w(x)\,dx\\ &\le \int_{\mathbb{H}}|g(y)|\bigg(\sum_{m=1}^{N}\rho^{-j\gamma}2^{-(m-1)\gamma}\\ &\qquad\times\int_{\rho^j2^{m-1}\le r(y^{-1}x)\le\rho^j2^m}\big|\Omega\big(\big(y^{-1}x\big)'\big)\big|w(x)\,dx\bigg)\,dy,\end{aligned}\tag{2.6}$$

where N is determined by $2^{N-1}<\rho^3\le 2^N$; therefore $N\sim\log\rho$. From this we can see that

$$\begin{aligned}\|f*\Delta_{k_1+j}*S_jK_0\|^2_{L^2(w)} &\le C(\log\rho)\|\Omega\|_1\big\||f*\Delta_{k_1+j}|^2*|S_jK_0|\big\|_{L^1(w)}\\ &\le C(\log\rho)^2\|\Omega\|_1\|f*\Delta_{k_1+j}\|^2_{L^2(M_{\tilde{\Omega}}(w))}.\end{aligned}\tag{2.7}$$

Combining the inequalities (2.5) and (2.7), we have

$$\|U_{k_1,k_2}f\|_{L^2(w)} \leq C(\log\rho)\|\Omega\|_1^{1/2}\bigg(\sum_j \|f*\Delta_{k_1+j}\|^2_{L^2(M_{\tilde\Omega}(w))}\bigg)^{1/2}$$

$$\leq C(\log\rho)\|\Omega\|_1^{1/2}\|f\|_{L^2(M_\beta M_{\tilde\Omega}(w))}, \tag{2.8}$$

where the last inequality follows from (2.3) of Lemma 2.2 with $p=2$ and the fact that $M_\beta M_{\tilde\Omega}(w)$ is in A_1 and hence in A_2, if it is finite a.e. Let $0<\theta<1$. Interpolating with change of measures between the estimates in Lemma 2.3 and (2.8), we get

$$\|U_{k_1,k_2}f\|_{L^2(w^\theta)} \leq Cq'2^{-(1-\theta)\varepsilon(|k_1|+|k_2|)}\|\Omega\|_1^{\theta/2}\|\Omega\|_q^{1-\theta}\|f\|_{L^2(M_\beta M_{\tilde\Omega}(w)^\theta)}.$$

Thus by (2.4)

$$\|Tf\|_{L^2(w^\theta)} \leq \sum \|U_{k_1,k_2}f\|_{L^2(w^\theta)} \leq Cq'\|\Omega\|_1^{\theta/2}\|\Omega\|_q^{1-\theta}\|f\|_{L^2(M_\beta M_{\tilde\Omega}(w)^\theta)} \tag{2.9}$$

for any $w\in A_2$ and $\theta\in(0,1)$. For $w\in A_2$ we choose $\theta\in(0,1)$ sufficiently close to 1 such that $w^{1/\theta}\in A_2$ and $\beta\theta>1$. Then by (2.9) with $w^{1/\theta}$ and $\beta\theta$ in place of w and β, respectively, we have

$$\|Tf\|_{L^2(w)} \leq Cq'\|\Omega\|_1^{\theta/2}\|\Omega\|_q^{1-\theta}\|f\|_{L^2(M_\beta M_{\tilde\Omega,1/\theta}(w))}.$$

Since $\|\Omega\|_1\leq S(\Sigma)^{1/q'}\|\Omega\|_q$, from this and (1.5) the conclusion of Theorem 1.3 follows.

Applying extrapolation methods using Theorem 1.3, we can show Theorem 1.5 as follows. Decompose $\Omega\in L\log L(\Sigma)$ as $\Omega=\sum_{k=1}^\infty c_k\Omega_k$, where each Ω_k satisfies (1.1), $\sup_{k\geq 1}\|\Omega_k\|_{1+1/k}\leq 1$ and $\{c_k\}$ is a sequence of non-negative real numbers such that $\sum_{k=1}^\infty kc_k<\infty$ (see [23]). By Theorem 1.3 we have

$$\|Tf\|_{L^2(w)} \leq C\sum_{k=1}^\infty kc_k\|\Omega_k\|_{1+1/k}^{1-1/(2\beta)}\|\Omega_k\|_1^{1/(2\beta)}\|f\|_{L^2(M_\beta M_{\tilde\Omega_k/\|\Omega_k\|_1,\beta}(w))}$$

$$\leq C\bigg(\sum_{k=1}^\infty kc_k\bigg)\|f\|_{L^2(M_\beta M_\beta^*(w))}.$$

Taking the infimum over the sequences $\{c_k\}$, we can get the conclusion of Theorem 1.5, since it is not difficult to show the infimum is equivalent to the norm of Ω in $L\log L$.

3. Proofs of Theorems 1.1 and 1.4

For $\eta>0$, $\Omega\in L^q(\Sigma)$ and a positive integer s, let

$$\Omega_{\eta,s}(x')=\Omega(x')\chi_{\{|\Omega|/\|\Omega\|_q>2^{\eta s}\}}(x'). \tag{3.1}$$

For a weight w, define $w_{\Omega,\eta}=\sum_{s\geq 1} M_{\tilde\Omega_{\eta,s}}(w)$. Theorem 1.1 will be deduced from the following.

Proposition 3.1. *Let $w\in A_2$, $\beta\in(1,\infty)$. Suppose that $\Omega\in L^q(\Sigma)$ for some $q>1$ and T is as in (1.4). Then, there exist positive constants C and η independent of Ω such that*

$$\|Tf\|_{L^{1,\infty}(w)}\leq C\|f\|_{L^1(W)}$$

with

$$W=\|\Omega\|_q^{1/\beta'}M_\beta M_{\tilde\Omega,\beta}(w)+\|\Omega\|_q M_\beta(w)+M(w_{\Omega,\eta}).$$

Indeed, note that

$$|\widetilde\Omega_{\eta,s}|\leq 2^{\eta s(1-q)}|\widetilde\Omega_{\eta,s}|^q\|\Omega\|_q^{1-q}$$

and hence $\|\widetilde\Omega_{\eta,s}\|_1\leq 2^{\eta s(1-q)}\|\Omega\|_q$. Thus by (1.5)

$$M_{\tilde\Omega_{\eta,s}}(w)\leq C2^{\eta s(1-q)/\beta'}\|\Omega\|_q^{1/\beta'}M_{\tilde\Omega,\beta}(w).$$

Consequently, by summation in s, $w_{\Omega,\eta}\leq C\|\Omega\|_q^{1/\beta'}M_{\tilde\Omega,\beta}(w)$, which will be used to get the conclusion of Theorem 1.1.

Assuming that f is smooth and compactly supported, we shall show that

$$w\big(\{x\in\mathbb{H}:|Tf(x)|>\lambda\}\big)\leq C\lambda^{-1}\|f\|_{L^1(W)}\quad\text{for all }\lambda>0,$$

where W is as in Proposition 3.1. To prove this we may assume that $\|\Omega\|_q=1$ and $\lambda=1$. By the Calderón-Zygmund decomposition at height 1, we have a family $\mathcal{F}$ of disjoint dyadic balls B, an associated family $\{Q_B\}_{B\in\mathcal{F}}$ of disjoint sets and functions g, b such that

$$f=g+b; \tag{3.2}$$

$$B\subset Q_B\subset B^*\quad\text{with } B^*=\varkappa B\text{ for some }\varkappa\geq 1; \tag{3.3}$$

$$c|B|\leq\int_B|f|\leq\int_{B^*}|f|\leq C|B|\quad\text{for some }c,C>0; \tag{3.4}$$

$$\|g\|_\infty\leq C,\quad \|g\|_{L^1(v)}\leq C\|f\|_{L^1(M(v))}; \tag{3.5}$$

$$b=\sum_{B\in\mathcal{F}} b_B, \quad \|b\|_{L^1(v)}\le C\|f\|_{L^1(M(v))}; \tag{3.6}$$

$$\operatorname{supp}(b_B)\subset Q_B, \quad \int b_B=0, \quad \|b_B\|_1\le C|B|, \tag{3.7}$$

where v is any weight function (see [18, Section 2], [8], [13] and [20]). We may assume without loss of generality that $\mathcal{F}$ has a finite cardinality.

We have

$$\{x\in\mathbb{H}: |Tf(x)|>1\}\subset O_1\cup O_2\cup O_3,$$

where

$$O_1=\{x\in\mathbb{H}: |Tg(x)|>1/3\},$$

$$O_2=\Big\{x\in\mathbb{H}: \sum_{s\le C_0}\Big|\sum_{B\in\mathcal{F}} b_B*S_{k(B)+s}K_0(x)\Big|>1/3\Big\},$$

$$O_3=\Big\{x\in\mathbb{H}: \sum_{s>C_0}\Big|\sum_{B\in\mathcal{F}} b_B*S_{k(B)+s}K_0(x)\Big|>1/3\Big\}.$$

Here S_j is defined as in (2.1) with $\rho=2$ and we recall that $k(B)$ denotes the radius of B. We assume that the positive constant C_0 is sufficiently large. This may imply that $b_B*S_{k(B)+s}K_0$ is supported in an annulus $\{c_1 2^{k(B)+s}\le r(x_B^{-1}x)\le c_2 2^{k(B)+s+3}\}$ (c_1, $c_2>0$), if $s>C_0$.

Applying Theorem 1.3, by Chebyshev's inequality and the first part of (3.5) we have

$$w(O_1)\le C\|Tg\|_{L^2(w)}^2\le C\|g\|_{L^2(W_1)}^2\le C\|g\|_{L^1(W_1)},$$

where $W_1=M_\beta M_{\tilde{\Omega},\beta}(w)$. Since $W_1\in A_1$ if it is finite a.e., using the second part of (3.5), we see that

$$w(O_1)\le C\|g\|_{L^1(W_1)}\le C\|f\|_{L^1(M(W_1))}\le C\|f\|_{L^1(W_1)}. \tag{3.8}$$

Note that O_2 is contained in $E=\bigcup_B CB$ for some $C>0$, since $\operatorname{supp}(S_jK_0)$ is contained in $\{2^j\le r(x)\le 2^{j+3}\}$ and $\operatorname{supp}(b_B)$ in B^* (see (3.7) and (3.3)). Therefore, by (3.4) we have

$$\begin{aligned} w(O_2)&\le w(E)\le \sum_B w(CB)\\ &\le C\sum_B |B|\inf_B M(w)\le C\sum_B\int_B |f(x)|M(w)(x)\,dx\\ &\le C\|f\|_{L^1(M(w))}, \end{aligned} \tag{3.9}$$

where $\inf_B M(w)=\inf_{x\in B} M(w)(x)$.

To prove Proposition 3.1, it remains to show $w(O_3)\le C\|f\|_{L^1(W)}$ with W described there. In the following arguments, s is a positive integer greater than C_0, where C_0 is as in the definition of O_3.

Lemma 3.2. *Let $\eta>0$ and*

$$L_s(x)=\chi_{[1,2]}\big(r(x)\big)r(x)^{-\gamma}\Omega^*_{\eta,s}(x'),\quad \Omega^*_{\eta,s}(x')=\Omega(x')\chi_{\{|\Omega|\le 2^{\eta s}\}}(x').$$

Then, if η is sufficiently small, we have

$$\left|\left\{x\in\mathbb{H}:\left|\sum_{B\in\mathcal{F}'} b_B*S_{k(B)+s}L_s(x)\right|>1\right\}\right|\le C2^{-\varepsilon s}\sum_{B\in\mathcal{F}'}|B|$$

for some constants $C,\varepsilon>0$, where $\mathcal{F}'$ is any subset of $\mathcal{F}$.

Lemma 3.3. *Let $L_s,\mathcal{F}'$ be as in Lemma 3.2. Then*

$$w\left(\left\{x\in\mathbb{H}:\left|\sum_{B\in\mathcal{F}'} b_B*S_{k(B)+s}L_s(x)\right|>1\right\}\right)\le C2^{\eta s}\sum_{B\mathcal{F}'}|B|\inf_B M(w).$$

In proving Lemma 3.2 we need a result of [29]. To recall the result, we introduce a function ψ_B defined as

$$\psi_B(x)=\psi_0\big(A_{2^{-k(B)}}(x_B^{-1}x)\big)$$

with a non-negative, smooth function ψ_0 on $\mathbb{H}$ such that $\operatorname{supp}(\psi_0)\subset\{d^{-1}\le r(x)\le d\}$, $\psi_0(x)=1$ if $2/d\le r(x)\le d/2$ for a sufficiently large positive number d and $\|\psi_0\|_\infty\le 1$. Also, let $\mathcal{B}$ be a finite family of disjoint dyadic balls such that

$$\sum_{B\in\mathcal{B}}|B|\le 1. \tag{3.10}$$

Then, the following result is shown in [29].

Proposition 3.4. *Let $\mathcal{B}$ be as in (3.10) and let b_B be a function satisfying (3.7) for each $B\in\mathcal{B}$. Suppose $1<p<2$. Then, there exist a positive number ε_0 and a set $E_s\subset\mathbb{H}$ such that*

$$|E_s|\le C2^{-\varepsilon_0 s};$$

$$\left\|\sum_{B\in\mathcal{B}}\psi_{2^sB}(b_B*S_{k(B)+s}f_B)\right\|_{L^p(\mathbb{H}\setminus E_s)}\le C2^{-\varepsilon_0 s}\left(\sum_{B\in\mathcal{B}}|B|\|f_B\|_2^2\right)^{1/2}$$

for any functions f_B in $L^2(\mathbb{H})$.

Proof of Lemma 3.2. We observe that by dilation invariance we may assume $c<\sum_{B\in\mathcal{F}'}|B|\le 1$ for some positive constant c. Then, if d is sufficiently large so that ψ_{2^sB} is identically 1 on the support of $b_B*S_{k(B)+s}L_s$, using Proposition 3.4 with $\mathcal{B}=\mathcal{F}'$ and $f_B=L_s$ for all B along with Chebyshev's inequality, we get

$$\begin{aligned}\Bigl|\Bigl\{x\in\mathbb{H}:\Bigl|\sum_{B\in\mathcal{F}'} b_B*S_{k(B)+s}L_s(x)\Bigr|>1\Bigr\}\Bigr|&\\ &\le |E_s|+\Bigl|\Bigl\{x\in\mathbb{H}\setminus E_s:\Bigl|\sum_{B\in\mathcal{F}'} b_B*S_{k(B)+s}L_s(x)\Bigr|>1\Bigr\}\Bigr|\\ &\le C2^{-\varepsilon_0 s}+C2^{-p\varepsilon_0 s}\Bigl(C2^{2\eta s}\sum_{B\in\mathcal{F}'}|B|\Bigr)^{p/2}\\ &\le C2^{-\varepsilon_0 s}+C2^{-p\varepsilon_0 s}C^{p/2}2^{p\eta s},\end{aligned}$$

which will prove Lemma 3.2, if η is small enough, since $\sum_{B\in\mathcal{F}'}|B|>c$. $\square$

Proof of Lemma 3.3. If we apply (2.6) with $\rho=2$ and $b_B*S_{k(B)+s}L_s$ in place of $g*S_jK_0$, since $|\Omega^*_{\eta,s}|\le 2^{\eta s}$, we see that

$$\begin{aligned}&\|b_B*S_{k(B)+s}L_s\|_{L^1(w)}\\ &\qquad\le C2^{\eta s}\int_{B^*}|b_B(y)|\Bigl(2^{-(k(B)+s)\gamma}\int_{r(y^{-1}x)\le 2^{k(B)+s+3}} w(x)\,dx\Bigr)\,dy.\end{aligned}\tag{3.11}$$

If $z\in B, y\in B^*$ and $r(y^{-1}x)\le 2^{k(B)+s+3}$, then

$$r(z^{-1}x)\le N_2 r(z^{-1}x_B)+N_2^2 r(x_B^{-1}y)+N_2^2 r(y^{-1}x)\le C2^{k(B)+s},$$

which implies

$$\sup_{y\in B^*}2^{-(k(B)+s)\gamma}\int_{r(y^{-1}x)\le 2^{k(B)+s+3}} w(x)\,dx\le C\inf_B M(w).$$

Using this in (3.11) and applying the last property of b_B in (3.7), we see that

$$\|b_B*S_{k(B)+s}L_s\|_{L^1(w)}\le C2^{\eta s}\inf_B M(w)\int_{B^*}|b_B(y)|\,dy\le C2^{\eta s}|B|\inf_B M(w).\tag{3.12}$$

Now Lemma 3.3 follows from (3.12) and Chebyshev's inequality. $\square$

Lemmas 3.2 and 3.3 are used to prove the following estimates.

Lemma 3.5. *For* $\alpha > 0$, *put*

$$E^s_\alpha = \left\{ x \in \mathbb{H} : \left| \sum_{B \in \mathcal{F}} b_B * S_{k(B)+s} L_s(x) \right| > \alpha \right\}.$$

Then we have

$$\int_{E^s_1} \min\big(w(x), u\big)\, dx \le C 2^{\eta s} \sum_{B \in \mathcal{F}} |B| \min\Big(u 2^{-\varepsilon s}, \inf_B M(w)\Big) \tag{3.13}$$

for all $u > 0$, *where* ε *is as in Lemma 3.2 and* w *is any weight function.*

Proof. For $u > 0$, set

$$\mathcal{F}_u = \left\{ B \in \mathcal{F} : \inf_B M(w) < u 2^{-\varepsilon s} \right\}$$

and $\mathcal{F}^c_u = \mathcal{F} \setminus \mathcal{F}_u$. Let $\alpha > 0$ and

$$E_{u,\alpha} = \left\{ x \in \mathbb{H} : \left| \sum_{B \in \mathcal{F}_u} b_B * S_{k(B)+s} L_s(x) \right| > \alpha \right\},$$

$$E'_{u,\alpha} = \left\{ x \in \mathbb{H} : \left| \sum_{B \in \mathcal{F}^c_u} b_B * S_{k(B)+s} L_s(x) \right| > \alpha \right\}.$$

Then, $E^s_\alpha \subset E_{u,\alpha/2} \cup E'_{u,\alpha/2}$, and hence

$$\begin{aligned} \int_{E^s_1} \min\big(w(x), u\big)\, dx &\le \int_{E_{u,1/2}} \min\big(w(x), u\big)\, dx + \int_{E'_{u,1/2}} \min\big(w(x), u\big)\, dx \\ &\le \int_{E_{u,1/2}} w(x)\, dx + \int_{E'_{u,1/2}} u\, dx. \end{aligned}$$

From Lemma 3.3, we easily see that

$$\int_{E_{u,1/2}} w(x)\, dx \le C 2^{\eta s} \sum_{B \in \mathcal{F}_u} |B| \inf_B M(w) = C 2^{\eta s} \sum_{B \in \mathcal{F}_u} |B| \min\Big(u 2^{-\varepsilon s}, \inf_B M(w)\Big).$$

Also, Lemma 3.2 implies that

$$\int_{E'_{u,1/2}} u\, dx \le C u 2^{-\varepsilon s} \sum_{B \in \mathcal{F}^c_u} |B| = C \sum_{B \in \mathcal{F}^c_u} |B| \min\Big(u 2^{-\varepsilon s}, \inf_B M(w)\Big).$$

Combining these estimates, we get the conclusion of Lemma 3.5. $\square$

Multiply both sides of the inequality in (3.13) by $u^{-1+\theta}$ ($\theta\in(0,1)$), and integrate them over $(0,\infty)$ with respect to the measure du/u. Interchanging the order of integration on the left hand side, performing termwise integration on the right hand side, and using the formula

$$\int_0^\infty \min(\Xi,u)u^{-1+\theta}\frac{du}{u}=c_\theta\Xi^\theta \quad (\Xi>0),$$

we then get

$$\begin{aligned}\int_{E_1^s} w(x)^\theta\,dx &\le C2^{\eta s}\sum_{B\in\mathcal{F}}|B|2^{-(1-\theta)\varepsilon s}\inf_B M(w)^\theta\\ &\le C2^{\eta s}2^{-(1-\theta)\varepsilon s}\sum_{B\in\mathcal{F}}\inf_B M(w)^\theta\int_B\big|f(x)\big|\,dx\\ &\le C2^{\eta s}2^{-(1-\theta)\varepsilon s}\int\big|f(x)\big|M(w)(x)^\theta\,dx,\end{aligned}$$

where the second inequality follows from (3.4). Substituting $w^{1/\theta}$ for w and reducing η, if necessary, we get

$$w\big(E_1^s\big)\le C2^{-(1-\theta)\varepsilon s/2}\int\big|f(x)\big|M_{1/\theta}(w)(x)\,dx. \tag{3.14}$$

Similarly, we have for any $\theta\in(0,1)$

$$w\big(E^s_{c_\delta 2^{-\delta s}}\big)\le C2^{-\tau s}\|f\|_{L^1(M_{1/\theta}(w))}, \tag{3.15}$$

where δ and τ are sufficiently small positive constants depending on θ and c_δ is chosen so that $\sum_{s>C_0}c_\delta 2^{-\delta s}=1$. This can be achieved by applying the proof of (3.14) to a version of E_1^s, where $\Omega^*_{\eta,s}$ is replaced by $c_\delta^{-1}2^{\delta s}\Omega^*_{\eta,s}$.

We note that

$$\Big\{x\in\mathbb{H}:\sum_{s>C_0}\Big|\sum_{B\in\mathcal{F}}b_B*S_{k(B)+s}L_s(x)\Big|>1\Big\}\subset\bigcup_{s>C_0}E^s_{c_\delta 2^{-\delta s}},$$

since $\sum_{s>C_0}c_\delta 2^{-\delta s}=1$. Therefore, (3.15) implies that

$$\begin{aligned}w\Big(\Big\{x\in\mathbb{H}:\sum_{s>C_0}\Big|\sum_{B\in\mathcal{F}}b_B*S_{k(B)+s}L_s(x)\Big|>1\Big\}\Big)&\\ \le\sum_{s>C_0}w\big(E^s_{c_\delta 2^{-\delta s}}\big)&\\ \le C\sum_{s>C_0}2^{-\tau s}\|f\|_{L^1(M_{1/\theta}(w))}\le C\|f\|_{L^1(M_{1/\theta}(w))}.&\end{aligned} \tag{3.16}$$

Next, let

$$R_s(x) = \chi_{[1,2]}\big(r(x)\big)r(x)^{-\gamma}\Omega_{\eta,s}\big(x'\big),$$

where $\Omega_{\eta,s}$ is as in (3.1). Arguing similarly to the proof of (3.12) in view of (2.6) (with $\rho=2$), we have

$$\|b_B * S_{k(B)+s}R_s\|_{L^1(w)} \le C\int_{Q_B}\big|b_B(y)\big|M_{\tilde{\Omega}_{\eta,s}}(w)(y)\,dy.$$

Combined with (3.6), this implies

$$\begin{aligned}\sum_{s>C_0}\sum_{B\in\mathfrak{F}}\|b_B * S_{k(B)+s}R_s\|_{L^1(w)} &\le C\int\big|b(y)\big|\sum_{s>C_0}M_{\tilde{\Omega}_{\eta,s}}(w)(y)\,dy \\ &\le C\|f\|_{L^1(M(w_{\Omega,\eta}))}.\end{aligned} \tag{3.17}$$

Noticing $K_0=L_s+R_s$ for all s, by (3.16) and (3.17) with Chebyshev's inequality, we have

$$w(O_3) \le C\|f\|_{L^1(M_\beta(w))} + C\|f\|_{L^1(M(w_{\Omega,\eta}))} \tag{3.18}$$

for any $\beta\in(1,\infty)$. Thus we have $w(O_3)\le C\|f\|_{L^1(W)}$ as claimed, which combined with (3.8), (3.9) completes the proof of Proposition 3.1.

We now proceed to the proof of Theorem 1.4. We may assume that $\|\Omega\|_{L\log L}=1$. Furthermore, arguing as in the proof of Proposition 3.1, the proof of Theorem 1.4 can be also reduced to the estimates of $w(O_i)$, $i=1,2,3$, where each O_i is defined similarly. By Theorem 1.5 we have

$$w(O_1) \le C\|f\|_{L^1(M_\beta M_\beta^*(w))}. \tag{3.19}$$

Also

$$w(O_2) \le C\|f\|_{L^1(M(w))}. \tag{3.20}$$

Further, similarly to the proof of (3.18), we obtain

$$w(O_3) \le C\int_{\mathbb{H}}\big|f(x)\big|\big[M_\beta(w)(x) + M(w_{\Omega,\eta})(x)\big]\,dx, \tag{3.21}$$

where $w_{\Omega,\eta}$ is defined as above from $\Omega_{\eta,s}$ in (3.1) with $\|\Omega\|_q$ replaced by $\|\Omega\|_{L\log L}$ and η is sufficiently small. We observe that

$$w_{\Omega,\eta} \le CM^*(w)\sum_{s\ge 1}\|\widetilde{\Omega}_{\eta,s}\|_1 \le CM^*(w), \tag{3.22}$$

since we assume that $\|\Omega\|_{L\log L}=1$. Using this in (3.21) we have

$$w(O_3) \le C\|f\|_{L^1(M_\beta(w))} + C\|f\|_{L^1(MM^*(w))}. \tag{3.23}$$

The conclusion follows from (3.19), (3.20) and (3.23).

4. Singular integrals on $\mathbb{R}^2$ with the generalized homogeneity

In this section we consider singular integrals on the Euclid space $\mathbb{R}^n$, with the usual addition, associated with the non-isotropic dilations $A_t=\exp((\log t)P)$, where P is not restricted to diagonal matrices. Similarly to the case of $\mathbb{R}^n$ considered as a homogeneous group with a diagonal matrix P, we can also define the maximal operators M, M_s, M_Ω, $M_{\Omega,s}$, M^*, M_s^* in this context with a norm function r related to $\{A_t\}$. Also, the Muckenhoupt class $A_p(\mathbb{R}^n)$ is defined as in Section 1.

We have results on $\mathbb{R}^2$ analogous to Theorems 1.1 and 1.4 in Section 1.

Theorem 4.1. *Suppose that $w\in A_2(\mathbb{R}^2)$, $\beta\in(1,\infty)$ and $\Omega\in L^q(\Sigma)$ for some q, $1<q\le\infty$. Let T be as in (1.2) with $n=2$. Then*

$$\|Tf\|_{L^{1,\infty}(w)}\le C\int_{\mathbb{R}^2}|f(x)|\big(\|\Omega\|_q^{1/\beta'}M_\beta M_{\tilde\Omega,\beta}(w)(x)+\|\Omega\|_qM_\beta(w)(x)\big)\,dx$$

for some positive constant C independent of Ω.

Theorem 4.2. *Let $\Omega\in L\log L(\Sigma)$ and let T be as in (1.2) with $n=2$. If $w\in A_2(\mathbb{R}^2)$ and $\beta\in(1,\infty)$, then*

$$\|Tf\|_{L^{1,\infty}(w)}\le C\|\Omega\|_{L\log L}\|f\|_{L^1(M_\beta M_\beta^*(w))}$$

with a constant C independent of Ω.

We also have an analogue of Corollary 1.2 in the present context.

To describe results on L^2 estimates, we introduce a kernel L on $\mathbb{R}^n$ defined by

$$L(y)=h\big(r(y)\big)K(y),\qquad K(y)=r(y)^{-\gamma}\Omega\big(y'\big),$$

where h is a bounded function on $\mathbb{R}_+=(0,\infty)$ and K is a homogeneous kernel as in (1.2). We consider a singular integral operator S on $\mathbb{R}^n$ defined by

$$Sf(x)=\text{p.v.}\int_{\mathbb{R}^n}f(x-y)L(y)\,dy. \tag{4.1}$$

Then, we have the following results, which are stated more generally than needed for the proofs of Theorems 4.1 and 4.2.

Theorem 4.3. *Let $\Omega\in L^q(\Sigma)$, $1<q\le\infty$, $h\in L^\infty(\mathbb{R}_+)$ and let S be as in (4.1). Suppose that $w\in A_2(\mathbb{R}^n)$ and $\beta\in(1,\infty)$. Then*

$$\|Sf\|_{L^2(w)}\le Cq'\|h\|_\infty\|\Omega\|_q^{1-1/(2\beta)}\left(\int_{\mathbb{R}^n}|f(x)|^2M_\beta M_{\tilde\Omega,\beta}(w)(x)\,dx\right)^{1/2}$$

for a positive constant C independent of q, Ω and h.

Theorem 4.4. *Suppose that* $w \in A_2(\mathbb{R}^n)$ *and* $\beta \in (1, \infty)$. *Let* S *be as in* (4.1) *with* $\Omega \in L \log L(\Sigma)$ *and* $h \in L^\infty(\mathbb{R}_+)$. *Then, there is a constant* C *independent of* Ω *and* h *such that*

$$\|Sf\|_{L^2(w)} \leq C\|h\|_\infty \|\Omega\|_{L \log L} \|f\|_{L^2(M_\beta M_\beta^*(w))}.$$

A result similar to Theorem 4.3 can be found in [17]. Theorem 4.4 can be derived from Theorem 4.3 by an extrapolation argument similar to the one that proves Theorem 1.5 from Theorem 1.3.

We give a proof of Theorem 4.3 using Fourier transform estimates, which differs from the proof of Theorem 1.3 in Section 2 in that it allows the presence of the function h in the kernel L.

Proof of Theorem 4.3. We apply methods of [10]. Let $E_j = \{x \in \mathbb{R}^n : 2^j < r(x) \leq 2^{j+1}\}$, $j \in \mathbb{Z}$. Set

$$L_j(x) = L(x)\chi_{E_j}(x).$$

Let $A_t^* = \exp((\log t)P^*)$, where B^* denote the adjoint of a matrix B. A norm function $s(\xi)$ will be defined from $\{A_t^*\}$ in the same way as $r(x)$ is defined from $\{A_t\}$. Let φ be a non-negative function in $C^\infty(\mathbb{R}_+)$ such that

$$\operatorname{supp}(\varphi) \subset [2^{-1}, 2], \qquad \sum_{k=-\infty}^{\infty} \varphi(2^k t)^2 = 1, \quad t > 0.$$

Define the operator D_k by

$$(D_k f)\hat{\ }(\xi) = \varphi(2^k s(\xi))\hat{f}(\xi),$$

where $\hat{f}(\xi) = \int f(x) e^{-2\pi i \langle x, \xi \rangle}\, dx$ is the Fourier transform. We write

$$Sf = \sum_{k=-\infty}^{\infty} U_k f,$$

where

$$U_k f = \sum_{j=-\infty}^{\infty} D_{j+k}(L_j * D_{j+k} f). \tag{4.2}$$

Then, it is known that

$$\|U_k f\|_2 \leq C\|h\|_\infty \|\Omega\|_q 2^{-\varepsilon|k|/q'} \|f\|_2 \tag{4.3}$$

for some $\varepsilon>0$ (see [10], [22]). Since we also have weighted Littlewood-Paley inequalities with operators $\{D_k\}$ analogous to Lemma 2.2, arguing similarly to the proof of (2.8), we have

$$\|U_k f\|_{L^2(w)} \le C\|h\|_\infty \|\Omega\|_1^{1/2} \|f\|_{L^2(M_\beta M_{\tilde{\Omega}}(w))}. \tag{4.4}$$

Applying interpolation with change of measures between (4.3) and (4.4) and arguing similarly to the proof of Theorem 1.3, we can obtain the conclusion. □

To prove Theorem 4.1, we show a version of Proposition 3.1 relevant to Theorem 4.1 by using a Calderón-Zygmund decomposition $f=g+b$ analogous to (3.2) and arguing similarly to the proof of Proposition 3.1. To treat Tg we apply Theorem 4.3 with $n=2$ and h identically 1. A version of the set O_2 of Section 3 can be handled similarly. Also, to prove an analogue of (3.16), we apply Proposition 2.1 of [24], which is analogous to Proposition 3.4, in the same way as Proposition 3.4 is used in proving (3.16), along with an interpolation argument with change of measures similar to the one used in the proof of (3.16). Finally, it is obvious that an analogue of (3.17) can be shown also in the present context. Combining results, we complete the proof of the result analogous to Proposition 3.1, which implies Theorem 4.1.

Also, we can prove Theorem 4.2 arguing similarly to the proof of Theorem 4.1 with suitable modifications using Theorem 4.4 with $n=2$ and $h=1$ and with an observation similar to (3.22), as we prove Theorem 1.4 from the procedure of the proof of Theorem 1.1 with suitable adjustments in Section 3.

Remark 4.5. To prove Theorem 4.3 we applied a Littlewood-Paley decomposition adapted to a fixed lacunary sequence. On the other hand, the Littlewood-Paley decomposition used in the proof of Theorem 1.3 is adapted to a lacunary sequence depending on q. This is needed to get the required estimates of Theorem 1.3 through the two parameter decomposition in (2.4).

5. Weighted weak type estimates with Ω in $L \log L$

In this section, we review Theorem 1.4 for the case of $\mathbb{R}^n$ with the usual addition, the isotropic dilation and the Euclidean norm.

Let $A_1(\mathbb{R}_+)$ be the A_1 class on $\mathbb{R}_+$. We recall a weight class introduced by [9]. Define

$$\tilde{A}_1(\mathbb{R}^n) = \{w(x)=v(|x|) : v \text{ is in } A_1(\mathbb{R}_+) \text{ and is decreasing or } v^2 \in A_1(\mathbb{R}_+)\}.$$

For $\theta\in\Sigma=S^{n-1}$, let

$$M_\theta(f)(x)=\sup_{t>0} t^{-1}\int_0^t \left|f(x+t\theta)\right| dt.$$

Then, it was noted in [9] that $M_\theta(w)\le Cw$ uniformly in $\theta\in S^{n-1}$ if $w\in\tilde{A}_1(\mathbb{R}^n)$, by the arguments based on results of [4]. Thus it follows that

$$\begin{aligned} M_F(w)(x) &= \sup_{t>0} t^{-n}\int_{|y|<t} w(x-y)\left|F(y')\right| dy \\ &\le C\int_{S^{n-1}} M_\theta(w)(x)\left|F(\theta)\right| d\sigma(\theta) \le C\|F\|_1 w(x), \end{aligned}$$

and hence $M^*(w)\le Cw$ whenever $w\in\tilde{A}_1(\mathbb{R}^n)$. Therefore, since it is easy to see that $w^\tau\in\tilde{A}_1(\mathbb{R}^n)$ for some $\tau>1$ if $w\in\tilde{A}_1(\mathbb{R}^n)$, Theorem 1.4 implies the following.

Corollary 5.1. *Let*

$$Tf(x)=\text{p.v.}\int_{\mathbb{R}^n} f(x-y)\frac{\Omega(y')}{|y|^n}\,dy$$

with $\Omega\in L\log L(S^{n-1})$ *satisfying* $\int_{S^{n-1}}\Omega\,d\sigma=0$. *Suppose that* $w\in\tilde{A}_1(\mathbb{R}^n)$. *Then*

$$\|Tf\|_{L^{1,\infty}(w)}\le C\|\Omega\|_{L\log L}\|f\|_{L^1(w)},$$

where C *is a constant independent of* Ω.

This is foreseen in [9, p. 879, (e)]. If $w_\alpha(x)=|x|^\alpha$, then $w_\alpha\in\tilde{A}_1(\mathbb{R}^n)$ for $-1<\alpha\le 0$. So, T is bounded from $L^1(w_\alpha)$ to $L^{1,\infty}(w_\alpha)$ if $-1<\alpha\le 0$ (the range of α is sharp). This result on the power weights follows from [27] combined with [26]; see [16] for the two dimensional case. Our treatment of Theorem 1.4 provides a different proof.

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