Ark. Mat., **52** (2014), 1–10
DOI: 10.1007/s11512-013-0187-y

Artinianness of local cohomology modules

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Abstract. Some uniform theorems on the artinianness of certain local cohomology modules are proven in a general situation. They generalize and imply previous results about the artinianness of some special local cohomology modules in the graded case.

1. Introduction

Throughout, A is a commutative noetherian ring. As a general reference to homological and commutative algebra we use [7] and [11]. The main problems in the study of local cohomology modules are to determine when they are artinian, finite, zero and non-zero and when their sets of associated primes are finite. Recently some results have been proved about the artinianness of graded local cohomology modules in [4], [5], [14] and [15].

In those papers the local cohomology modules $\mathrm{H}^i_{R_+}(M)$ of a finitely generated graded module $M=\bigoplus_{n\in\mathbb{Z}} M_n$ over a noetherian homogeneous ring $R=\bigoplus_{n=0}^{\infty} R_n$ are studied. Here R_+ is the irrelevant homogeneous ideal and the base ring $(R_0, \mathfrak{m}_0)$ is assumed to be local.

We prove, with uniform proofs, some general results about artinianness of local cohomology modules in the context of an arbitrary noetherian ring A. They have the special cases in the above references as immediate consequences.

In [15, Theorem 2.4] Sazeedeh showed that if n is a non-negative integer such that $\mathrm{H}^i_{R_+}(M)$ is artinian for all $i>n$, then $\mathrm{H}^n_{R_+}(M)/\mathfrak{m}_0\mathrm{H}^n_{R_+}(M)$ is an artinian R-module.

We generalize this in two ways. First we replace the graded ring R by an arbitrary noetherian ring A and consider ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $A/(\mathfrak{a}+\mathfrak{b})$ is artinian, in Corollary 2.3. We get this result as a corollary of a general theorem, Theorem 2.1, in which we consider arbitrary Serre subcategories of the category of A-modules.

A Serre subcategory of the category of A-modules is a full subcategory closed under taking submodules, quotient modules and extensions. An example is given by the class of artinian A-modules. A useful method to prove that a certain module belongs to such a Serre subcategory is to apply the homological principle of [13, Corollary 3.2]:

• Suppose we are given a (strongly) connected sequence $\{T^i\}_{i=0}^{\infty}$ of functors between the abelian categories $\mathcal{A}$ and $\mathcal{B}$ and a Serre subcategory $\mathcal{S}$ of $\mathcal{B}$ (i.e., a full subcategory such that whenever $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{B}$, then X and Z both belong to $\mathcal{S}$ if and only if Y belongs to $\mathcal{S}$), then for $f\colon X \to Y$, a morphism in $\mathcal{A}$ and $i \geq 0$, if $T^i \operatorname{Coker} f$ and $T^{i+1} \operatorname{Ker} f$ are in $\mathcal{S}$, then also $\operatorname{Coker} T^i f$ and $\operatorname{Ker} T^{i+1} f$ are in $\mathcal{S}$.

Let A be a noetherian ring and $\mathfrak{a}$ be an ideal of A. An A-module M is called $\mathfrak{a}$-*cofinite* if $\operatorname{Supp}_A(M) \subset \mathrm{V}(\mathfrak{a})$ and $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ are finite A-modules (i.e., finitely generated A-modules) for all i. This notion was introduced by Hartshorne in [9]. For more information about cofiniteness with respect to an ideal, see [10], [8] and [13].

2. Main results

Theorem 2.1. *Let $\mathcal{S}$ be a Serre subcategory of the category of A-modules. Let M be a finite A module, $\mathfrak{a}$ be an ideal of A and n be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ belong to $\mathcal{S}$ for all $i > n$. If $\mathfrak{b}$ is an ideal of A such that $\mathrm{H}^n_{\mathfrak{a}}(M/\mathfrak{b}M)$ belongs to $\mathcal{S}$, then the module $\mathrm{H}^n_{\mathfrak{a}}(M)/\mathfrak{b}\,\mathrm{H}^n_{\mathfrak{a}}(M)$ belongs to $\mathcal{S}$.*

Proof. Let $\mathfrak{b} = (b_1, \ldots, b_r)$ and consider the map $f\colon M^r \to M$ that is defined by $f(x_1, \ldots, x_r) = \sum_{i=1}^{r} b_i x_i$. Then $\operatorname{Im} f = \mathfrak{b}M$ and $\operatorname{Coker} f = M/\mathfrak{b}M$. Since $\mathrm{H}^i_{\mathfrak{a}}(M)$ is in $\mathcal{S}$ for all $i > n$ and $\operatorname{Supp}_A \operatorname{Ker} f \subset \operatorname{Supp}_A M$ it follows from [1, Theorem 3.1] that $\mathrm{H}^{n+1}_{\mathfrak{a}}(\operatorname{Ker} f)$ is also in $\mathcal{S}$. By hypothesis $\mathrm{H}^n_{\mathfrak{a}}(\operatorname{Coker} f)$ belongs to $\mathcal{S}$. Hence by [13, Corollary 3.2] $\operatorname{Coker} \mathrm{H}^n_{\mathfrak{a}}(f)$, which equals $\mathrm{H}^n_{\mathfrak{a}}(M)/\mathfrak{b}\mathrm{H}^n_{\mathfrak{a}}(M)$, is in $\mathcal{S}$. □

Corollary 2.2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of A. Let M be a finite A-module and n be a non-negative integer. If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian for $i > n$ and $\mathrm{H}^n_{\mathfrak{a}}(M/\mathfrak{b}M)$ is artinian, then $\mathrm{H}^n_{\mathfrak{a}}(M)/\mathfrak{b}\,\mathrm{H}^n_{\mathfrak{a}}(M)$ is artinian.*

Proof. In Theorem 2.1 take $\mathcal{S}$ as the category of artinian A-modules. □

The next corollary generalizes [15, Theorem 2.4], as said in the introduction.

Corollary 2.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of A such that $A/(\mathfrak{a}+\mathfrak{b})$ is artinian and let M be a finite A-module and n be a non-negative integer. If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian for $i > n$, then $\mathrm{H}^n_{\mathfrak{a}}(M)/\mathfrak{b}\,\mathrm{H}^n_{\mathfrak{a}}(M)$ is artinian.*

Proof. Note that $\mathrm{H}^n_{\mathfrak{a}}(M/\mathfrak{b}M)\cong\mathrm{H}^n_{\mathfrak{a}+\mathfrak{b}}(M/\mathfrak{b}M)$, which is artinian. $\square$

Corollary 2.4. *Let M be a finite A-module such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is artinian for $i>n$, where n is a positive integer, then $\mathrm{H}^n_{\mathfrak{a}}(M)/\mathfrak{a}\mathrm{H}^n_{\mathfrak{a}}(M)$ is artinian.*

Proof. Note that $\mathrm{H}^n_{\mathfrak{a}}(M/\mathfrak{a}M)=0$ when $n\geq 1$. $\square$

Remark 2.5. In Corollary 2.4 we must assume that $n\geq 1$. Take any ideal $\mathfrak{a}$ in a ring A such that $A/\mathfrak{a}$ is not artinian and let $M=A/\mathfrak{a}$. Then $\mathrm{H}^i_{\mathfrak{a}}(M)=0$ for $i\geq 1$, and $\Gamma_{\mathfrak{a}}(M)=M$. On the other hand $M/\mathfrak{a}M\cong M$. Thus $\mathrm{H}^0_{\mathfrak{a}}(M)/\mathfrak{a}\,\mathrm{H}^0_{\mathfrak{a}}(M)$ is not artinian.

Corollary 2.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of A and let M be a finite A-module and n be a non-negative integer. If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite artinian (resp. has finite support) for $i>n$, and $\mathrm{H}^n_{\mathfrak{a}}(M/\mathfrak{b}M)$ is $\mathfrak{a}$-cofinite artinian (resp. has finite support) then $\mathrm{H}^n_{\mathfrak{a}}(M)/\mathfrak{b}\,\mathrm{H}^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite artinian (resp. has finite support). In particular, if $n\geq 1$ then $\mathrm{H}^n_{\mathfrak{a}}(M)/\mathfrak{a}\,\mathrm{H}^n_{\mathfrak{a}}(M)$ has finite length (resp. has finite support).*

Proof. The category of $\mathfrak{a}$-cofinite artinian modules and the category of modules with finite support are Serre subcategories of the category of A-modules. $\square$

Corollary 2.7. *If $c=\mathrm{cd}(\mathfrak{a},M)>0$, then $\mathrm{H}^c_{\mathfrak{a}}(M)=\mathfrak{a}\,\mathrm{H}^c_{\mathfrak{a}}(M)$.*

Proof. Take $\mathcal{S}$ to consist of the zero module. $\square$

As a corollary we recover Yoshida's theorem [16, Proposition 3.1].

Corollary 2.8. *If $c=\mathrm{cd}(\mathfrak{a},M)>0$, then $\mathrm{H}^c_{\mathfrak{a}}(M)$ is not finite.*

Proof. From Corollary 2.7, we get that $\mathrm{H}^c_{\mathfrak{a}}(M)=\mathfrak{a}^n\mathrm{H}^c_{\mathfrak{a}}(M)$ for all n. But if the module $\mathrm{H}^c_{\mathfrak{a}}(M)$ is finite, then it is annihilated by $\mathfrak{a}^n$ for some n and we get the contradiction $\mathrm{H}^c_{\mathfrak{a}}(M)=0$. $\square$

Proposition 2.9. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A with $A/(\mathfrak{a}+\mathfrak{b})$ artinian. If the module M is $\mathfrak{a}$-cofinite, then $\mathrm{H}^i_{\mathfrak{b}}(M)$ is artinian for all i.*

Proof. $\mathrm{Supp}_A(\mathrm{H}^i_{\mathfrak{b}}(M))$ is contained in $\mathrm{V}(\mathfrak{a}+\mathfrak{b})$, which consists of finitely many maximal ideals $\mathfrak{m}$. Since M is $\mathfrak{a}$-cofinite, $\mathrm{Ext}^i_A(A/\mathfrak{m},M)$ is finite for each such $\mathfrak{m}$. From [13, Theorem 5.5, (i)$\Leftrightarrow$(iv)] we get that the module $\mathrm{H}^i_{\mathfrak{b}}(M)$ is artinian for all i. $\square$

With the notation as in the introduction, the modules $\mathrm{H}^i_{\mathfrak{m}_0 R}(\mathrm{H}^j_{R_+}(M))$ were shown to be artinian for all i and j when $\dim R_0{=}1$ in [4, Theorem 2.5(b)], or when the ideal R_+ is principal in [14, Proposition 2.6]. This is obtained as a special case of the following general result.

Corollary 2.10. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A such that $A/(\mathfrak{a}{+}\mathfrak{b})$ is artinian. Let M be a finite module such that $\dim M/\mathfrak{a}M{\le}1$ or $\operatorname{cd}\mathfrak{a}{=}1$. Then $\mathrm{H}^i_{\mathfrak{b}}(\mathrm{H}^j_{\mathfrak{a}}(M))$ is an artinian module for all i and j.*

Proof. The modules $\mathrm{H}^j_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite by [3, Corollary 2.7] resp. [13, Corollary 3.14]. $\square$

In [5, Proposition 5.10] Brodmann, Rohrer and Sazeedeh (with the notation as in the introduction) showed that $\mathrm{H}^1_{\mathfrak{m}_0 R}(\mathrm{H}^i_{R_+}(M))$ is artinian for all i if $\dim R_0{\le}2$. They also give an example [5, Example 5.11] where this does not hold if $\dim R_0{>}2$ and in [4, Example 4.2] there is an example with $\dim R_0{=}2$ but where $\Gamma_{\mathfrak{m}_0 R}(\mathrm{H}^2_{R_+}(M))$ not artinian.

In the next theorem we generalize this result.

Theorem 2.11. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A such that $A/(\mathfrak{a}{+}\mathfrak{b})$ is artinian. Let M be a finite module such that $\dim M/\mathfrak{a}M{\le}2$. Then $\mathrm{H}^1_{\mathfrak{b}}(\mathrm{H}^i_{\mathfrak{a}}(M))$ is an artinian module for all i.*

Proof. Take $x{\in}\mathfrak{b}$ outside all prime ideals $\mathfrak{p}{\supset}\mathfrak{a}{+}\operatorname{Ann}(M)$, such that $\dim A/\mathfrak{p}{=}2$. Then $\dim M/(\mathfrak{a}{+}xA)M{\le}1$.

Consider for a fixed i the exact sequence [6, Proposition 8.1.2]

$$\mathrm{H}^i_{\mathfrak{a}+xA}(M) \xrightarrow{g} \mathrm{H}^i_{\mathfrak{a}}(M) \xrightarrow{f} \mathrm{H}^i_{\mathfrak{a}}(M)_x \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{a}+xA}(M).$$

Put $K{=}\operatorname{Ker} f$ and $L{=}\operatorname{Ker} g$. From the short exact sequence

$$0 \longrightarrow L \longrightarrow \mathrm{H}^i_{\mathfrak{a}+xA}(M) \longrightarrow K \longrightarrow 0$$

we get the exact sequence

$$\mathrm{H}^1_{\mathfrak{b}}(\mathrm{H}^i_{\mathfrak{a}+xA}(M)) \longrightarrow \mathrm{H}^1_{\mathfrak{b}}(K) \longrightarrow \mathrm{H}^2_{\mathfrak{b}}(L).$$

Since

$$\operatorname{Supp}_A L \subset \mathrm{V}(\mathfrak{a}{+}xA{+}\operatorname{Ann} M)$$

and $\dim(A/(\mathfrak{a}{+}xA{+}\operatorname{Ann} M)){\le}1$, we get $\mathrm{H}^2_{\mathfrak{b}}(L){=}0$. Hence $\mathrm{H}^1_{\mathfrak{b}}(K)$ is a homomorphic image of $\mathrm{H}^1_{\mathfrak{b}}(\mathrm{H}^i_{\mathfrak{a}+xA}(M))$, which is artinian by Corollary 2.10. Moreover,

$\Gamma_{\mathfrak{b}}(\operatorname{Coker} f)$ is isomorphic to a submodule of $\Gamma_{\mathfrak{b}}(H^{i+1}_{\mathfrak{a}+xA}(M))$, which is artinian by Corollary 2.10.

Since we now have shown that $\mathrm{H}^1_{\mathfrak{b}}(\operatorname{Ker} f)$ and $\Gamma_{\mathfrak{b}}(\operatorname{Coker} f)$ are artinian, we can apply [13, Lemma 3.1] in order to deduce that the kernel of $\mathrm{H}^1_{\mathfrak{b}}(f)$ is artinian. However, $x \in \mathfrak{b}$, so the codomain of $\mathrm{H}^1_{\mathfrak{b}}(f)\colon \mathrm{H}^1_{\mathfrak{b}}(\mathrm{H}^i_{\mathfrak{a}}(M)) \to \mathrm{H}^1_{\mathfrak{b}}(\mathrm{H}^i_{\mathfrak{a}}(M)_x)$ is 0. Hence $\operatorname{Ker} \mathrm{H}^1_{\mathfrak{b}}(f) = \mathrm{H}^1_{\mathfrak{b}}(\mathrm{H}^i_{\mathfrak{a}}(M))$, which therefore is artinian. $\square$

The following lemma is a generalization of [12, Proposition 1.8].

Lemma 2.12. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A such that $A/(\mathfrak{a}+\mathfrak{b})$ is artinian. If M is a module such that $\operatorname{Supp}_A(M) \subset \mathrm{V}(\mathfrak{a})$ and $0 :_M \mathfrak{a}$ is finite, then $\Gamma_{\mathfrak{b}}(M)$ is $\mathfrak{a}$-cofinite artinian.*

Proof. The module $0 :_{\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}$ is contained in $0 :_M \mathfrak{a}$ and is therefore finite. But the support of $0 :_{\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}$ is contained in $\mathrm{V}(\mathfrak{a}+\mathfrak{b})$, and therefore $0 :_{\Gamma_{\mathfrak{b}}(M)} \mathfrak{a}$ has finite length. Moreover, $\operatorname{Supp}_A(\Gamma_{\mathfrak{b}}(M)) \subset \operatorname{Supp}_A(M) \subset \mathrm{V}(\mathfrak{a})$. We conclude that $\Gamma_{\mathfrak{b}}(M)$ is $\mathfrak{a}$-cofinite artinian, by [13, Proposition 4.1]. $\square$

Corollary 2.13. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A such that $A/(\mathfrak{a}+\mathfrak{b})$ is artinian. If M is a module such that*

$$\operatorname{Ext}^n_A(A/\mathfrak{a}, M) \quad \text{and} \quad \operatorname{Ext}^{n+1-j}_A(A/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$$

are finite for all $j<n$, then $\Gamma_{\mathfrak{b}}(\mathrm{H}^n_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-cofinite artinian.

Proof. Note that by [2, Theorem 4.1(b)], $\operatorname{Hom}_A(A/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M))$ is finite. Hence by Lemma 2.12, $\Gamma_{\mathfrak{b}}(\mathrm{H}^n_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-cofinite artinian. $\square$

Next we generalize [15, Theorem 2.9], where it was shown that (with the notation as in the introduction) if $\mathrm{H}^i_{R_+}(M)$ is R_+-cofinite for $i<s$, then $\Gamma_{\mathfrak{m}_0 R}(\mathrm{H}^i_{R_+}(M))$ is artinian for $i<s$.

Corollary 2.14. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of A such that $A/(\mathfrak{a}+\mathfrak{b})$ is artinian. If M is a finite module such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for $i<n$, then $\Gamma_{\mathfrak{b}}(\mathrm{H}^i_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-cofinite artinian for all $i \leq n$.*

Proof. The hypothesis in Corollary 2.13 is satisfied. $\square$

The next theorem is a generalization of [14, Theorem 2.2], where it is shown (with the notation as in the introduction) that $\mathrm{H}^1_{\mathfrak{m}_0 R}(\mathrm{H}^1_{R_+}(M))$ is artinian.

Theorem 2.15. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A such that $A/(\mathfrak{a}+\mathfrak{b})$ is artinian. For each finite module M, the modules $\Gamma_{\mathfrak{b}}(\mathrm{H}^1_{\mathfrak{a}}(M))$ and $\mathrm{H}^1_{\mathfrak{b}}(\mathrm{H}^1_{\mathfrak{a}}(M))$ are artinian.*

Proof. Corollary 2.14 with $n{=}1$ implies that $\Gamma_{\mathfrak{b}}(\mathrm{H}^1_{\mathfrak{a}}(M))$ is artinian. We may assume that $\Gamma_{\mathfrak{a}}(M){=}0$, so there is an M-regular element x in $\mathfrak{a}$. From the exact sequence $0\to M\xrightarrow{x} M\to M/xM\to 0$, we get the exact sequence

$$0\longrightarrow \Gamma_{\mathfrak{a}}(M/xM)\longrightarrow \mathrm{H}^1_{\mathfrak{a}}(M)\xrightarrow{\ f\ }\mathrm{H}^1_{\mathfrak{a}}(M)\longrightarrow \mathrm{H}^1_{\mathfrak{a}}(M/xM),$$

where the map f is defined as multiplication with x on $\mathrm{H}^1_{\mathfrak{a}}(M)$.

We get

$$\mathrm{H}^1_{\mathfrak{b}}(\operatorname{Ker} f)\cong \mathrm{H}^1_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(M/xM))\cong \mathrm{H}^1_{\mathfrak{a}+\mathfrak{b}}(\Gamma_{\mathfrak{a}}(M/xM)),$$

which is artinian. We also get the exact sequence

$$0\longrightarrow \Gamma_{\mathfrak{b}}(\operatorname{Coker} f)\longrightarrow \Gamma_{\mathfrak{b}}(\mathrm{H}^1_{\mathfrak{a}}(M/xM)).$$

Since $\Gamma_{\mathfrak{b}}(\mathrm{H}^1_{\mathfrak{a}}(M/xM))$ is artinian by Corollary 2.14, $\Gamma_{\mathfrak{b}}(\operatorname{Coker} f)$ is artinian. We use [13, Lemma 3.1] with $S{=}\Gamma_{\mathfrak{b}}(-)$ and $T{=}\mathrm{H}^1_{\mathfrak{b}}(-)$ to conclude that $\operatorname{Ker}\mathrm{H}^1_{\mathfrak{b}}(f)$ is artinian. But $\mathrm{H}^1_{\mathfrak{b}}(f)$ is multiplication by the element $x{\in}\mathfrak{a}$ on $\mathrm{H}^1_{\mathfrak{b}}(\mathrm{H}^1_{\mathfrak{a}}(M))$. Again using [13, Lemma 3.1], we conclude that $\mathrm{H}^1_{\mathfrak{b}}(\mathrm{H}^1_{\mathfrak{a}}(M))$ is artinian. □

Recall that the *arithmetic rank*, $\operatorname{ara}(\mathfrak{a})$, of an ideal $\mathfrak{a}$ in a noetherian ring A is defined as the least number of elements of A required to generate an ideal with the same radical as $\mathfrak{a}$.

In [14, Theorem 2.3] it was shown that (with the notation given in the introduction) if $\operatorname{ara}(R_+){=}2$, then for $i{\geq}0$, the module $\mathrm{H}^i_{\mathfrak{m}_0R}(\mathrm{H}^2_{R_+}(M))$ is artinian if and only if $\mathrm{H}^{i+2}_{\mathfrak{m}_0R}(\mathrm{H}^1_{R_+}(M))$ is artinian. Before we state and prove a result of this kind in general form, we need a homological lemma, which might also have independent interest.

Lemma 2.16. *Let $\{T^i\}_{i=0}^{\infty}$ be a strongly connected sequence of functors between the abelian categories $\mathcal{A}$ and $\mathcal{B}$ and let $\mathcal{S}$ be a Serre subcategory of $\mathcal{B}$. Suppose $f\colon L\to M$ is a morphism in $\mathcal{A}$ and i is a positive integer such that*

$$\operatorname{Ker} T^{i+1}f,\quad \operatorname{Ker} T^{i}f,\quad \operatorname{Coker} T^{i}f\quad \textit{and}\quad \operatorname{Coker} T^{i-1}f$$

are all in $\mathcal{S}$. Then $T^{i+1}(\operatorname{Ker} f)$ is in $\mathcal{S}$ if and only if $T^{i-1}(\operatorname{Coker} f)$ is in $\mathcal{S}$.

Proof. Factorize f as $f=h\circ g$, where $0\to K\to L\xrightarrow{g}I\to 0$ and $0\to I\xrightarrow{h}M\to C\to 0$ are exact. Thus there are exact sequences

$$T^{i-1}I \xrightarrow{T^{i-1}h} T^{i-1}M \longrightarrow T^{i-1}C \longrightarrow T^iI \xrightarrow{T^ih} T^iM$$

and

$$T^iL \xrightarrow{T^ig} T^iI \longrightarrow T^{i+1}K \longrightarrow T^{i+1}L \xrightarrow{T^{i+1}g} T^{i+1}I.$$

Hence we get the exact sequences

$$0 \longrightarrow \operatorname{Coker} T^{i-1}h \longrightarrow T^{i-1}C \longrightarrow \operatorname{Ker} T^ih \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Coker} T^ig \longrightarrow T^{i+1}K \longrightarrow \operatorname{Ker} T^{i+1}g \longrightarrow 0.$$

From the compositions $T^jf=T^jh\circ T^jg$, where j is $i-1$, i or $i+1$, we get the exact sequences

$$\operatorname{Coker} T^{i-1}f \longrightarrow \operatorname{Coker} T^{i-1}h \longrightarrow 0,$$

$$\operatorname{Ker} T^if \longrightarrow \operatorname{Ker} T^ih \longrightarrow \operatorname{Coker} T^ig \longrightarrow \operatorname{Coker} T^if$$

and

$$0 \longrightarrow \operatorname{Ker} T^{i+1}g \longrightarrow \operatorname{Ker} T^{i+1}f.$$

Applying our hypothesis we get that $\operatorname{Coker} T^{i-1}h$ and $\operatorname{Ker} T^{i+1}g$ are in $\mathcal{S}$. Moreover we get that $\operatorname{Ker} T^ih$ is in $\mathcal{S}$ if and only if $\operatorname{Coker} T^ig$ is in $\mathcal{S}$. Comparing this with the two exact sequences we have previously obtained, we conclude that $T^{i-1}C$ is in $\mathcal{S}$ if and only if $\operatorname{Ker} T^ih$ is in $\mathcal{S}$, and $T^{i+1}K$ is in $\mathcal{S}$ if and only if $\operatorname{Coker} T^ig$ is in $\mathcal{S}$. Summing up, $T^{i-1}C$ is in $\mathcal{S}$ precisely when $T^{i+1}K$ is. $\square$

Theorem 2.17. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of A such that $A/(\mathfrak{a}+\mathfrak{b})$ is artinian. Assume that $\operatorname{ara}(\mathfrak{a})=2$ and that M is a finite A-module. Then for a given $i>0$, the module $\mathrm{H}^{i-1}_{\mathfrak{b}}(\mathrm{H}^2_{\mathfrak{a}}(M))$ is artinian if and only if the module $\mathrm{H}^{i+1}_{\mathfrak{b}}(\mathrm{H}^1_{\mathfrak{a}}(M))$ is artinian.*

Proof. We may assume that $\Gamma_{\mathfrak{a}}(M)=0$. Then $\mathfrak{a}$ can be generated by two M-regular elements x and y, [11, Exercise 16.8]. Therefore there is an exact sequence [6, Proposition 8.1.2]

$$0 \longrightarrow \mathrm{H}^1_{\mathfrak{a}}(M) \longrightarrow \mathrm{H}^1_{xA}(M) \xrightarrow{f} \mathrm{H}^1_{xA}(M_y) \longrightarrow \mathrm{H}^2_{\mathfrak{a}}(M) \longrightarrow 0.$$

In order to apply Lemma 2.16, we prove that $\operatorname{Ker} \mathrm{H}^j_{\mathfrak{b}}(f)$ and $\operatorname{Coker} \mathrm{H}^j_{\mathfrak{b}}(f)$ are artinian for all j.

By [6, Proposition 8.1.2], there is an exact sequence

$$\mathrm{H}^j_{\mathfrak{b}+yA}(\mathrm{H}^1_{xA}(M)) \longrightarrow \mathrm{H}^j_{\mathfrak{b}}(\mathrm{H}^1_{xA}(M)) \xrightarrow{\mathrm{H}^j_{\mathfrak{b}}(f)} \mathrm{H}^j_{\mathfrak{b}}(\mathrm{H}^1_{xA}(M)_y) \longrightarrow \mathrm{H}^{j+1}_{\mathfrak{b}+yA}(\mathrm{H}^1_{xA}(M)).$$

The outer modules are artinian, by Corollary 2.10. Hence $\operatorname{Ker}\mathrm{H}^j_{\mathfrak{b}}(f)$ and $\operatorname{Coker}\mathrm{H}^j_{\mathfrak{b}}(f)$ are artinian for all j. $\square$

We now extend [14, Proposition 2.8], where it was shown (for notation see the introduction) that if $\dim R_0{=}d$ and $\mathrm{H}^i_{R_+}(M){=}0$ for all $i{>}c$, then $\mathrm{H}^d_{\mathfrak{m}_0R}(\mathrm{H}^c_{R_+}(M))$ is artinian. We will now again use Lemma 2.16 in our proof.

Theorem 2.18. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of A such that $A/(\mathfrak{a}{+}\mathfrak{b})$ is artinian. Let M be a finite A-module and n be a non-negative integer such that $\mathrm{H}^j_{\mathfrak{a}}(M){=}0$ for all $j{>}n$. If $\dim A/\mathfrak{a}{=}d$, then the modules $\mathrm{H}^d_{\mathfrak{b}}(\mathrm{H}^n_{\mathfrak{a}}(M))$ and $\mathrm{H}^{d-1}_{\mathfrak{b}}(\mathrm{H}^n_{\mathfrak{a}}(M))$ are artinian. Moreover $\mathrm{H}^{d-2}_{\mathfrak{b}}(\mathrm{H}^n_{\mathfrak{a}}(M))$ is an artinian module if and only if $\mathrm{H}^d_{\mathfrak{b}}(\mathrm{H}^{n-1}_{\mathfrak{a}}(M))$ is artinian.*

Proof. Since $\dim A/\mathfrak{a}{=}d$, $\mathrm{H}^j_{\mathfrak{b}}(X){=}0$ for all $j{>}d$ and each A-module X with $\operatorname{Supp}_A(X){\subset}\mathrm{V}(\mathfrak{a})$. Hence if $X'{\to}X{\to}X''{\to}0$ is an exact sequence, where the modules have support in $\mathrm{V}(\mathfrak{a})$, then the sequence $\mathrm{H}^d_{\mathfrak{b}}(X'){\to}\mathrm{H}^d_{\mathfrak{b}}(X){\to}\mathrm{H}^d_{\mathfrak{b}}(X''){\to}0$ is exact.

Let $0{\to}M{\to}E^0\xrightarrow{\partial^0}E^1\xrightarrow{\partial^1}E^2{\to}\dots$ be a minimal injective resolution of M. The modules $\Gamma_{\mathfrak{a}+\mathfrak{b}}(E^i)$ are artinian, since they are finite direct sums of modules of the form $\mathrm{E}(A/\mathfrak{m})$, where $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{a}{+}\mathfrak{b}$.

Let $L{=}\operatorname{Ker}\partial^n$ and $N{=}\operatorname{Ker}\partial^{n-1}$. Since $\mathrm{H}^j_{\mathfrak{a}}(M){=}0$ for $j{>}n$,

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(L) \longrightarrow \Gamma_{\mathfrak{a}}(E^n) \longrightarrow \Gamma_{\mathfrak{a}}(E^{n+1}) \longrightarrow \dots$$

is an injective resolution of $\Gamma_{\mathfrak{a}}(L)$. For each j, the module $\mathrm{H}^j_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(L))$ is a subquotient of $\Gamma_{\mathfrak{a}+\mathfrak{b}}(E^{n+j})$ and is therefore an artinian module. Also the modules $\mathrm{H}^i_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(E^{n-1})$ are artinian for all i, actually zero for $i{>}0$.

Thus we can apply Lemma 2.16 to the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow \Gamma_{\mathfrak{a}}(E^{n-1}) \xrightarrow{f} \Gamma_{\mathfrak{a}}(L) \longrightarrow \mathrm{H}^n_{\mathfrak{a}}(M) \longrightarrow 0.$$

Consequently, for each i the module $\mathrm{H}^i_{\mathfrak{b}}(\mathrm{H}^n_{\mathfrak{a}}(M))$ is artinian if and only if the module $\mathrm{H}^{i+2}_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(N))$ is artinian. Since the latter module even vanishes if $i{=}d$ or $i{=}d{-}1$, we conclude that the former is artinian if $i{=}d$ or $i{=}d{-}1$.

The exact sequence $\Gamma_{\mathfrak{a}}(E^{n-2}){\to}\Gamma_{\mathfrak{a}}(N){\to}\mathrm{H}^{n-1}_{\mathfrak{a}}(M){\to}0$ gives the exact sequence $\mathrm{H}^d_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(E^{n-2})){\to}\mathrm{H}^d_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(N)){\to}\mathrm{H}^d_{\mathfrak{b}}(\mathrm{H}^{n-1}_{\mathfrak{a}}(M)){\to}0$, by the right exactness of $\mathrm{H}^d_{\mathfrak{b}}(-)$

on modules with support at $\mathrm{V}(\mathfrak{a})$. Since the first module is artinian the next two modules are artinian simultaneously. However we showed above that $\mathrm{H}_{b}^{d}(\Gamma_{a}(N))$ is artinian if and only if the module $\mathrm{H}_{\mathfrak{b}}^{d-2}(\mathrm{H}_{\mathfrak{a}}^{n}(M))$ is artinian. $\square$

Acknowledgement. We thank the referee for the careful reading of our paper.

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Received May 6, 2011
in revised form August 29, 2013
published online December 5, 2013