Ark. Mat., **48** (2010), 79–95
DOI: 10.1007/s11512-008-0089-6

A multi-dimensional Markov chain and the Meixner ensemble

Kurt Johansson

Abstract. We show that the transition probability of the Markov chain $(G(i,1),...,G(i,n))_{i\geq 1}$, where the $G(i,j)$'s are certain directed last-passage times, is given by a determinant of a special form. An analogous formula has recently been obtained by Warren in a Brownian motion model. Furthermore we demonstrate that this formula leads to the Meixner ensemble when we compute the distribution function for $G(m,n)$. We also obtain the Fredholm determinant representation of this distribution, where the kernel has a double contour integral representation.

1. Introduction

The starting point for the present paper are some nice results from the interesting paper [18] by J. Warren. Let $B_k(t)$, $1\leq k\leq n$, be independent Brownian motions started at the origin and define $X_k(t)$, $k=1,...,n$, recursively by

$$X_k(t)=\sup_{0\leq s\leq t}(X_{k-1}(s)+B_k(t)-B_k(s)), \tag{1}$$

$t\geq 0$. The multi-dimensional Markov process $\mathbf{X}(t)=(X_1(t),...,X_n(t))$ at a fixed time is closely related to the largest eigenvalues of successive principal submatrices of a Gaussian unitary ensemble (GUE) matrix. In fact, let $H=(h_{ij})_{1\leq i,j\leq n}$, be an $n\times n$ GUE matrix, i.e. distributed according to the probability measure $Z_n^{-1}\exp(-\operatorname{Tr}H^2)\,dH$ on the space of $n\times n$ Hermitian matrices, and let $H_k=(h_{ij})_{1\leq i,j\leq k}$, $1\leq k\leq n$, be the principal submatrices. Then, if $\lambda_{\max}(M)$ denotes the largest eigenvalue of the Hermitian matrix M, we have $X\left(\frac{1}{2}\right)=(\lambda_{\max}(H_1),...,\lambda_{\max}(H_n))$ in distribution, [1], [6], [18]. Furthermore, there is a nice formula for the transition function of the Markov process $\mathbf{X}(t)$, [18],

$$\mathbb{P}[\mathbf{X}(t)=y\,|\,\mathbf{X}(s)=x]=\det(D^{j-i}\phi_{t-s}(y_j-x_i))_{1\leq i,j\leq n}, \tag{2}$$

Supported by the Göran Gustafsson Foundation (The Royal Swedish Acadamy of Sciences), the ESF-network MISGAM and the EU network ENIGMA.

if $x_1 \le \ldots \le x_n$ and $y_1 \le \ldots \le y_n$. Here D denotes ordinary differentiation, D^{-1} is antiderivation,

$$D^{-k} f(x) = \int_{-\infty}^{x} \frac{(x-y)^{k-1}}{(k-1)!} f(y)\, dy, \tag{3}$$

and $\phi_t(x) = (2\pi t)^{-1/2} \exp(-x^2/2t)$ is the transition density for Brownian motion.

Let $F_{\mathrm{GUE}(n)}(\eta)$ be the distribution function for the largest eigenvalue of an $n \times n$ Hermitian matrix H from GUE. It follows easily from (2) that

$$F_{\mathrm{GUE}(n)}(\eta) = \det(D^{j-i+1} \phi_{1/2}(\eta))_{1 \le i,j \le n}. \tag{4}$$

This formula, given in [18], can also be obtained directly from the GUE eigenvalue measure, see Proposition 2.3 below.

We will show in this paper, starting from the definitions, that we have analogous formulas for the vector $\mathbf{G}(i) = (G(i,1), \ldots, G(i,n))$, $i \ge 0$, of certain last-passage times defined as follows, [7]. Let $w(i,j)$, $(i,j) \in \mathbb{Z}_+^2$, be independent geometric random variables with parameter q, $0<q<1$,

$$\mathbb{P}[w(i,j) = k] = (1-q) q^k \tag{5}$$

and define

$$G(m,n) = \max_{\pi} \sum_{(i,j) \in \pi} w(i,j), \tag{6}$$

where the maximum is over all up/right paths from $(1,1)$ to (m,n). It is clear that these random variables satisfy the recursion relation

$$G(m,n) = \max\{G(m-1,n), G(m,n-1)\} + w(m,n), \tag{7}$$

where $G(0,n) = G(n,0) = 0$ for $n \ge 1$. If we use this recursion relation repeatedly we see that

$$G(m,n) = \max_{1 \le j \le n} \left(G(j,n-1) + \sum_{i=1}^{m} w(i,n) - \sum_{i=1}^{j-1} w(i,n) \right),$$

which looks like a discrete version of (1). From this it is reasonable to expect that there should be a formula for the transition function for the Markov chain $(\mathbf{G}(i))_{i \ge 0}$ similar to (2). This is indeed the case in a very natural way where the differentiation operator is replaced by a finite difference operator, see Theorem 2.1. This will imply a formula similar to (4) for the distribution function of $G(m,n)$, see Theorem 2.2. We will also show how we can go from this formula (10) to the known expressions [14], [9], for this distribution function in terms of the Meixner ensemble, [7], and as a Fredholm determinant with a double contour integral expression for the kernel, [14], [9]. The Fredholm determinant formula has the advantage that it is much better suited for computation of asymptotics. The argument in this paper leading to (15) gives an alternative approach, starting from the definitions, to this formula.

Results related to the transition probability (9) for $(\mathbf{G}(i))_{i\geq 0}$, but in the case of $w(i,j)$ exponentially distributed, go back to the work of G. Schütz, [16], where the totally asymmetric exclusion process (TASEP) is studied using the Bethe ansatz, see e.g. [7] for a discussion of the relation to $G(m,n)$. This is not exactly the same Markov chain, but the results of Schütz can also be used to derive the expression for the distribution function for $G(m,n)$ in terms of the Laguerre ensemble, see [15] and also [13]. In [15] the case of geometric random variables is also considered and the formula for the distribution function for $G(m,n)$ in terms of the Meixner ensemble derived. This is based on results from [4] on a discrete TASEP-type model. More general formulas for the asymmetric exclusion process (ASEP) have been proved recently in [17]. Results generalizing the formula (2) to discrete models have also been given independently by Dieker and Warren in [5].

2. Results

For $x\in\mathbb{Z}$ we define $w(x)=(1-q)q^xH(x)$, $0<q<1$, where H is the Heaviside function, $H(x)=0$ if $x<0$ and $H(x)=1$ if $x\geq 0$. The m-fold convolution of w with itself is then the negative binomial distribution,

$$w_m(x)=(1-q)^m\binom{x+m-1}{x}q^xH(x), \tag{8}$$

as is not difficult to see using generating functions.

For a function $f\colon\mathbb{Z}\to\mathbb{C}$ we denote by Δ the usual finite difference operator, $\Delta f(x)=f(x+1)-f(x)$. We also set

$$(\Delta^{-1}f)(x)=\sum_{y=-\infty}^{x-1}f(y)$$

provided the sum is convergent, and $\Delta^0 f=f$. Note that $\Delta(\Delta^{-1}f)=\Delta^{-1}(\Delta f)=f$.

Let $W_n=\{x\in\mathbb{Z}^n;x_1\leq\ldots\leq x_n\}$, and let $\mathbf{G}(i)$, $i\geq 0$, be the Markov chain defined in the introduction.

Theorem 2.1. *If $x,y\in W_n$, then for $m\geq l$,*

$$\mathbb{P}[\mathbf{G}(m)=y\,|\,\mathbf{G}(l)=x]=\det(\Delta^{j-i}w_{m-l}(y_j-x_i))_{1\leq i,j\leq n}. \tag{9}$$

We postpone the proof to Section 3. The proof first shows (9) in the case $m=l+1$ using (7) and an induction argument in n, and then establishes a convolution formula for the determinants involved using the generalized Cauchy–Binet identity (12).

Taking $\mathbf{G}(0)=0$ it is not difficult to show, see Section 3, that we have the following consequence of (9), which is analogous to (4).

Theorem 2.2. *For any $\eta\in\mathbb{N}$, $m\geq n\geq 1$,*

$$\mathbb{P}[G(m,n)\leq\eta]=\det(\Delta^{j-i-1}w_m(\eta+1))_{1\leq i,j\leq n}. \tag{10}$$

As stated in the introduction it is possible to relate the expression in the right-hand side of (4) directly to the expression for the distribution function coming from the GUE eigenvalue measure. Let

$$\Delta_n(x)=\det(x_i^{j-1})_{1\leq i,j\leq n}$$

be the Vandermonde determinant.

Proposition 2.3. *We have the following identity for any $\eta\in\mathbb{R}$, $n\geq 1$,*

$$\det(D^{j-i+1}\phi_{1/2}(\eta))_{1\leq i,j\leq n}=\frac{1}{Z_n}\int_{(-\infty,\eta]^n}\Delta_n(x)^2\prod_{j=1}^n e^{-x_j^2}\,d^nx, \tag{11}$$

where Z_n is the appropriate normalization constant.

Proof. Let H_j, $j\geq 0$, be the standard Hermite polynomials. Then $H_j(x)=2^jp_j(x)$, where p_j is a monic polynomial, and we have Rodrigues' formula

$$D^je^{-x^2}=(-1)^jH_j(x)e^{-x^2}.$$

Hence,

$$\begin{aligned}D^{j-i-1}\phi_{1/2}(\eta)=D^{-i}(D^{j-1}\phi_{1/2})(\eta)&=\int_{-\infty}^{\eta}\frac{(\eta-x)^{i-1}}{(i-1)!}D^{j-1}\phi_{1/2}(x)\,dx\\&=\frac{2^{j-1}(-1)^{i+j}}{(i-1)!\sqrt{\pi}}\int_{-\infty}^{\eta}(x-\eta)^{i-1}p_j(x)e^{-x^2}\,dx.\end{aligned}$$

Using row and column operations we obtain

$$\begin{aligned}\det(D^{j-i-1}\phi_{1/2}(\eta))_{1\leq i,j\leq n}&=\prod_{j=0}^{n-1}\frac{2^j}{j!\sqrt{\pi}}\det\left(\int_{-\infty}^{\eta}x^{i-1}x^{j-1}e^{-x^2}\,dx\right)_{1\leq i,j\leq n}\\&=\frac{1}{Z_n}\int_{(-\infty,\eta]^n}\Delta_n(x)^2\prod_{j=1}^n e^{-x_j^2}\,d^nx,\end{aligned}$$

with $Z_n = 2^{-n(n-1)/2}\pi^{n/2}\prod_{j=0}^{n-1} j!$. In the last equality we have used the generalized Cauchy–Binet identity,

$$\det\left(\int_X \phi_i(x)\psi_j(x)\,d\mu(x)\right) = \frac{1}{n!}\int_{X^n} \det(\phi_i(x_j))\det(\psi_i(x_j))\prod_{j=1}^{n} d\mu(x_j), \tag{12}$$

where all determinants are $n\times n$. $\square$

We have a similar identity relating the right-hand side of (10) to the Meixner ensemble. The proof is a little more involved and we postpone it to Section 3.

Proposition 2.4. *For any $\eta\in\mathbb{N}$, $m\geq n\geq 1$,*

$$\det(\Delta^{j-i-1}w_m(\eta+1))_{1\leq i,j\leq n} = \frac{1}{Z_{m,n}}\sum_{0\leq x_i\leq \eta+n-1}\Delta_n(x)^2\prod_{j=1}^{n}\binom{x_j+m-n}{x_j}q^{x_j}. \tag{13}$$

For asymptotic analysis it is more useful to have a representation of the distribution function for $G(m,n)$ as a Fredholm determinant with an appropriate kernel. It is possible to go to such a formula using Meixner polynomials and a standard random matrix theory computation as was done in [7]. There is also another formula for the kernel as a double contour integral which can be obtained from the Schur measure, see [14], [9] or [10]. It is actually possible to go directly to a Fredholm determinant formula with a double contour integral formula for the kernel starting from the expression in the right-hand side of (10).

Let γ_r denote a circle centered at the origin with radius $r>0$. Let $1<r_2<r_1<1/q$ and define

$$K_{m,n}(x,y) = \frac{1}{(2\pi i)^2}\int_{\gamma_{r_2}}\int_{\gamma_{r_1}} \frac{w}{w-z}\,\frac{z^{x+n}}{w^{y+n}}\,\frac{(1-qz)^m(1-w)^n}{(1-z)^n(1-qw)^n}\,\frac{dw}{w}\,\frac{dz}{z} \tag{14}$$

for $x,y\in\mathbb{Z}$.

Proposition 2.5. *For any integer $\eta\geq 0$, $m\geq n\geq 1$,*

$$\mathbb{P}[G(m,n)\leq\eta] = \det(I-K_{m,n})_{\ell^2(\{\eta+1,\eta+2,\dots\})}. \tag{15}$$

The proof will be given in the next section.

Remark 2.6. It would be interesting to understand the joint distribution of $G(m_i,n_i)$, $i=1,\dots,p$, in order to understand the fluctuations of the "last-passage times surface", $\mathbb{Z}_+^2\ni(m,n)\mapsto G(m,n)$. If $(m_1,n_1),\dots,(m_p,n_p)$ form a right/down path then the joint distribution of $G(m_1,n_1),...,G(m_p,n_p)$ can be expressed as a Fredholm determinant, see [10] and [3], and it is possible to investigate the asymptotic fluctuations. However, the asymptotic correlation between for example $G(m,m)$ and $G(n,n)$, $m<n$, is not known and their is no nice expression for their joint distribution. Using (9) we can write down an expression for their joint distribution, which was one of the motivations for the present work. We have

$$\mathbb{P}[G(m,m)\le\eta_1, G(n,n)\le\eta_2] = \sum_{\substack{x\in W_n\\ x_m\le\eta_1}}\sum_{\substack{y\in W_n\\ y_n\le\eta_2}} \det(\Delta^{j-i}w_m(x_j))_{1\le i,j\le n}\det(\Delta^{j-i}w_{n-m}(y_j-x_i))_{1\le i,j\le n}. \tag{16}$$

However, we have not been able to rewrite this in a form useful for asymptotic computations.

Remark 2.7. The case when the $w(i,j)$'s are exponential random variables can be treated in a completely analogous way or by taking the appropriate limit of the formula above, $q=1-\alpha/L$, $G(m,n)\mapsto G(m,n)/L$ and $L\to\infty$.

Remark 2.8. Random permutations can be obtained as a limit of the above model, $q=\alpha/n^2$, $n\to\infty$, [8]. The random variable $G(n,n)$ then converges to $L(\alpha)$, the Poissonized version of the length l_N of a longest increasing subsequence of a random permutation from S_N. We can take this limit in the formulas (14) and (15) and this leads to a formula for $\mathbb{P}[L(\alpha)\le\eta]$ as a Fredholm determinant involving the discrete Bessel kernel, [8], [2]. Hence, we obtain a new proof of this result which does not involve directly the Robinson–Schensted–Knuth correspondence.

3. Proofs

3.1. Proofs of Theorems 2.1 and 2.2

To prove Theorem 2.1 we first consider the case $m-l=1$. The transition function from $\mathbf{G}(l)$ to $\mathbf{G}(m)$ is, by (7),

$$\mathbb{P}[\mathbf{G}(l+1)=y\,|\,\mathbf{G}(l)=x]=\prod_{k=1}^{n} w(y_k-\max\{x_k,y_{k-1}\}), \tag{17}$$

where we have set $y_0=0$ and $x,y\in W_n$. Note also that it is clear from (7) that $(\mathbf{G}(i))_{i\ge0}$ is a Markov chain. The right-hand side can be written as a determinant by the following lemma.

Lemma 3.1. *If* $x,y\in W_n$, *then*

$$\prod_{k=1}^{n} w(y_k-\max\{x_k,y_{k-1}\})=\det(\Delta^{j-i}w(y_j-x_i))_{1\le i,j\le n}. \tag{18}$$

Proof. We use induction with respect to n. The claim is trivial for $n=1$. Assume that it is true up to $n-1$. Expand the determinant in (18) along the last row,

$$\begin{aligned}(19)\quad &\det(\Delta^{j-i}w(y_j-x_i))_{1\le i,j\le n}\\ &\quad=\sum_{k=1}^{n-2}(-1)^{k+n}\Delta^{k-n}w(y_k-x_n)\det(\Delta^{j-i}w(y_j-x_i))_{1\le i<n,1\le j\le n,j\ne k}\\ &\quad\quad-\Delta^{-1}w(y_{n-1}-x_n)\det(\Delta^{j-i}w(y_j-x_i))_{1\le i<n,1\le j\le n,j\ne n-1}\\ &\quad\quad+w(y_n-x_n)\det(\Delta^{j-i}w(y_j-x_i))_{1\le i,j\le n-1}.\end{aligned}$$

We will first show that each term in the sum from $k=1$ to $n-2$ in the right-hand side of (19) is zero. Let Δ_y denote the difference operator with respect to the variable y. The fact that $\Delta(\Delta^{-1}w)=w$, then gives

$$\det(\Delta^{j-i}w(y_j-x_i))_{1\le i<n,1\le j\le n,j\ne k}=\Delta_{y_{k+1}}...\Delta_{y_n}\det(\Delta^{j-i}w(\tilde y_j-x_i))_{1\le i,j\le n-1},$$

where $\tilde y_j=y_j$ if $1\le j\le k-1$ and $\tilde y_j=y_{j+1}$ if $k\le j\le n-1$. By the induction assumption this equals

$$\Delta_{y_{k+1}}...\Delta_{y_n}\left[\prod_{j=1}^{k-1}w(y_j-\max\{x_j,y_{j-1}\})\right]w(y_{k+1}-\max\{x_k,y_{k-1}\})\\ \times\prod_{j=k+1}^{n-1}w(y_{j+1}-\max\{x_j,y_j\}).$$

If $y_k<x_n$, then $\Delta^{k-n}w(y_k-x_n)=0$ since $\Delta^{-j}w(x)=0$ if $x<j$. In this case the kth term in the sum in (19) is zero. Assume that $y_k\ge x_n$. Then $y_n\ge...\ge y_k\ge x_n$ and we obtain

$$\det(\Delta^{j-i}w(y_j-x_i))_{1\le i<n,1\le j\le n,j\ne k}\\ =\prod_{j=1}^{k-1}w(y_j-\max\{x_j,y_{j-1}\})(1-q)q^{-\max\{x_k,y_{k-1}\}}\Delta_{y_{k+1}}...\Delta_{y_n}\prod_{j=k+1}^{n-1}(1-q)q^{y_{j+1}-y_j}.$$

But,

$$\Delta_{y_{k+1}}...\Delta_{y_n}\prod_{j=k+1}^{n-1}(1-q)q^{y_{j+1}-y_j}=(1-q)^{n-k-1}\Delta_{y_{k+1}}...\Delta_{y_n}q^{y_n}=0$$

since $k+1<n$. Hence, each term in the sum from $k=1$ to $n-2$ in (19) is zero and we obtain

$$\det(\Delta^{j-i}w(y_j-x_i))_{1\le i,j\le n} \tag{20}$$
$$=-\Delta^{-1}w(y_{n-1}-x_n)\det(\Delta^{j-i}w(y_j-x_i))_{1\le i<n,1\le j\le n,j\ne n-1}$$
$$+w(y_n-x_n)\det(\Delta^{j-i}w(y_j-x_i))_{1\le i,j\le n-1}.$$

If $y_{n-1}<x_n$, then $\Delta^{-1}w(y_{n-1}-x_n)=0$ and the right-hand side of (20) is

$$w(y_n-x_n)\prod_{j=1}^{n-1}w(y_j-\max\{x_j,y_{j-1}\}),$$

by the induction assumption. Since $y_{n-1}<x_n$, $w(y_n-x_n)=w(y_n-\max\{x_n,y_{n-1}\})$ and we get exactly the right-hand side of (18).

Assume now that $y_{n-1}\ge x_n$. Note that $\Delta^j w(x)=(q-1)\Delta^{j-1}w(x)$ if $j\ge 1$ and $x\ge 0$. The determinant

$$\det(\Delta^{j-i}w(y_j-x_i))_{1\le i<n,1\le j\le n,j\ne n-1}$$

has $\Delta^{n-i}w(y_n-x_i)=(q-1)\Delta^{n-1-i}w(y_n-x_i)$, $1\le i<n$, in the last column, since $y_n\ge y_{n-1}\ge x_n\ge ...\ge x_1$. By the induction assumption this determinant equals

$$(q-1)\prod_{j=1}^{n-2}w(y_j-\max\{x_j,y_{j-1}\})w(y_n-\max\{x_{n-1},y_{n-2}\}).$$

Thus, the right-hand side of (20) equals

$$-(1-q)(q^{y_{n-1}-x_n}-1)\prod_{j=1}^{n-2}w(y_j-\max\{x_j,y_{j-1}\})w(y_n-\max\{x_{n-1},y_{n-2}\}) \tag{21}$$
$$+w(x_n-y_n)\prod_{j=1}^{n-1}w(y_j-\max\{x_j,y_{j-1}\}),$$

where we have used the fact that $(\Delta^{-1}w)(y_{n-1}-x_n)(q-1)=(1-q)(q^{y_{n-1}-x_n}-1)$ for $y_{n-1}-x_n\ge 0$. Since $y_n\ge y_{n-1}\ge x_n\ge ...\ge x_1$, the expression in (21) can be written as

$$(1-q)^2[-(q^{y_n-x_n-1}-1)q^{y_n-\max\{x_{n-1},y_{n-2}\}}+q^{y_n-x_n}q^{y_{n-1}-\max\{x_{n-1},y_{n-2}\}}]$$
$$\times\prod_{j=1}^{n-2}w(y_j-\max\{x_j,y_{j-1}\})$$
$$=(1-q)q^{y_n-y_{n-1}}\prod_{j=1}^{n-1}w(y_j-\max\{x_j,y_{j-1}\})=\prod_{j=1}^{n}w(y_j-\max\{x_j,y_{j-1}\}). \quad \square$$

Theorem 2.1 follows from Lemma 3.2 and the following convolution-type formula for determinants of the form we have.

Lemma 3.2. *Assume that $f,g\colon\mathbb{Z}\to\mathbb{C}$ are such that $f(x)=g(x)=0$ if $x<M$ for some $M\in\mathbb{Z}$. Then,*

$$\sum_{y\in W_n}\det(\Delta^{j-i}f(y_j-x_i))\det(\Delta^{j-i}g(z_j-y_i))=\det((\Delta^{j-i}f*g)(z_j-x_i)), \tag{22}$$

where all determinants are $n\times n$.

Proof. We will make use of the following summation by parts formula

$$\begin{aligned}\sum_{y=a}^{b}\Delta u(y-x)v(z-y)& \\ =\sum_{y=a}^{b}u(y-x)\Delta v(z-y)&+u(b+1-x)v(z-b)-u(a-x)v(z+1-a).\end{aligned} \tag{23}$$

The first step is to show that

$$\begin{aligned}\sum_{y\in W_n}\det(\Delta^{j-i}f(y_j-x_i))\det(\Delta^{j-i}g(z_j-y_i))& \\ =\sum_{y\in W_n}\det(\Delta^{1-i}f(y_j-x_i))\det(\Delta^{i-1}g(z_i-y_j))&\end{aligned} \tag{24}$$

by repeated summation by parts. The left-hand side of (24) can be written as

$$\begin{aligned}\sum_{y\in W_n}\Delta_{y_n}\det(\Delta^{1-i}f(y_1-x_i)...\Delta^{n-1-i}f(y_{n-1}-x_i)\Delta^{n-1-i}f(y_n-x_i))& \\ \times\det(\Delta^{i-1}g(z_i-y_1)...\Delta^{i-n}g(z_i-y_n)).&\end{aligned}$$

Here we have taken out one finite difference operator in the first determinant and taken the transpose of the second determinant. Next we use the summation by parts formula (23) to sum y_n between y_{n-1} and ∞. The terms coming from $u(b+1-x)v(z-b)-u(a-x)v(z+1-a)$ in (23) with $b\to\infty$ and $a=y_{n-1}$ give 0 since $\Delta^{i-n}g(z_i-b)=0$ if b is large enough and the other term gives rise to a determinant with two equal columns containing $\Delta^{n-1-i}f(y_{n-1}-x_i)$. The result is

$$\begin{aligned}\sum_{y\in W_n}\det(\Delta^{1-i}f(y_1-x_i)...\Delta^{n-1-i}f(y_{n-1}-x_i)\Delta^{n-1-i}f(y_n-x_i))& \\ \times\det(\Delta^{i-1}g(z_i-y_1)...\Delta^{i-n+1}g(z_i-y_{n-1})\Delta^{i-n+1}g(z_i-y_n)).&\end{aligned}$$

We can now repeat this procedure with $y_{n-1}, y_{n-2}, ..., y_2$, which gives

$$\sum_{y\in W_n} \det(\Delta^{1-i}f(y_1-x_i)\Delta^{1-i}f(y_2-x_i)\Delta^{2-i}f(y_3-x_i)...\Delta^{n-1-i}f(y_n-x_i))$$
$$\times \det(\Delta^{i-1}g(z_i-y_1)\Delta^{i-1}g(z_i-y_2)\Delta^{i-2}g(z_i-y_3)...\Delta^{i-n+1}g(z_i-y_n)).$$

Again we repeat the summation by parts procedure with $y_n, ..., y_3$, then with $y_n, ..., y_4$ and so on until we get the right-hand side of (24).

Next, we apply the generalized Cauchy–Binet identity (12) to the right-hand side of (24). This gives

$$\det\left(\sum_{y\in\mathbb{Z}} \Delta^{1-i}f(y-x_i)\Delta^{j-1}g(z_j-y)\right).$$

To prove the lemma it remains to show that

$$\sum_{y\in\mathbb{Z}} \Delta^{1-i}f(y-x)\Delta^{j-1}g(z-y) = \Delta^{j-i}(f*g)(z-x).$$

If we set $h(x)=H(x-1)$, then $\Delta^{-1}f(x)=h*f(x)$ and hence

$$\sum_{y\in\mathbb{Z}} \Delta^{1-i}f(y-x)\Delta^{j-1}g(z-y) = \Delta_z^{j-1}(h^{*(i-1)}*f)*g(z-x)$$
$$= \Delta_z^{j-1}(\Delta^{1-i}(f*g))(z-x) = \Delta^{j-i}(f*g)(z-x).$$

Here h^{*j} denotes the j-fold convolution of h with itself. □

To prove Theorem 2.2 we note that by Theorem 2.1,

$$\mathbb{P}[G(m,n)\le\eta] = \sum_{x_1\le\ldots\le x_n\le\eta} \det(\Delta^{j-i}w_m(x_j))$$
$$= \sum_{x_1\le\ldots\le x_{n-1}\le\eta} \sum_{x_n=x_{n-1}}^{\eta} \det(\Delta^{j-i}w_m(x_j)).$$

Now,

$$\sum_{x_n=x_{n-1}}^{\eta} \det(\Delta^{j-i}w_m(x_j))$$
$$= \det(\Delta^{1-i}w_m(x_1)...\Delta^{n-1-i}w_m(x_{n-1})\Delta^{n-1-i}w_m(\eta+1) - \Delta^{n-1-i}w_m(x_{n-1}))$$
$$= \det(\Delta^{1-i}w_m(x_1)...\Delta^{n-1-i}w_m(x_{n-1})\Delta^{n-1-i}w_m(\eta+1)).$$

Repeated use of this argument proves Theorem 2.2.

3.2. Proofs of Propositions 2.4 and 2.5

The generating function for $w(x)$ is

$$\sum_{x\in\mathbb{Z}} w(x)z^x = \frac{1-q}{1-qz}$$

and hence, since w_m is the m-fold convolution of w with itself,

$$w_m(x) = \frac{(1-q)^m}{2\pi i}\int_{\gamma_r} \frac{dz}{(1-qz)^m z^{x+1}},$$

where the radius r of the circle γ_r, centered at the origin, satisfies $0<r<1/q$, and i throughout the paper in the expression $2\pi i$ denotes the imaginary unit, whereas it elsewhere does not.

It follows that

$$\Delta^k w_m(x) = \frac{(1-q)^m}{2\pi i}\int_{\gamma_r} \frac{(1-z)^k}{(1-qz)^m z^{x+k+1}}\, dz \tag{25}$$

for $k\geq 0$ (and also for $k<0$ if $r<1$).

As observed above, if $h(x)=H(x-1)$, then $\Delta^{-1}f(x)=h*f(x)$ and therefore $\Delta^{-k}f(x)=(h^{*k}*f)(x)$. Using e.g. generating functions it is not difficult to see that

$$h^{*k}(x) = \frac{(x-1)^{[k-1]}}{(k-1)!} H(x-k), \tag{26}$$

where $y^{[k]}=y(y-1)...(y-k+1)$ is the factorial power. We can write

$$\Delta^{j-i-1}w_m(x) = \Delta^{-i}(\Delta^{j-1}w_m)(x) = \sum_{y\in\mathbb{Z}} h^{*i}(x-y)\Delta^{j-1}w_m(y).$$

Note that $h^{*i}(x-y)=0$ if $x-y<i$ by (26) and hence $h^{*i}(x-y)=0$ if $y>x-1$ for any $i\geq 1$. We obtain

$$\Delta^{j-i-1}w_m(\eta+1) = \sum_{y=-\infty}^{\eta} \frac{(\eta-y)^{[i-1]}}{(i-1)!}\Delta^{j-1}w_m(y). \tag{27}$$

Fix $L\geq n-1$. By (27) and some row operations we find that

(28)

$$\begin{aligned}\det(\Delta^{j-i-1}w_m(\eta+1))_{1\leq i,j\leq n} &= \det\left(\sum_{y=-\infty}^{\eta} \frac{(y+L)^{i-1}}{(i-1)!}(-1)^{i-1}\Delta^{j-1}w_m(y)\right)_{1\leq i,j\leq n}\\ &= \det\left(\sum_{y=-L}^{\eta} \frac{(y+L)^{i-1}}{(i-1)!}(-1)^{j-1}\Delta^{j-1}w_m(y)\right)_{1\leq i,j\leq n},\end{aligned}$$

since it follows from (25) that $\Delta^{j-1}w_m(y)=0$ if $y\leq -n$ for $1\leq j\leq n$.

If we choose $L=n-1$ it follows from Theorem 2.2, (28) and the generalized Cauchy–Binet identity (12) that

(29)
$$\mathbb{P}[G(m,n)\le\eta]=\prod_{j=1}^{n-1}\frac{1}{j!}\sum_{y_1,\dots,y_n=0}^{\eta+n-1}\Delta_n(y)\det((-1)^{j-1}\Delta^{j-1}w_m(y_i-n+1))_{1\le i,j\le n}.$$

If $0\le k<n$ and $m\ge n$, then

$$(-1)^k\Delta^k w_m(y)=s_k(y)\prod_{l=k+1}^{m-1}(y+l)q^yH(y+n-1), \tag{30}$$

where s_k is a polynomial of degree k. To see this we can use (25). We see that the integral in (25) is zero if $x\le-(k+1)$, and in particular if $x\le-n$ for all k, $0\le k<n$. If we make the change of variables $z\mapsto 1/z$ and assume that $x\ge 1-m$, we find that

$$\begin{aligned}\Delta^k w_m(y)&=\frac{(1-q)^m}{2\pi i}\int_{\gamma_{1/r}}\frac{(z-1)^k}{(z-q)^m}z^{y+m-1}\,dz\\ &=\sum_{r=0}^{y+m-1}\binom{y+m-1}{r}\frac{(1-q)^m}{2\pi i}\int_{\gamma_{1/r}}(z-1)^k(z-q)^{r-m}q^{y+m-1-r}\,dz.\end{aligned}$$

This is a polynomial of degree $m-1$ in y times q^y, and this polynomial has zeros at $-(k+1),\dots,1-m$. Consequently (30) follows.

Furthermore we have the following determinantal identity. Let p_j, $j=0,\dots,n-1$, be polynomials of degree j and $A_2,\dots,A_{n-1}$ constants. Then there is a constant B such that

$$\det\Bigg(p_{j-1}(x_i)\prod_{k=j+1}^{n}(x_i+A_k)\Bigg)_{1\le i,j\le n}=B\Delta_n(x). \tag{31}$$

This is not hard to see. Choose c_{rj} so that

$$\sum_{r=1}^{n}x^{r-1}c_{rj}=p_{j-1}(x)\prod_{k=j+1}^{n}(x+A_k).$$

Then, the left-hand side of (31) is

$$\det\Bigg(\sum_{r=1}^{n}x_i^{r-1}c_{rj}\Bigg)_{1\le i,j\le n}=\Delta_n(x)\det C,$$

and we have proved (31) with $B = \det C$. Actually, according to [12] we have

$$B = \prod_{j=1}^{n} (-1)^{j-1} p_{j-1}(-A_j),$$

but we will not need this result.

By (29), (30) and (31) we obtain

$$\begin{aligned}\mathbb{P}[G(m,n) \le \eta] &= C_{m,n} \sum_{y_1,\dots,y_n=0}^{\eta+n-1} \Delta_n(y)^2 \prod_{j=1}^{n} q^{y_j} \prod_{k=n}^{m-1} (y_j+k+1-n) \\ &= \frac{1}{Z_{m,n}} \sum_{y_1,\dots,y_n=0}^{\eta+n-1} \Delta_n(y)^2 \prod_{j=1}^{n} \binom{y_j+m-n}{y_j} q^{y_j}\end{aligned}$$

for some constants $C_{m,n}$ and $Z_{m,n}$. If we let $\eta \to \infty$ we see that $Z_{m,n}$ must be exactly the normalization constant in the Meixner ensemble. This proves Proposition 2.4.

We now turn to the proof of Proposition 2.5. Write $K = m-n+1$ and define, for $0 \le j < n$ and $x \in \mathbb{Z}$,

$$a_j(x) = \frac{q-1}{2\pi i} \int_{\gamma_{r_2}} z^{x-1} \frac{(qz-1)^{j+K-1}}{(z-1)^{j+1}}\, dz$$

and

$$b_j(x) = \frac{1}{2\pi i} \int_{\gamma_{r_1}} \frac{(w-1)^j}{w^x (qw-1)^{j+k}}\, dw,$$

where $1 < r_2 < r_1 < 1/q$. If $x \ge 0$ and $0 \le j < n$, then

$$a_j(x) = \frac{(q-1)^{j+K}}{j!} p_j(x-n), \tag{32}$$

where p_j is a monic polynomial of degree 1. This follows from the computation

$$\begin{aligned}a_j(x+n) &= \frac{q-1}{2\pi i} \int_{\gamma_{r_2}} z^{x+n-1} \frac{(qz-1)^{j+K-1}}{(z-1)^{j+1}}\, dz \\ &= \frac{q-1}{2\pi i} \int_{\gamma_{r_2}} \sum_{r=0}^{x+n-1} \binom{x+n-1}{r} (z-1)^{r-j-1} (qz-1)^{j+K-1}\, dz \\ &= \frac{q-1}{2\pi i} \int_{\gamma_{r_2}} \sum_{r=0}^{j} \binom{x+n-1}{r} (z-1)^{r-j-1} (qz-1)^{j+K-1}\, dz.\end{aligned}$$

We see that this is a polynomial of degree j in x with leading coefficient $(q-1)^{j+K}/j!$.

It follows from Theorem 2.2 and (28) with $L=n$ that

$$\mathbb{P}[G(m,n)\le\eta]=\det\left(\sum_{y=0}^{\eta+n}\frac{y^i}{i!}(-1)^j\Delta^j w_m(y-n)\right)_{0\le i,j<n}, \tag{33}$$

where, by (25),

$$(-1)^j\Delta^j w_m(y-n)=\frac{(1-q)^m}{2\pi i}\int_{\gamma_{r_1}}\frac{(z-1)^j}{(1-qz)^{n+K-1}z^{y+j-n+1}}\,dz. \tag{34}$$

Set

$$c_{jl}=\binom{n-l-1}{j-l}H(j-l).$$

Then,

$$\sum_{j=0}^{n-1}\frac{(-1)^j}{(q-1)^{j+K}}\Delta^j w_m(y-n)c_{jl}=b_l(y). \tag{35}$$

To show this it is sufficient to show that

$$\sum_{j=0}^{n-1}\frac{(1-q)^{m-j-K}(-1)^{j+K}(z-1)^j}{(1-qz)^{n+K-1}z^{y+j-n+1}}c_{jl}=\frac{(z-1)^l}{(qz-1)^{l+K}z^y} \tag{36}$$

by (34) and the definition of b_l. The identity (36) can be rewritten as

$$\sum_{j=0}^{n-1}\left(1-\frac{1}{z}\right)^j(1-q)^{m-j-K}(-1)^{j+K}c_{jl}=(-1)^K(1-qz)^{n-l-1}(1-z)^l\frac{1}{z^{n-1}}. \tag{37}$$

Set $\zeta=1-1/z$. Then (37) is equivalent to

$$\sum_{j=0}^{n-1}\zeta^j(1-q)^{n-j-1}(-1)^j c_{jl}=(1-q-\zeta)^{n-l-1}\zeta^l(-1)^l.$$

By the binomial theorem

$$\begin{aligned}(1-q-\zeta)^{n-l-1}\zeta^l(-1)^l&=\sum_{r=0}^{n-l-1}\binom{n-l-1}{r}(-\zeta)^{r+l}(1-q)^{n-l-1-r}\\&=\sum_{j=l}^{n-1}\binom{n-l-1}{j-l}(-1)^j(1-q)^{n-j-1}\zeta^j,\end{aligned}$$

and we have proved (35). Note that $\det(c_{jl})=1$ and hence by (32), (33) and (35),

$$\begin{aligned}(38)\quad \mathbb{P}[G(m,n)\le\eta] &= \det\left(\sum_{y=0}^{\eta+n}\frac{(-1)^{i+K}}{i!}y^i\frac{(-1)^j}{(q-1)^{j+K}}\Delta^j w_m(y-n)\right)_{0\le i,j<n}\\ &= \det\left(\sum_{y=0}^{\eta+n}a_i(y)\sum_{j=0}^{n-1}\frac{(-1)^j}{(q-1)^{j+K}}\Delta^j w_m(y-n)c_{jl}\right)_{0\le i,l<n}\\ &= \det\left(\sum_{y=0}^{\eta+n}a_i(y)b_j(y)\right)_{0\le i,j<n}.\end{aligned}$$

We will now use the fact that

$$\sum_{y=0}^{\infty}a_j(y)b_k(y)=\delta_{jk}. \tag{39}$$

To prove this we use the definitions of a_j and b_k,

$$\begin{aligned}\sum_{x=0}^{\infty}a_j(x)b_k(x) &= \frac{q-1}{(2\pi i)^2}\int_{\gamma_{r_2}}\int_{\gamma_{r_1}}\left(\sum_{x=0}^{\infty}\left(\frac{z}{\zeta}\right)^x\right)\frac{(qz-1)^{j+K-1}(\zeta-1)^k}{(z-1)^{j+1}(q\zeta-1)^{k+K}}\,d\zeta\,\frac{dz}{z}\\ &= \frac{q-1}{(2\pi i)^2}\int_{\gamma_{r_2}}\int_{\gamma_{r_1}}\frac{\zeta}{z}\frac{1}{\zeta-z}\frac{(qz-1)^{j+K-1}(\zeta-1)^k}{(z-1)^{j+1}(q\zeta-1)^{k+K}}\,d\zeta\,dz.\end{aligned}$$

Since $1<r_2<r_1<1/q$ we see that $\zeta=z$ is the only pole in the ζ-integral and hence by the residue theorem this equals

$$\frac{q-1}{2\pi i}\int_{\gamma_{r_2}}(qz-1)^{j-k-1}(z-1)^{k-j-1}\,dz.$$

If $j<k$ the integral is zero by Cauchy's theorem. If $j>k$ we make the change of variables $z'=1/z$ and we see again that the integral is zero. When $j=k$ we get

$$\frac{q-1}{2\pi i}\int_{\gamma_{r_2}}(qz-1)^{-1}(z-1)^{-1}\,dz=1.$$

Using (39) in (38) we get

$$\begin{aligned}\mathbb{P}[G(m,n)\le\eta] &= \det\left(\delta_{ij}-\sum_{y=\eta+n+1}^{\infty}a_i(y)b_j(y)\right)_{0\le i,j<n}\\ &= \det(I-K_{m,n})_{\ell^2(\{\eta+1,\eta+2,\dots\})},\end{aligned}$$

where

$$K_{m,n}(x,y)=\sum_{j=0}^{n-1}a_j(x+n)b_j(y+n)$$
$$=\frac{q-1}{(2\pi i)^2}\int_{\gamma_{r_2}}\int_{\gamma_{r_1}}\frac{z^{x+n}}{\zeta^{y+n}}\frac{(qz-1)^{K-1}}{(q\zeta-1)^K(z-1)}\sum_{j=0}^{n-1}\left[\frac{(\zeta-1)(qz-1)}{(z-1)(q\zeta-1)}\right]^j d\zeta\,\frac{dz}{z}$$
$$=\frac{1}{(2\pi i)^2}\int_{\gamma_{r_2}}\int_{\gamma_{r_1}}\frac{z^{x+n}}{\zeta^{y+n}}\frac{(1-qz)^{K-1}}{(1-q\zeta)^{K-1}}\left[\frac{(1-\zeta)^n(1-qz)^n}{(1-z)^n(1-q\zeta)^n}-1\right]\frac{1}{\zeta-z}\,d\zeta\,\frac{dz}{z}$$
$$=\frac{1}{(2\pi i)^2}\int_{\gamma_{r_2}}\int_{\gamma_{r_1}}\frac{z^{x+n}}{\zeta^{y+n}}\frac{(1-qz)^{K-1}}{(1-q\zeta)^{K-1}}\frac{(1-\zeta)^n(1-qz)^n}{(1-z)^n(1-q\zeta)^n}\frac{\zeta}{\zeta-z}\frac{d\zeta}{\zeta}\frac{dz}{z}.$$

Here we have used the fact that by Cauchy's theorem the -1 term in the third line gives a zero contribution in the z-integral since $r_1>r_2$. This proves Proposition 2.5.

Remark 3.3. The Meixner ensemble is related to Meixner polynomials since these polynomials are orthogonal on $\mathbb{N}$ with respect to the negative binomial weight. These polynomials are not used explicitly in the computations above but figure in the background. This can be seen from the Rodrigues' formula,

$$p_j^{(m,q)}(x)q^{j+x}\binom{x+m-1}{x}=\Delta^j\left[\binom{x+m-1}{x}q^x\prod_{k=0}^{j-1}(x-k)\right],$$

and the integral representation

$$p_j^{(m,q)}(x)=\frac{j!}{2\pi i}\int_{\gamma_r}\frac{(1-z/q)^x}{(1-z)^{x+K}}\frac{dz}{z^{j+1}},$$

with $0<r<1$, where $p_j^{(m,q)}(x)$ are the standard Meixner polynomials, [11].

Acknowledgements. I thank Jon Warren for a discussion and for sending me a preliminary version of [5].

References

1. Baryshnikov, Y., GUEs and queues, *Probab. Theory Related Fields* **119** (2001), 256–274.
2. Borodin, A., Okounkov, A. and Olshanski, G., Asymptotics of Plancherel measures for symmetric groups, *J. Amer. Math. Soc.* **13** (2000), 481–515.
3. Borodin, A. and Olshanski, G., Stochastic dynamics related to Plancherel measure on partitions, in *Representation Theory, Dynamical Systems, and Asymptotic Combinatorics*, Amer. Math. Soc. Transl. **217**, pp. 9–21, Amer. Math. Soc., Providence, RI, 2006.

4. Brankov, J. G., Priezzhev, V. B. and Shelest, R. V., Generalized determinant solution of the discrete-time totally asymmetric exclusion process and zero-range process, *Phys. Rev. E* **69** (2004), 066136, 9.
5. Dieker, A. B. and Warren, J., Determinantal transition kernels for some interacting particles on the line, *Preprint*, 2007. `arXiv:0707.1843`.
6. Gravner, J., Tracy, C. A. and Widom, H., A growth model in a random environment, *Ann. Probab.* **30**:3 (2002), 1340–1368.
7. Johansson, K., Shape fluctuations and random matrices, *Comm. Math. Phys.* **209** (2000), 437–476.
8. Johansson, K., Discrete orthogonal polynomial ensembles and the Plancherel measure, *Ann. of Math.* **153** (2001), 259–296.
9. Johansson, K., Discrete polynuclear growth and determinantal processes, *Comm. Math. Phys.* **242** (2003), 277–329.
10. Johansson, K., Random matrices and determinantal processes, in *Mathematical Statistical Physics, Session* **83** (*Les Houches, 2005*), Chapter **1**, Elsevier, Amsterdam, 2006.
11. Koekoek, R. and Swarttouw, R., The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics, Report no. 98-17, 1998 available at
`http://fa.its.tudelft.nl/~koekoek/askey/`.
12. Krattenthaler, C., Advanced determinant calculus, *Séminaire Lotharingien de Combinatoire* **B42q** (1999), 67pp.
13. Nagao, T. and Sasamoto, T., Asymmetric simple exclusion process and modified random matrix ensembles, *Nuclear Phys. B* **699** (2004), 487–502.
14. Okounkov, A., Infinite wedge and random partitions, *Selecta Math.* **7** (2001), 57–81.
15. Rákos, A. and Schütz, G. M., Current distribution and random matrix ensembles for an integrable asymmetric fragmentation process, *J. Stat. Phys.* **118** (2005), 511–530.
16. Schütz, G. M., Exact solution of the master equation for the asymmetric exclusion process, *J. Stat. Phys.* **88** (1997), 427–445.
17. Tracy, C. A. and Widom, H., Integral formulas for the asymmetric simple exclusion process, *Comm. Math. Phys.* **279** (2008), 815–844.
18. Warren, J., Dyson's Brownian motions, intertwining and interlacing, *Electron. J. Probab.* **12** (2007), 573–590.

Kurt Johansson
Department of Mathematics
Royal Institute of Technology
SE-100 44 Stockholm
Sweden
kurtj@kth.se

Received November 15, 2007
published online November 6, 2008