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Parallel mean curvature surfaces in symmetric spaces

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Abstract. We present a reduction-of-codimension theorem for surfaces with parallel mean curvature in symmetric spaces.

1. Introduction

Surfaces whose mean curvature is a parallel section of the normal bundle (i.e. parallel mean curvature surfaces) have been considered by many geometers. Yau [5] studied them when the ambient space is a space form. He showed that such a surface must be either minimal in a round sphere, or its image must be contained in a three-dimensional totally geodesic, or totally umbilic, submanifold. This result was recently extended to maps into $E^n(c)\times\mathbb{R}$ by Alencar, do Carmo and Tribuzy [1], where $E^n(c)$ is a space form with constant sectional curvature $c\neq 0$. They proved that the image of the surface is either minimal in a totally umbilic hypersurface of $E^n(c)$, or has constant mean curvature in a three-dimensional totally geodesic, or totally umbilic, submanifold of $E^n(c)$, or lies in $E^4\times\mathbb{R}$.

Fetcu [3] has shown that a surface immersed in $\mathbb{CP}^n$ with parallel mean curvature must be either pseudo-umbilic and totally real or its image must be contained in $\mathbb{CP}^5$. We recall that an immersion $\varphi\colon M\to N$ between two Riemannian manifolds is said to be *pseudo-umbilic* if the H-Weingarten operator A_H is a multiple of the identity, where H denotes the mean curvature. When the target manifold is a non-flat cosymplectic space form, Fetcu and Rosenberg [4] proved that either the immersion is pseudo-umbilic or lies in a totally geodesic submanifold with dimension less than or equal to 11.

The aim of this work is to generalize the above results to immersions into symmetric spaces.

From now on we let S denote a Riemannian symmetric space and R its curvature tensor. For $p \in S$, we say that a subspace $K \subset T_pS$ is *invariant by the curvature tensor* R_p, if $R_p(u,v)w \in K$, whenever $u, v, w \in K$. When $X \subset T_pS$, $R_p(X)$ will represent the least vector subspace of T_pS, containing X, invariant by the curvature tensor R_p at p.

Let $\varphi\colon M \to S$ be an isometric immersion from a Riemann surface M. For each $x \in M$, the nth osculating space of φ at x will be denoted by $O_x^n(\varphi)$.

Our main result is the following.

Theorem 1. *If φ has parallel mean curvature $H \neq 0$, one of the following conditions holds*:

(i) *φ is pseudo-umbilic*;

(ii) *the dimension d of $R_{\varphi(x)}(O_x^2)$ is independent of $x \in M$ and there exists a totally geodesic submanifold $S' \subset S$, with dimension d, such that $\varphi(M) \subset S'$.*

2. Proof of the main theorem

The following theorem dealing with reduction of codimension of maps is fundamental to our purpose. For a proof see [2].

Theorem 2. *Let $\varphi\colon M \to S$ be an isometric immersion from a Riemannian manifold into a symmetric space. If there exists a parallel fiber bundle L, over M, such that $R(L)=L$ and $TM \subset L$, then there exists a totally geodesic submanifold S' of S with $\varphi(M) \subset S'$ and $L_x = T_{\varphi(x)}S'$ for $x \in M$.*

Let α denote the second fundamental form of the immersion. We remark that, for each $x \in M$, $O_x^2(\varphi) = T_xM + N_1(x)$, where $N_1(x) = \{\alpha(u,v) : u, v \in T_xM\}$. We will show that either H is an umbilic direction, or $L = R(O^2(\varphi))$ is a parallel fiber bundle.

Let U be an open subset of M where $L_x = R_x(O_x^2(\varphi))$ has maximal dimension. We will prove that, when H is not an umbilic direction, L is parallel on U. Using analyticity arguments we may conclude that its dimension is constant on M.

Lemma 3. *Whenever $X, Y, W \in \Gamma(L)$ and $\nabla X, \nabla Y, \nabla W \in \Gamma(L)$, we have that $R(X,Y)W \in \Gamma(L)$.*

Proof. It follows straightforwardly from the parallelism of R. $\square$

We remark that L is constructed from $O^2(\varphi)$ by successive applications of R to its elements. Thus it is enough to show that $\nabla\xi \in \Gamma(L)$ for every $\xi = \alpha(U,V)$ to conclude that L is parallel.

Let $\{\varepsilon_1, \varepsilon_2\}$ be a local orthonormal frame field defined on an open subset V of U. Notice that $R_x(\varepsilon_1, \varepsilon_2)H \in O_x^2$ for $x \in V$. Let $\pi_{N_1(x)}: T_{\varphi(x)}S \to N_1(x)$ denote the orthogonal projection and consider $n_1(x) = \pi_{N_1(x)}(R_x(\varepsilon_1, \varepsilon_2)H)$.

Lemma 4. *Assume n_1 vanishes identically on V. Then L is parallel on V.*

Proof. From Ricci equations we get

$$0 = \langle R^\perp(\varepsilon_1, \varepsilon_2)H, \eta\rangle = -\langle [A_H, A_\eta]\varepsilon_1, \varepsilon_2\rangle,$$

so that A_H commutes with A_η for every section η of the normal bundle. Therefore, without loss of generality, $\{\varepsilon_1, \varepsilon_2\}$ may be chosen in such a way that $\alpha(\varepsilon_1, \varepsilon_2) = 0$ and it is enough to show that

$$\nabla_{\varepsilon_i}\alpha(\varepsilon_j, \varepsilon_j) \in \Gamma(L).$$

But this is equivalent to proving that

$$(\nabla_{\varepsilon_i}\alpha)(\varepsilon_j, \varepsilon_j) \in \Gamma(L),$$

since

$$(\nabla_{\varepsilon_i}\alpha)(\varepsilon_j, \varepsilon_j) = \nabla_{\varepsilon_i}\alpha(\varepsilon_j, \varepsilon_j) - 2\alpha(\nabla_{\varepsilon_i}\varepsilon_j, \varepsilon_j).$$

From Codazzi equations we have

$$(\nabla_{\varepsilon_i}\alpha)(\varepsilon_j, \varepsilon_j) = (\nabla_{\varepsilon_j}\alpha)(\varepsilon_i, \varepsilon_j) + (R(\varepsilon_i, \varepsilon_j)\varepsilon_j)^\perp,$$

where $(\,\cdot\,)^\perp$ represents the orthogonal projection onto the normal bundle. But $(R(\varepsilon_i, \varepsilon_j)\varepsilon_j)^\perp$ sits in L since

$$(R(\varepsilon_i, \varepsilon_j)\varepsilon_j)^\perp = R(\varepsilon_i, \varepsilon_j)\varepsilon_j - A,$$

with $A = (R(\varepsilon_i, \varepsilon_j)\varepsilon_j)^T \in \Gamma(L)$, where $(\,\cdot\,)^T$ stands for the orthogonal projection onto TV.

Of course $\nabla_{\varepsilon_i}\alpha(\varepsilon_j, \varepsilon_j) \in \Gamma(L)$ if $i \neq j$. When $i = j$,

$$\nabla_{\varepsilon_i}\alpha(\varepsilon_i, \varepsilon_i) = \nabla_{\varepsilon_i}(2H - \alpha(\varepsilon_l, \varepsilon_l)) = -\nabla_{\varepsilon_i}\alpha(\varepsilon_l, \varepsilon_l) + B \in \Gamma(L),$$

with $l \neq i$ and $B = 2(\nabla_{\varepsilon_i}H)^T$. □

We proceed now with the proof of Theorem 1, considering the case where n_1 does not vanish identically.

Let us consider an open subset V of U where $n_1(x) \neq 0$ for $x \in V$.

We remark that n_1 and H are linearly independent, $\nabla n_1 \in \Gamma(L)$ and $\nabla H \in \Gamma(L)$. Remember that, due to the parallelism of R, $\nabla n_1 = R(\nabla\varepsilon_1, \varepsilon_2)H + R(\varepsilon_1, \nabla\varepsilon_2)H + R(\varepsilon_1, \varepsilon_2)\nabla H$.

Let Z denote the sub-bundle of the normal bundle spanned by n_1 and H. Consider a unitary section γ orthogonal to Z and sitting in N_1 and take an orthonormal frame field $\{\varepsilon_1, \varepsilon_2\}$ diagonalizing the Weingarten operator A_γ.

We observe that $\alpha(X,Y) = \alpha(X,Y)_Z + \alpha(X,Y)_\gamma$, where α_Z and α_γ are the Z and $\langle\gamma\rangle$ components of α, respectively. Here $\langle\gamma\rangle$ denotes the sub-bundle spanned by γ.

Now, to show $\nabla\alpha(X,Y) \in \Gamma(L)$, it is enough to prove that $\nabla\alpha_\gamma(X,Y) \in \Gamma(L)$, since $\alpha_Z(X,Y) \in \Gamma(L)$. Hence we get that, for any $i,j,k \in \{1,2\}$, $\nabla_{\varepsilon_i}\alpha_\gamma(\varepsilon_j, \varepsilon_k) \in \Gamma(L)$ if and only if $\nabla_{\varepsilon_i}\alpha(\varepsilon_j, \varepsilon_k) \in \Gamma(L)$, which is equivalent to $(\nabla_{\varepsilon_i}\alpha)(\varepsilon_j, \varepsilon_k) \in \Gamma(L)$.

Clearly $\nabla_{\varepsilon_i}\alpha_\gamma(\varepsilon_j, \varepsilon_k) = 0$, whenever $j \neq k$.

When $j = k \neq i$, using Codazzi equations, we have

$$(\nabla_{\varepsilon_i}\alpha)(\varepsilon_j, \varepsilon_j) = (\nabla_{\varepsilon_j}\alpha)(\varepsilon_i, \varepsilon_j) + A,$$

where $A \in \Gamma(L)$, and, since $(\nabla_{\varepsilon_j}\alpha_\gamma)(\varepsilon_i, \varepsilon_j) \in \Gamma(L)$, we obtain $(\nabla_{\varepsilon_i}\alpha)(\varepsilon_j, \varepsilon_j) \in \Gamma(L)$.

If $i = j = k$, then

$$\nabla_{\varepsilon_i}\alpha_\gamma(\varepsilon_i, \varepsilon_i) = \nabla_{\varepsilon_i}(2H_\gamma - \alpha_\gamma(\varepsilon_l, \varepsilon_l)) = -\nabla_{\varepsilon_i}\alpha_\gamma(\varepsilon_l \varepsilon_l),$$

where $l \neq i$ and we have used the fact that $H_\gamma = 0$. Hence $\nabla_{\varepsilon_i}\alpha_\gamma(\varepsilon_i, \varepsilon_i) \in \Gamma(L)$.

3. Remarks

3.1. When $S = E^n(c)$ is a space form of constant sectional curvature c, it follows directly from Ricci equations that, for each section η of the normal bundle, A_H and A_η commute. Therefore A_H is either a multiple of the identity, or there exists a basis diagonalizing, at each point, the second fundamental form. Hence the second normal space has dimension less than or equal to 4. Of course the curvature tensor leaves this space invariant and we get that $\varphi(M) \subset E^4(c)$ [5].

3.2. When $S = \mathbb{CP}^n$ the knowledge of the explicit formula of the curvature tensor allows us to get that

$$R(O^2(\varphi)) \subset O^2(\varphi) + JO^2(\varphi),$$

where J denotes the standard complex structure of $\mathbb{CP}^n$. Therefore, if φ is not pseudo-umbilic, we must clearly have $\varphi(M) \subset \mathbb{CP}^5$. In the case where H is an umbilic direction, a direct computation gives $R(X,Y)H = 0$, whenever $X, Y \in \Gamma(TM)$. Thus, again from the explicit formula of the curvature tensor, we obtain straightforwardly that JTM is orthogonal to TM, i.e. φ is totally real. This case was studied by Fetcu [3].

3.3. Assume now that the target manifold is a product $S_1 \times S_2$ of symmetric spaces. We have the following result.

Corollary 5. *Let $\varphi\colon M \to S_1 \times S_2$ be an immersion from a Riemann surface with parallel mean curvature $H \neq 0$. Then, one of the following conditions holds:*

(i) *φ is pseudo-umbilic;*

(ii) *$\varphi(M) \subset S_1' \times S_2'$, where, for each $i \in \{1,2\}$, $S_i' \subset S_i$ is a symmetric space totally geodesically embedded in S_i and $\dim S_i' = \dim R_i((O^2(\varphi))^{T_i})$. Here R_i represents the curvature tensor of S_i and $(\,\cdot\,)^{T_i}$ stands for the orthogonal projection onto TS_i.*

Proof. Clearly

$$R(O^2(\varphi)) \subset R_1((O^2(\varphi))^{TS_1}) + R_2((O^2(\varphi))^{TS_2}).$$

Therefore it is enough to prove that each $R_i((O^2(\varphi))^{TS_i})$ is parallel, when φ is not pseudo-umbilic. Let us take an open subset of M where the dimension of $O^2(\varphi)^{T_i}$ is maximal. By Lemma 3, it suffices to show that $\nabla \mu_i \in \Gamma(R((O^2(\varphi))^{T_i}))$, whenever $\mu_i \in \Gamma((O^2(\varphi))^{T_i})$. This follows from the fact that there exists $\mu \in \Gamma(O^2(\varphi))$ such that $\mu_i = (\mu)^{T_i}$. Since $\nabla \mu = (\nabla \mu_1, \nabla \mu_2) \in \Gamma(R(O^2(\varphi)))$ we obtain that $\nabla \mu_i \in \Gamma(R_i((O^2(\varphi))^{T_i}))$. $\square$

In the particular situation where $S_2 = \mathbb{R}$, it follows from Corollary 5 that either φ is pseudo-umbilic, or $\varphi(M) \subset S_1' \times \mathbb{R}$, where $\dim S_1' = \dim R((O^2(\varphi))^{TS_1})$ and R denotes the curvature tensor of S_1. For instance, if $S_1 = E^n(c)$, clearly $S_1' = E^4(c)$, like in [1].

When the target manifold is $E^n(c) \times E^m(d)$, either φ is pseudo-umbilic, or $\varphi(M) \subset E^4(c) \times E^4(d)$.

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