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A complete classification of homogeneous plane continua

by

Logan C. Hoehn
Nipissing University
North Bay, ON, Canada

Lex G. Oversteegen
University of Alabama at Birmingham
Birmingham, AL, U.S.A.

Dedicated to Andrew Lelek on the occasion of his 80th birthday.

1. Introduction

By a *compactum*, we mean a compact metric space, and by a *continuum*, we mean a compact connected metric space. A continuum is *non-degenerate* if it contains more than one point. We refer to the space $\mathbb{R}^2$, with the Euclidean topology, as *the plane*. By a *map* we mean a continuous function.

A space X is (topologically) *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h: X \to X$ with $h(x)=y$. All homeomorphisms in this paper are onto.

The concept of topological homogeneity was first introduced by Sierpiński in [54]. Since the underlying/ambient space of many topological models is homogeneous, the classification of homogeneous spaces has a long and rich history. For example, all connected manifolds are homogeneous, and the Hilbert cube $[0,1]^{\mathbb{N}}$, which contains a homeomorphic copy of every compact metric space, is an example of an infinite-dimensional homogeneous continuum. Even for low dimensions, the classification of homogeneous Riemannian manifolds remains an active area of research today. Contrary to naive expectation, homogeneous continua do not necessarily have a simple local structure (in particular, they do not need to contain a manifold). As a consequence, even the classification of 1-dimensional homogeneous continua appears out of reach. This paper concerns the classification of homogeneous compact subsets of the plane.

In the first volume of Fundamenta Mathematicae in 1920, Knaster and Kuratowski [23] asked (Problème 2) whether the circle is the only (non-degenerate) homogeneous

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plane continuum. Mazurkiewicz [39] showed early on that the answer is yes if the continuum is locally connected. Cohen [6] showed that the answer is yes if the continuum is arcwise connected or, equivalently, pathwise connected, and Bing [4] proved more generally that the answer remains yes if the continuum simply contains an arc. A continuum X is *decomposable* if it is the union of two proper subcontinua and *indecomposable* otherwise. A continuum is *hereditarily decomposable* (*hereditarily indecomposable*) if every non-degenerate subcontinuum is decomposable (indecomposable, respectively). Hagopian [11] showed that the answer to the question of Knaster and Kuratowski is still yes if the continuum merely contains a hereditarily decomposable subcontinuum.

Problème 2 by Knaster and Kuratowski was formally solved by Bing [2] who showed in 1948 that the pseudo-arc, described in detail in §1.1, is another homogeneous plane continuum. The pseudo-arc is a one-dimensional fractal-like hereditarily indecomposable continuum (in particular it contains no arcs). This stunning example of a homogeneous continuum shows that homogeneity is possible at two extremes: one where the local structure is simple (e.g. for locally connected spaces) and one where the local structure is not simple (e.g. for not locally connected spaces). Since Bing's surprising solution, the question has been: What are all homogeneous plane continua? A third homogeneous plane continuum, called the circle of pseudo-arcs (since it admits an open map to the circle whose point preimages are all pseudo-arcs), was added by Bing and Jones [5] in 1954. We show in this paper that these three comprise the complete list of all homogeneous non-degenerate plane continua.

Even though hereditarily indecomposable continua seem to be obscure objects, they arise naturally in mathematics, for example as attractors in dynamical systems [21] (even for an open set of parameters).

Another hereditarily indecomposable continuum, the *pseudo-circle*, was considered to be a strong candidate to be an additional example of a homogeneous plane continuum. However, it was proved to be non-homogeneous independently by Fearnley [10] and Rogers [48].

This long-standing question of the classification of all homogeneous plane continua has been raised and/or addressed in several papers and surveys, including [18], [19], [20], [34], [35], [36], [51], [52], [53], and the "New Scottish Book" (Problem 920). The first explicit statement concerning this problem that we could find is in [15].

There exists a rich literature concerning homogeneous continua (including several excellent surveys, such as [37], [51], and [52]) so we will only briefly state some pertinent highlights here.

In 1954, Jones proved the following result.

Theorem A. ([16]) *If M is a decomposable homogeneous continuum in the plane, then M is a circle of mutually homeomorphic indecomposable homogeneous continua.*

The conclusion of this theorem implies that there is an indecomposable homogeneous continuum X (possibly a single point) and an open map from M to the circle all of whose point preimages are homeomorphic to X. Bing and Jones [5] constructed in 1954 such a continuum in the plane for which X is the pseudo-arc, and also proved that it is homogeneous. This example is known as the *circle of pseudo-arcs* (see §1.1).

It follows from this theorem of Jones that every decomposable homogeneous continuum in the plane separates the plane. Rogers [49] proved that conversely, every homogeneous plane continuum which separates the plane is decomposable.

Hagopian (see also [18]) obtained in 1976 the following result.

Theorem B. ([12]) *Every indecomposable homogeneous plane continuum is hereditarily indecomposable.*

A map $f\colon X\to Y$ is called an *ε-map* if, for each $y\in Y$, $\operatorname{diam}(f^{-1}(y))<\varepsilon$. A continuum X is *arc-like* (respectively, *tree-like*) provided that for each $\varepsilon>0$ there exists an ε-map from X to an arc (respectively, tree). Bing [3] proved in 1951 that the pseudo-arc is the only hereditarily indecomposable arc-like continuum. Hence, to show that an indecomposable homogeneous plane continuum is homeomorphic to the pseudo-arc, by the results of Hagopian and Bing, it suffices to show that it is arc-like.

The main idea of our proof is based on a generalization of the following simple fact, which is central to much work done with the pseudo-arc.

- Let $f\colon[0,1]\to[0,1]$ be a piecewise linear map. For any $\varepsilon>0$, if $g\colon[0,1]\to[0,1]$ is a sufficiently crooked map, then there is a map $h\colon[0,1]\to[0,1]$ such that the composition $f\circ h$ is ε-close to g.

See §1.1 for a formal definition of "crookedness". See also Theorems 8 and 20 below for related properties.

We will prove a generalization of the above statement, where instead of $[0,1]$ we consider graphs, and we restrict to a certain class of piecewise linear maps f. To describe how this result pertains to the study of homogeneous plane continua, we provide some context below.

It is in general a difficult task to prove that a given continuum is (or is not) arc-like. A closely related notion, introduced by Lelek in 1964 [29], is that of span zero. A continuum X has *span zero* if for any continuum C and any two maps $f,g\colon C\to X$ such that $f(C)\subseteq g(C)$, there exist $p\in C$ with $f(p)=g(p)$ (by [8] this is equivalent to the traditional definition of span zero where the images of f and g coincide). It is easy to see that every arc-like continuum has span zero [29]. Moreover, in some cases it is easier to show that a continuum X has span zero than to show that it is arc-like. For example, the following theorem was obtained in the early 1980s.

THEOREM C. ([45]) *Every homogeneous indecomposable plane continuum has span zero.*

It was a long standing open problem whether each continuum of span zero is arc-like. Unfortunately the answer was shown to be negative in [13]. The example given in [13] relied heavily on the existence of patterns which required the continuum to contain arcs. Such patterns are not possible for hereditarily indecomposable continua. Indeed, using our generalization of the above result about crooked maps between arcs, we in this paper prove the following theorem.

THEOREM 1. *A non-degenerate continuum X is homeomorphic to the pseudo-arc if and only if X is hereditarily indecomposable and has span zero.*

We suspect that this result will be useful in other contexts as well, for example, in the classification of attractors in certain dynamical systems.

It follows immediately from Theorems B and C above and Theorem 1 that every indecomposable non-degenerate homogeneous plane continuum is a pseudo-arc. Combining this with Theorem A above, we obtain the following classification of homogeneous plane continua.

THEOREM 2. *Up to homeomorphism, the only non-degenerate homogeneous continua in the plane are*

(1) *the circle*,

(2) *the pseudo-arc*, *and*

(3) *the circle of pseudo-arcs.*

Finally, if Y is a homogeneous compactum then by [41] (see also [1] and [42]) Y is homeomorphic to $X \times Z$, where X is a homogeneous continuum and Z is a 0-dimensional homogeneous compactum and, hence, either a finite set or the Cantor set. Thus we obtain the following corollary.

THEOREM 3. *Up to homeomorphism, the only homogeneous compact spaces in the plane are*

(1) *finite sets*,

(2) *the Cantor set*, *and*

(3) *the spaces $X \times Z$, where X is a circle*, *a pseudo-arc*, *or a circle of pseudo-arcs*, *and Z is either a finite set or the Cantor set.*

The paper is organized as follows. After fixing some definitions and notation in §2, we draw a connection in §3 between the property of span zero and sets in the product of a graph G and the interval $[0,1]$ which separate $G\times\{0\}$ from $G\times\{1\}$. For the rest of the paper after this, we focus our attention on these separators, rather than work with span directly. In §4, we characterize hereditarily indecomposable compacta in terms of simple piecewise linear functions between graphs.

In §5, we introduce a special type of separating set in the product of a graph with the interval, and prove that such separators are in a certain sense dense in the set of all separators. §6 is devoted to some technical results towards showing that such special separators can be "unfolded" by simple piecewise linear maps. Finally, in §7 we bring everything together and prove our main result, Theorem 1 above. §8 includes some discussion and open questions.

1.1. The pseudo-arc

In this subsection we give a brief introduction to the pseudo-arc, and describe some of its most important properties.

The pseudo-arc is the most well-known example of a hereditarily indecomposable continuum. It is a very exotic and complex space with many remarkable and strange properties, yet it is also in some senses ubiquitous and quite natural.

Most descriptions of the pseudo-arc involve some notion of "crookedness". We will appeal to the notion of a crooked map, as follows.

An onto map $g:[0,1]\to[0,1]$ is considered crooked if, roughly speaking, as x travels from 0 to 1, $g(x)$ goes back and forth many times, on large and on small scales in $[0,1]$. More precisely, given $\delta>0$, we say g is *δ-crooked* if there is a finite set $F\subset[0,1]$ which is a δ-net for $[0,1]$ (i.e. each point of $[0,1]$ is within distance δ from some point of F), such that whenever y_1,y_2,y_3,y_4 is an increasing or decreasing sequence of points in F, and $x_1,x_4\in[0,1]$ with $x_1<x_4$, $g(x_1)=y_1$ and $g(x_4)=y_4$, there are points $x_2,x_3\in[0,1]$ such that $x_1<x_2<x_3<x_4$ and $g(x_2)=y_3$, $g(x_3)=y_2$.

To construct the pseudo-arc, one should choose a sequence of onto maps $g_n:[0,1]\to[0,1]$, $n=1,2,...$, such that, for each n and each $1\leqslant k\leqslant n$, the composition $g_k\circ g_{k+1}\circ...\circ g_n$ is $(1/n)$-crooked. The *pseudo-arc* is then the inverse limit of this sequence, $\varprojlim([0,1],g_n)$.

The pseudo-arc, as constructed by this procedure, is a hereditarily indecomposable arc-like continuum. According to Bing's characterization theorem [3], any two continua which are both hereditarily indecomposable and arc-like are homeomorphic. Thus the pseudo-arc is the unique continuum with these properties. This also means that the particular choices of maps g_n in the above construction are not important—so long as the crookedness properties are satisfied, the resulting inverse limit will be the same space.

One can equivalently construct the pseudo-arc in the plane as the intersection of a nested sequence of "snakes" (homeomorphs of the closed unit disk) which are nested inside one another in a manner reminiscent of the crooked pattern for maps described above.

Because of the enormous extent of crookedness inherent in the pseudo-arc, it is impossible to draw an informative, accurate raster image of this space (see [38] for a detailed explanation). Nevertheless, the pseudo-arc is in some sense ubiquitous: in any manifold M of dimension at least 2, the set of subcontinua homeomorphic to the pseudo-arc is a dense G_δ subset of the set of all subcontinua of M (equipped with the Vietoris topology). The pseudo-arc is a universal object in the sense that it is arc-like, and every arc-like continua is a continuous image of it.

The pseudo-arc has an interesting history of discovery. It was first constructed by Knaster [22] in 1922 as the first example of a hereditarily indecomposable continuum. Moise [43] in 1948 constructed a similar example, which has the remarkable property that it is homeomorphic to each of its non-degenerate subcontinua. Moise named this space the "pseudo-arc", since the interval $[0,1]\subset\mathbb{R}$ is the only other known space which shares this same property. Also in 1948, Bing [2] constructed another similar example which he proved was homogeneous, and thus answering the original question of Knaster and Kuratowski about homogeneous continua in the plane. Shortly after this, in 1951 Bing published the characterization theorem stated above, from which it follows that all three of these examples are in fact the same space.

Not only is the pseudo-arc homogeneous, but in fact it satisfies the following stronger properties:

(1) given a collection of n points $x_1,...,x_n$, no two of which belong to any proper subcontinuum of the pseudo-arc, and given another such collection $y_1,...,y_n$, there is a homeomorphism h of the pseudo-arc to itself such that $h(x_i)=y_i$ for each $i=1,...,n$ [27];

(2) given two points x and y and an open subset U, if there is a subcontinuum of the pseudo-arc containing x and y which is disjoint from $\overline{U}$, then there is a homeomorphism h of the pseudo-arc to itself such that $h(x)=y$ and h is the identity on U [33].

These properties should be compared with similar ones satisfied by the circle $\mathbb{S}^1$:

(1′) given two sets of n points $x_1,...,x_n,y_1,...,y_n\in\mathbb{S}^1$, both arranged in circular order, there is a homeomorphism h of $\mathbb{S}^1$ to itself such that $h(x_i)=y_i$ for each $i=1,...,n$;

(2′) given two points x and y and an open subset U, if there is a subarc of $\mathbb{S}^1$ containing x and y which is disjoint from $\overline{U}$, then there is a homeomorphism h of $\mathbb{S}^1$ to itself such that $h(x)=y$ and h is the identity on U.

In 1954, Bing and Jones [5] constructed a space called the *circle of pseudo-arcs*. This is a circle-like continuum which admits an open map to the circle whose point preimages are pseudo-arcs (a continuum X is *circle-like* if for any $\varepsilon>0$ there exists an ε-map from X to the circle $\mathbb{S}^1$). Bing and Jones proved that the circle of pseudo-arcs is homogeneous and that it is unique, in the sense that it is the only continuum (up to homoemorphism) with the above properties. The circle of pseudo-arcs should not be confused with the product of the pseudo-arc with $\mathbb{S}^1$ (which is homogeneous but not embeddable in the plane), or with another related space called the *pseudo-circle* (which is a hereditarily indecomposable circle-like continuum in the plane, but is not homogeneous—see [10] and [48]).

2. Definitions and notation

An *arc* is a space homeomorphic to the interval $[0,1]$. A *graph* is a space which is the union of finitely many arcs which intersect at most in endpoints. Given a graph G and a point $x\in G$, x is an *endpoint* if x is not a cutpoint of any connected neighborhood of x in G, and x is a *branch point* if x is a cutpoint of order $\geqslant 3$ in some connected neighborhood of x in G.

The *Hilbert cube* is the space $[0,1]^{\mathbb{N}}$, with the standard product metric d. It has the property that any compact metric space embeds in it. For this reason, we will assume throughout this paper that any compacta we consider are embedded in $[0,1]^{\mathbb{N}}$, and use this same metric d for all of them.

Given two functions $f,g\colon X\to Y$ between compacta X and Y, we use the *supremum metric* to measure the distance between f and g, defined by

$$d_{\sup}(f,g)=\sup\{d(f(x),g(x)):x\in X\}.$$

Given two non-empty subsets A and B of a compactum X, the *Hausdorff distance* between A and B is

$$d_H(A,B)=\inf\{\varepsilon>0:A\subset B_\varepsilon \text{ and } B\subset A_\varepsilon\},$$

where A_ε (respectively, B_ε) is the ε-neighborhood of A (respectively, B). It is well known that the hyperspace of all non-empty compact subsets of X, equipped with the Hausdorff metric, is compact.

3. Span and separators

In this section, we draw a correspondance between the property of span zero and the existence of certain separating sets in the product of a graph and an arc which approximate a continuum.

As in the introduction, a continuum X has *span zero* if whenever $f, g: C \to X$ are maps of a continuum C to X with $f(C) \subseteq g(C)$, there is a point $p \in C$ such that $f(p) = g(p)$. This can equivalently be formulated as follows: X has span zero if every subcontinuum $Z \subseteq X \times X$ with $\pi_1(Z) \subseteq \pi_2(Z)$ meets the diagonal $\Delta X = \{(x, x) : x \in X\}$ (here π_1 and π_2 are the first and second coordinate projections $X \times X \to X$, respectively). By [8], this is equivalent to the traditional definition of span zero where one insists that $\pi_1(Z) = \pi_2(Z)$.

The proof of the following theorem is implicit in results of [46]. We include a self-contained proof here for completeness.

We remark that in fact the property of "span zero" in this theorem could be replaced by the weaker property of "surjective semispan zero", which has the same definition as span zero except that one insists that $\pi_1(Z) \subseteq \pi_2(Z) = X$ [30].

Theorem 4. *Let $X \subset [0,1]^{\mathbb{N}}$ be a continuum in the Hilbert cube with span zero. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $G \subset [0,1]^{\mathbb{N}}$ is a graph and $I \subset [0,1]^{\mathbb{N}}$ is an arc with endpoints p and q, such that the Hausdorff distance from X to each of G and I is less than δ, then the set $M = \{(x, y) \in G \times (I \setminus \{p, q\}) : d(x, y) < \varepsilon\}$ separates $G \times \{p\}$ from $G \times \{q\}$ in $G \times I$.*

Proof. If the theorem were false, then there would exist $\varepsilon > 0$ and a sequence of graphs $\langle G_n \rangle_{n=1}^{\infty}$ and arcs $\langle I_n \rangle_{n=1}^{\infty}$ with endpoints p_n and q_n in $[0,1]^{\mathbb{N}}$, both converging to X in the Hausdorff metric, and such that the set $M_n = \{(x, y) \in G_n \times (I_n \setminus \{p_n, q_n\}) : d(x, y) < \varepsilon\}$ does not separate $G_n \times \{p_n\}$ from $G_n \times \{q_n\}$ for each $n = 1, 2, \ldots$. This would mean (see e.g. [44, Theorem 5.2]) that for every $n = 1, 2, \ldots$, there is a continuum $Z_n \subset G_n \times I_n$ meeting $G_n \times \{p_n\}$ and $G_n \times \{q_n\}$ (hence the second coordinate projection of Z_n is all of I_n), such that $d(x, y) \geqslant \varepsilon$ for all $(x, y) \in Z_n$.

Since $G_n \times I_n$ converges to $X \times X$, the sequence of continua Z_n accumulates on a continuum $Z \subset X \times X$. Clearly $d(x, y) \geqslant \varepsilon$ for all $(x, y) \in Z$, and the second coordinate projection of Z is X since the second coordinate projection of Z_n is I_n for each $n = 1, 2, \ldots$. This means that $Z \cap \Delta X = \varnothing$ and $\pi_1(Z) \subseteq \pi_2(Z) = X$, and hence X does not have span zero, a contradiction. $\square$

4. Simple folds

Throughout the remainder of this paper, G will denote a (not necessarily connected) graph. A subset A of G will be called *regular* if A is closed and has finitely many components, each of which is non-degenerate. Note that a regular set always has finite boundary.

The following definition is adapted from [47].

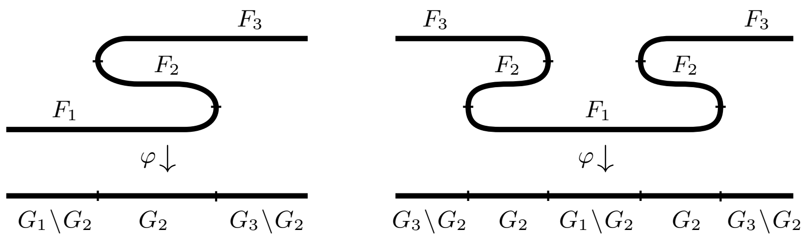

Figure 1. Two examples of simple folds $\varphi: F \to G$, where F and G are arcs. In both cases, the map φ is the vertical projection. Note that in the second example, the sets F_2, F_3, G_2, and G_3 are all disconnected (each has two components).

Definition 5. A *simple fold* on G is given by a graph $F = F_1 \cup F_2 \cup F_3$ and a function $\varphi: F \to G$, called the *projection*, which satisfy the following properties, where $G_i = \varphi(F_i)$ for $i = 1, 2, 3$:

(F1) G_1, G_2, and G_3 are non-empty regular subsets of G;

(F2) $G_1 \cup G_3 = G$, and $G_2 = G_1 \cap G_3$;

(F3) $\overline{G_1 \setminus G_2} \cap \overline{G_3 \setminus G_2} = \varnothing$;

(F4) $\varphi|_{F_i}$ is a homeomorphism of F_i onto G_i for each $i = 1, 2, 3$; and

(F5) $\partial G_1 = \varphi(F_1 \cap F_2)$, $\partial G_3 = \varphi(F_2 \cap F_3)$, and $F_1 \cap F_3 = \varnothing$.

Observe that property (F3) implies that $G_1 \cap G_3 \subseteq G_2$, so in (F2) we could replace the condition $G_2 = G_1 \cap G_3$ with $G_2 \subseteq G_1 \cap G_3$.

See Figure 1 for two examples of simple folds, in which both graphs F and G are arcs.

We record here some basic properties of simple folds. The proofs of these properties are left to the reader.

Lemma 6. *Let $F = F_1 \cup F_2 \cup F_3$ be a simple fold on G with projection $\varphi: F \to G$, and let $G_i = \varphi(F_i)$ for $i = 1, 2, 3$. Then, the following facts hold*:

(1) *$\partial G_2 = \partial G_1 \cup \partial G_3$ and $\partial G_1 \cap \partial G_3 = \varnothing$*;

(2) *$\partial(G_1 \setminus G_2) = \partial G_3$ and $\partial(G_3 \setminus G_2) = \partial G_1$*;

(3) *F_1, F_2, and F_3 are regular subsets of F*;

(4) *$F_1 \cap F_2$ and $F_2 \cap F_3$ are finite sets*;

(5) *$\partial F_1 = F_1 \cap F_2$, $\partial F_3 = F_2 \cap F_3$, and $\partial F_2 = \partial F_1 \cup \partial F_3$*;

(6) *∂F_1 separates $F_1 \setminus \partial F_1$ from $(F_2 \cup F_3) \setminus \partial F_1$ in F, and ∂F_3 separates $F_3 \setminus \partial F_3$ from $(F_1 \cup F_2) \setminus \partial F_3$ in F*;

(7) *$\varphi|_{F_i \setminus \partial F_i}: F_i \setminus \partial F_i \to G$ is an open map for each $i = 1, 2, 3$.*

To define a simple fold, it is enough to identify three subsets G_1, G_2, G_3 of G satisfying properties (F1)–(F3). Indeed, take spaces E_1, E_2, and E_3, with $E_i \approx G_i$, and take homeomorphisms $\varphi_i: E_i \to G_i$. Define $F = (E_1 \sqcup E_2 \sqcup E_3)/\sim$, where $\sim$ identifies pairs of the form $p \in E_i$, $q \in E_2$ with $\varphi_i(p) = \varphi_2(q) \in \partial G_i$ for $i = 1, 3$. Define F_i to be the projection of E_i in the quotient space F and define $\varphi\colon F\to G$ by $\varphi|_{F_i}=\varphi_i$, for each $i=1,2,3$. It is straightforward to see that this is a well defined simple fold, and if F' is another simple fold on G with projection φ' such that $\varphi'(F_i')=G_i$ for $i=1,2,3$, then there is a homeomorphism $\theta\colon F'\to F$ with $\theta(F_i')=F_i$ for $i=1,2,3$ and $\varphi'=\varphi\circ\theta$.

In general, even if G is connected, a simple fold F on G need not be connected. However, the next proposition shows that for connected G we can always reduce F to a connected simple fold.

Note that if G is connected and $\partial G_1=\varnothing$, then $G_1=G$, F is disconnected, and $\varphi|_{F_1}$ is a homeomorphism of F_1 onto G. Likewise, if $\partial G_3=\varnothing$, then $\varphi|_{F_3}$ is a homeomorphism of F_3 onto G. In light of this, we will assume $\partial G_1\neq\varnothing\neq\partial G_3$ in the following proposition.

Proposition 7. *Let $F=F_1\cup F_2\cup F_3$ be a simple fold on G with projection $\varphi\colon F\to G$, and let $G_i=\varphi(F_i)$ for $i=1,2,3$. Suppose that G is connected, and that $\partial G_1\neq\varnothing\neq\partial G_3$. Then there is a component C of F such that $\varphi(C)$ meets ∂G_1 and ∂G_3. Moreover, for any such component, $\varphi(C)=G$, and if we let $F_i'=F_i\cap C$ for $i=1,2,3$, then $F'=F_1'\cup F_2'\cup F_3'$ is also a simple fold on G, with projection map $\varphi|_{F'}\colon F'\to G$.*

Proof. We first prove that there exists a component C of F such that $\varphi(C)$ meets ∂G_1 and ∂G_3. By (F2) and (F3), and since G is connected, there is a component K of G_2 which meets both $\overline{G_1\setminus G_2}$ and $\overline{G_3\setminus G_2}$. By Lemma 6 (2), it follows that $K\cap\partial G_1\neq\varnothing$ and $K\cap\partial G_3\neq\varnothing$. Because $\varphi|_{F_2}$ is a homeomorphism of F_2 onto G_2 (by (F4)), we have that there is a component C of F such that $\varphi^{-1}(K)\cap F_2\subset C$. Then $\varphi(C)\supseteq K$, so $\varphi(C)\cap\partial G_1\neq\varnothing$ and $\varphi(C)\cap\partial G_3\neq\varnothing$.

Now fix any such component C of F.

Claim 7.1. *If $C'\subseteq C$ is any connected subset such that $\varphi(C')\subset G_2$, $\varphi(C')\cap\partial G_1\neq\varnothing$ and $\varphi(C')\cap\partial G_3\neq\varnothing$, then $\varphi^{-1}(\varphi(C'))\subset C$.*

Proof. Let $C'\subseteq C$ be a connected subset such that $\varphi(C')\subset G_2$, $\varphi(C')\cap\partial G_1\neq\varnothing$, and $\varphi(C')\cap\partial G_3\neq\varnothing$. As $G_2=G_1\cap G_3$ (by (F2)), we have that the intersections

$$\varphi^{-1}(\varphi(C'))\cap F_1,\quad \varphi^{-1}(\varphi(C'))\cap F_2,\quad\text{and}\quad \varphi^{-1}(\varphi(C'))\cap F_3$$

are all homeomorphic to $\varphi(C')$ by (F4); in particular they are all connected. Moreover, $\varphi^{-1}(\varphi(C'))\cap F_1\cap F_2\neq\varnothing$ since $\varphi(C')\cap\partial G_1\neq\varnothing$ and $\partial G_1=\varphi(F_1\cap F_2)$ by (F5). Likewise, $\varphi^{-1}(\varphi(C'))\cap F_2\cap F_3\neq\varnothing$. It follows that $\varphi^{-1}(\varphi(C'))$, which is the union of the sets

$$\varphi^{-1}(\varphi(C'))\cap F_1,\quad \varphi^{-1}(\varphi(C'))\cap F_2,\quad\text{and}\quad \varphi^{-1}(\varphi(C'))\cap F_3,$$

is connected. Thus $\varphi^{-1}(\varphi(C'))\subset C$. □

Since C is closed, $\varphi(C)$ is closed in G. To show that $\varphi(C)=G$, we will show that $\varphi(C)$ is also open; this suffices since G is connected. To this end, let $x\in\varphi(C)$, and let $p\in C$ be such that $\varphi(p)=x$. If $p\notin\partial F_2$ then, by Lemma 6 (7), φ is a homeomorphism in a neighborhood of p, so since C is open in F, $\varphi(C)$ contains a neighborhood of x.

Suppose now that $p\in\partial F_2$. Then $p\in\partial F_1\cup\partial F_3$ by Lemma 6 (5); say $p\in\partial F_1$. Let C' be the closure of the component of $C\setminus\varphi^{-1}(\partial G_3)$ containing p. Then, by the boundary bumping theorem (see e.g. [44, Theorem 5.4]), $C'\cap\varphi^{-1}(\partial G_3)\neq\varnothing$. Thus, by Claim 7.1, we have $\varphi^{-1}(\varphi(C'))\subset C$. In particular, the point $q=(\varphi|_{F_3})^{-1}(x)\in C$. But $q\notin\partial F_3$ (because $\varphi(q)=x\in\partial G_1$ and, by (F5) and Lemma 6 (5), $\varphi(\partial F_3)=\partial G_3$, which is disjoint from ∂G_1), thus $q\notin\partial F_2$, and so again as above, $\varphi(C)$ contains a neighborhood of $\varphi(q)=x$. The argument for $p\in\partial F_3$ is similar.

Therefore $\varphi(C)=G$. It is straightforward to check from the definition of a simple fold that if $C\subset F$ is a component with $\varphi(C)=G$, then $F'=F_1'\cup F_2'\cup F_3'$, where $F_i'=F_i\cap C$ for $i=1,2,3$, is a simple fold on G with projection map $\varphi|_{F'}$ (note that it may well happen that $G_i'=\varphi(F_i')$ is a proper subset of G_i for one or more $i=1,2,3$). ☐

The next result is related to Theorem 2 of [47], and it is alluded to in that paper though not treated in detail there. It should be considered as a translation to the setting of simple folds of the following result of Krasinkiewicz and Minc [24]: A continuum X is hereditarily indecomposable if and only if for any disjoint closed subsets A and B of X and any open sets U and V containing A and B, respectively, there exist three closed sets $X_1,X_2,X_3\subset X$ such that $X=X_1\cup X_2\cup X_3$, $A\subset X_1$, $B\subset X_3$, $X_1\cap X_2\subset V$, $X_2\cap X_3\subset U$, and $X_1\cap X_3=\varnothing$. We remark that one can replace "hereditarily indecomposable continuum" with "hereditarily indecomposable compactum" in this result; the proof is unchanged.

Theorem 8. *Let X be a compactum. Then the following are equivalent*:

(1) *X is hereditarily indecomposable*;

(2) *for any map $f\colon X\to G$ to a graph G, for any simple fold $\varphi\colon F\to G$, and for any $\varepsilon>0$, there exists a map $g\colon X\to F$ such that $d_{\sup}(f,\varphi\circ g)<\varepsilon$*;

(3) *for any map $f\colon X\to[0,1]$, for any simple fold $\varphi\colon F\to[0,1]$ where F is an arc, and for any $\varepsilon>0$, there exists a map $g\colon X\to F$ such that $d_{\sup}(f,\varphi\circ g)<\varepsilon$.*

Proof. To show that (1) ⇒ (2), suppose that (1) holds. Let G be a graph, $f\colon X\to G$ a map, $\varphi\colon F\to G$ be a simple fold, and fix $\varepsilon>0$. As in Definition 5, we set $G_i=\varphi(F_i)$ for $i=1,2,3$.

Suppose that $\partial(G_i\setminus G_2)=\{y_1^i,...,y_{m(i)}^i\}$ for $i=1,3$. Each of these points y_j^i is the vertex point of a finite fan $Y_j^i\subset G_2$ (meaning Y_j^i is the union of finitely many arcs, each having y_j^i as one endpoint, and which are otherwise pairwise disjoint) such that Y_j^i is the closure of an open neighborhood O_j^i of y_j^i in G_2, and the diameter of Y_j^i is less than ε.

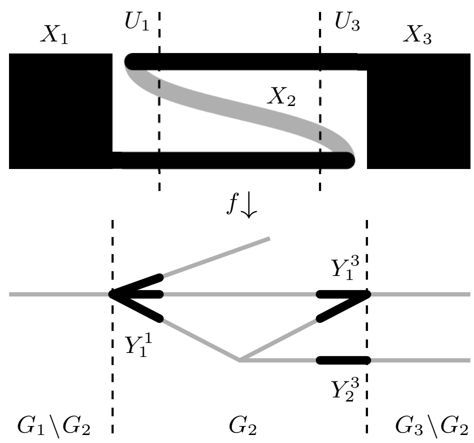

Figure 2. An illustration of the situation in the proof of Theorem 8.

Let $K_i = f^{-1}(\overline{G_i \setminus G_2})$. Then $K_1 \cap K_3 = \varnothing$ by (F3) of Definition 5.

For $i=1,3$, choose neighborhoods U_i of K_i so that $f(\overline{U_i \setminus K_i}) \subset \bigcup_{j=1}^{m(i)} O_j^i$. By [24], there exist closed sets X_i, $i=1,2,3$, such that

- $X = X_1 \cup X_2 \cup X_3$,
- $K_i \subset X_i$ for $i=1,3$,
- $X_1 \cap X_3 = \varnothing$,
- $X_1 \cap X_2 \subset U_3$, and
- $X_2 \cap X_3 \subset U_1$.

See Figure 2 for an illustration. Let

$$A = [X_1 \setminus U_3] \cup [X_2 \setminus (U_1 \cup U_3)] \cup [X_3 \setminus U_1],$$

and consider the restriction $f|_A$. Observe that

$$X \setminus A = [(X_2 \cup X_3) \cap U_1] \cup [(X_1 \cup X_2) \cap U_3].$$

We extend $f|_A$ to a map $h: X \to G$ as follows. Observe that

$$f((X_2 \cup X_3) \cap U_1) \subset \bigcup_{j=1}^{m(1)} Y_j^1.$$

In fact, for each $j = 1, \dots, m(1)$, since $[(X_2 \cup X_3) \cap U_1] \cap X_1 = \varnothing$ and

$$f^{-1}(y_j^1) \subset f^{-1}(\overline{G_1 \setminus G_2}) = K_1 \subset X_1,$$

we have that $f((X_2\cup X_3)\cap U_1)\subset\bigcup_{j=1}^{m(1)}(Y_j^1\setminus\{y_j^1\})$. Let L be an arc in Y_j^1 with one endpoint y_j^1 and the other endpoint equal to an endpoint of the fan Y_j^1. Let

$$L'=f^{-1}(L)\cap(X_2\cup X_3)\cap U_1.$$

Then L' is closed and open in $(X_2\cup X_3)\cap U_1$. By the Tietze extension theorem, we can define a continuous function $h_L\colon\bar{L}'\to L$ so that $h_L|_{\partial L'}=f|_{\partial L'}$ and $h_L(x)=y_j^1$ for all $x\in X_2\cap X_3\cap L'$. We then let $h|_{\bar{L}'}=h_L$, and do this for all such arcs L in the fans Y_j^1. We proceed similarly to define h on $(X_1\cup X_2)\cap U_3$.

In this way, we obtain a continuous function $h\colon X\to G$ such that

- $h|_A=f|_A$,
- $h(X_i)\subseteq G_i$ for $i=1,2,3$,
- $h(X_1\cap X_2)\subset\partial(G_3\setminus G_2)=\partial G_1$ (see Lemma 6 (2)), and
- $h(X_2\cap X_3)\subset\partial(G_1\setminus G_2)=\partial G_3$ (see Lemma 6 (2)).

Observe that $d_{\sup}(f,h)<\varepsilon$ since the diameters of the sets Y_j^i are less than ε.

Now define $g\colon X\to F$ by $g(x)=((\varphi|_{F_i})^{-1}\circ h)(x)$ if $x\in X_i$, for $i=1,2,3$. This is well defined and continuous because of the above properties of h and X_1, X_2, and X_3. Then g is as required so that (2) holds.

The implication (2) $\Rightarrow$ (3) is trivial.

To show that (3) $\Rightarrow$ (1), suppose that (3) holds. Let $A,B\subset X$ be disjoint closed sets, and let U be a neighborhood of A and V a neighborhood of B. By [24] it suffices to show that $X=X_1\cup X_2\cup X_3$ where X_1, X_2, and X_3 are closed subsets of X such that $A\subset X_1$, $B\subset X_3$, $X_1\cap X_3=\varnothing$, $X_1\cap X_2\subset U$, and $X_2\cap X_3\subset V$.

Let $f\colon X\to[0,1]$ be a map such that $f^{-1}(0)=A$ and $f^{-1}(1)=B$. Choose $0<u<v<1$ such that $f^{-1}([0,u])\subset U$ and $f^{-1}([v,1])\subset V$. Let $u'\in(0,u)$ and $v'\in(v,1)$. Construct a simple fold $\varphi\colon F\to[0,1]$, where $F=F_1\cup F_2\cup F_3$ is an arc, such that $\varphi(F_1)=[0,v']$, $\varphi(F_2)=[u',v']$ and $\varphi(F_3)=[u',1]$. Let $\varepsilon>0$ be small enough so that $(u'-\varepsilon,u'+\varepsilon)\subset[0,u)$ and $(v'-\varepsilon,v'+\varepsilon)\subset[v,1]$. By (3), there is a map $g\colon X\to F$ such that $d_{\sup}(f,\varphi\circ g)<\varepsilon$.

Put $X_i=g^{-1}(F_i)$ for $i=1,2,3$. Then $X=X_1\cup X_2\cup X_3$, and clearly $X_1\cap X_3=\varnothing$. To see that $X_1\cap X_2\subset V$, let $x\in X_1\cap X_2$. Then $(\varphi\circ g)(x)=v'$, and since $d_{\sup}(f,\varphi\circ g)<\varepsilon$, we have $f(x)\in[v,1]$ and, hence, $x\in V$. Similarly, $X_2\cap X_3\subset U$. By [24], X is hereditarily indecomposable. $\square$

We now introduce notions which will be relevant when considering structured separators in the next section.

Definition 9. Let $A\subset G$ be regular, and let $B\subset\partial A$.

- A has *consistent complement relative to* B if for each component C of $G\setminus A$, either $\partial C\subseteq B$ or $\partial C\cap B=\varnothing$.

• The A *side of* B, denoted $\sigma_B(A)$, is the closure of the union of all components of $G\setminus B$ meeting A.

If B is empty, then $\sigma_B(A)$ is simply equal to the union of all components of G which A intersects. In particular, if $A=\varnothing$ then $\sigma_B(A)=\varnothing$.

Suppose that A and B are both non-empty. Observe that if G is connected and A has consistent complement relative to B, then in fact, for any neighborhood V of B, $\sigma_B(A)$ is equal to the closure of the union of all components of $G\setminus B$ meeting $A\cap V$. In fact, $\sigma_B(A)$ can be characterized as the unique closed (regular) set $D\subset G$ such that $\partial D=B$ and $D\cap V=A\cap V$ for some neighborhood V of B.

Proposition 10. *Let G be connected, let $A, A'\subset G$ be non-empty regular sets, and let $B\subseteq\partial A\cap\partial A'$. If A and A' each have consistent complement relative to B, and if there is a neighborhood V of B such that $A\cap V=A'\cap V$, then $\sigma_B(A)=\sigma_B(A')$. Moreover, if C is a component of $G\setminus A$ with $\overline{C}\cap B\neq\varnothing$, then C is also a component of $G\setminus A'$.*

Proof. The fact that $\sigma_B(A)=\sigma_B(A')$ follows immediately from the observations after Definition 9. For the last statement, let C be a component of $G\setminus A$ with $\overline{C}\cap B\neq\varnothing$. Since $C\cap A=\varnothing$ and $A\cap V=A'\cap V$ (where V is the neighborhood of B described in the statement of this proposition), we have $(C\cap V)\cap A'=\varnothing$. Let C' be a component of $G\setminus A'$ meeting $C\cap V$.

Obviously $C'\subseteq C$, since $\partial C\subseteq B$ and $C'\cap B=\varnothing$. If $C'\neq C$, then there must be a point $x\in\partial C'\cap C$. But since A' has consistent complement relative to B, we must have $x\in B$, so $\varnothing\neq C\cap B\subset C\cap A$, a contradiction. Therefore $C'=C$. □

Proposition 11. *Let G be connected, let $A\subset G$ be regular and non-empty, and let $B_1, B_2\subseteq\partial A$ with $B_1\cup B_2=\partial A$ and $B_1\cap B_2=\varnothing$. Suppose A has consistent complement relative to B_1 and to B_2. Let $G_1=\sigma_{B_1}(A)$, $G_2=A$, and $G_3=\sigma_{B_2}(A)$. Then G_1, G_2, and G_3 define a simple fold on G (i.e. they satisfy properties* (F1)–(F3)).

Proof. Note that if $A=G$, then $G_1=G_2=G_3=G$, which define a simple fold. We suppose therefore that $A\neq G$, in which case at least one of B_1 and B_2 is non-empty.

Clearly G_1, G_2, and G_3 are all regular subsets of G, so (F1) holds.

Consider (F2). By definition, it is clear that $A\subseteq\sigma_{B_1}(A)$ and $A\subseteq\sigma_{B_2}(A)$, thus $G_2\subseteq G_1\cap G_3$. For the reverse inclusion, suppose that $x\in G\setminus G_2=G\setminus A$, and let C be the component of $G\setminus A$ containing x. Because G is connected, $\overline{C}\cap A\neq\varnothing$, and either $\partial C\subseteq B_1$ or $\partial C\subseteq B_2$, since A has consistent complement relative to B_1 and to B_2. In the former case we have $C\cap\sigma_{B_1}(A)=\varnothing$, and in the latter case we have $C\cap\sigma_{B_2}(A)=\varnothing$. In either case, $x\notin G_1\cap G_3$. Thus $G_1\cap G_3\subseteq G_2$.

To see that $G_1\cup G_3=G$, let $x\in G$, and assume that $x\notin A$ (since $A=G_2=G_1\cap G_3$). Let C be the component of $G\setminus A$ containing x. Again $\overline{C}\cap A\neq\varnothing$, and either $\partial C\subseteq B_1$ or

$\partial C \subseteq B_2$. If $\partial C \subseteq B_1$, then since $\sigma_{B_2}(A) \supset A \supset B_1$, it is clear that $C \subset \sigma_{B_2}(A)$. Similarly, if $\partial C \subseteq B_2$, then $C \subset \sigma_{B_1}(A)$. Thus in any case, $x \in G_1 \cup G_3$.

For property (F3), let $x \in \overline{G_1 \setminus G_2}$. If $x \in A$, then we must have $x \in B_2$, and in this case $x \notin \overline{\sigma_{B_2}(A) \setminus A} = \overline{G_3 \setminus G_2}$, since one can find a neighborhood of x which meets only A and components of $G \setminus A$ whose closures meet B_2. On the other hand, if $x \notin A$, then $x \notin \sigma_{B_2}(A) = G_3$, as $x \in \sigma_{B_1}(A)$ and $\sigma_{B_1}(A) \cap \sigma_{B_2}(A) = A$. Thus, in any case, $x \notin \overline{G_3 \setminus G_2}$. Therefore $\overline{G_1 \setminus G_2} \cap \overline{G_3 \setminus G_2} = \varnothing$. $\square$

We remark that, if $\partial A = B_1 \cup B_2$ and $B_1 \cap B_2 = \varnothing$, and if A has consistent complement relative to B_1, then A automatically has consistent complement relative to B_2 as well.

5. Stairwells

We pause here to give an outline of the remainder of the proof of Theorem 1, which is presented in full in §7. Beginning with a hereditarily indecomposable continuum X with span zero, in the Hilbert cube $[0,1]^{\mathbb{N}}$, we fix some $\varepsilon > 0$ and let $I \approx [0,1]$ be an arc which is close to X in Hausdorff distance. Our task is to produce an ε-map from X to I, which would imply X is arc-like, and hence X is homeomorphic to the pseudo-arc by Bing's characterization [3].

Because X has span zero, it is tree-like [31], so we can choose a tree $T \subset [0,1]^{\mathbb{N}}$ and a map $f: X \to T$ such that $d(x, f(x))$ is small for all $x \in X$. According to Theorem 4, the set $M = \left\{(x,y) \in T \times I : d(x,y) < \frac{1}{2}\varepsilon\right\}$ separates $T \times \{0\}$ from $T \times \{1\}$ in $T \times I$ provided T and I are chosen close enough to X. If we can find a map $h: X \to M$, $h(x) = (h_1(x), h_2(x))$, such that $d(h_1(x), f(x))$ is small for all $x \in X$, then, by the definition of M and choice of f, it follows that $h_2(x)$ is close to x for all $x \in X$, and so h_2 will be an ε-map once appropriate care is taken with constants.

To obtain this map $h: X \to M$, we use our assumption that X is hereditarily indecomposable. According to Theorem 8, the map $f: X \to T$ can be (approximately) factored through any simple fold $\phi: F \to T$. Our method is to inspect the structure of the separator M and use a sequence of simple folds to match the continuum X and map f with the pattern of M and the first coordinate projection π_1.

In order to accomplish this, we introduce in this section a special type of separator (one with a "stairwell structure") which has a positive integer measure of complexity (the "height" of the stairwell). It follows from Theorem 15 below that M contains a subset which is a separator with a stairwell structure. We then prove in the next section that one can use a sequence of simple folds to effectively reduce the height of a stairwell. The proof of Theorem 1 is then completed by induction (note from Definition 13 below

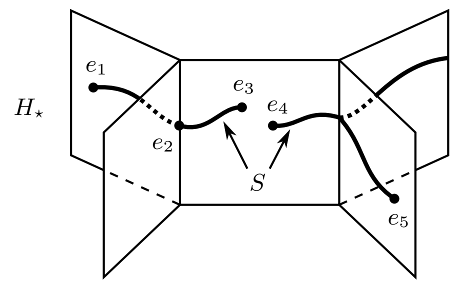

Figure 3. An example of a straight set S in $H_\star = H \times [0,1]$, where H is a graph homeomorphic to the letter "H". In this example, S has two connected components. The end set of S is $\mathcal{E}(S) = \{e_1, e_2, e_3, e_4, e_5\}$.

that if S has a stairwell structure of height 1, then $\pi_1|_S$ is one-to-one and, hence, a homeomorphism).

Given a set X, let $X_\star = X \times [0,1]$. Define $\pi: X_\star \to X$ by $\pi(x,t) = x$. Given a function $f: X \to Y$, define $f_\star: X_\star \to Y_\star$ by $f_\star(x,t) = (f(x), t)$.

Definition 12. (1) A collection $\langle B_1, ..., B_n \rangle$ of finite subsets of G is *generic* if B_i is disjoint from the set of branch points and endpoints of G for each i, and $B_i \cap B_j = \varnothing$ whenever $i \neq j$.

(2) A subset $S \subset G_\star$ is *straight* if S is closed, π is one-to-one on S, and $\pi(S)$ is regular. The *end set* of a straight subset $S \subset G_\star$ is $\mathcal{E}(S) = S \cap \pi^{-1}(\partial \pi(S))$.

See Figure 3 for an example of a straight set and its end set.

Observe that, if $S \subset G_\star$ is straight then π, restricted to $S \setminus \mathcal{E}(S)$, is an open mapping from $S \setminus \mathcal{E}(S)$ to G (see Figure 3).

Definition 13. Let $S \subset G_\star$. A *stairwell structure for S of height k* is a tuple $\langle S_1, ..., S_k \rangle$ satisfying the following properties:

(S1) $S_1, ..., S_k$ are non-empty straight subsets of $G_\star$ with $S = S_1 \cup ... \cup S_k$;

(S2) for each $i = 1, ..., k$, $\mathcal{E}(S_i) = \alpha_i \cup \beta_i$, where α_i and β_i are disjoint finite sets, with $\alpha_1 = \varnothing = \beta_k$, and $\beta_i = \alpha_{i+1}$ for each $i = 1, ..., k-1$;

(S3) for each $i = 1, ..., k-1$, there is a neighborhood V of $\pi(\beta_i) = \pi(\alpha_{i+1})$ such that $\pi(S_i) \cap V = \pi(S_{i+1}) \cap V$;

(S4) for each $i = 1, ..., k$, $\pi(S_i)$ has consistent comple ment relative to $\pi(\alpha_i)$ and to $\pi(\beta_i)$;

(S5) the family $\langle \pi(\alpha_2), ..., \pi(\alpha_k) \rangle$ (which is equal to $\langle \pi(\beta_1), ..., \pi(\beta_{k-1}) \rangle$) is generic in G.

See Figure 4 for a simple example of a set with a stairwell structure.

Note that even though the sets $S_1, ..., S_k$ are all non-empty, we do allow for the possibility that $\alpha_i = \varnothing$ for some values of $i \in \{2, ..., k\}$.

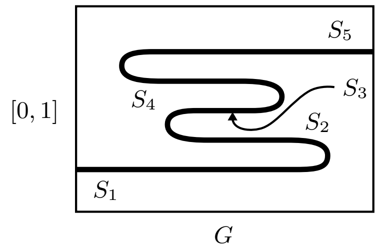

Figure 4. An example of a set with a stairwell structure of height 5 in $G_\star$, where G is an arc.

We make the following observation: if $S\subset G_\star$ is a set with a stairwell structure, and if C is a component of G such that $S_i\cap C_\star\neq\varnothing$ for each $i=1,...,k$, then $S\cap C_\star$ has a stairwell structure obtained by intersecting each of the sets S_i, α_i, and β_i, with $C_\star$.

Note that there is no requirement that a set $S\subset G_\star$ with a stairwell structure of height k will satisfy $\pi(S)=G$. Indeed, if k is even this need not be the case. However, it will follow from the next proposition (in fact from Claim 14.1) that if k is odd then $\pi(S)=G$. Here it is crucial that $\alpha_1=\beta_k=\varnothing$ (see (S2)). The reader is encouraged to draw a couple of examples of sets with stairwell structures of even and odd heights in $G_\star$, for G a simple graph such as an arc, circle, or simple triod, to explore these possibilities.

Though we will not technically need the next proposition in the sequel, it serves to clarify the connection between separators in $G_\star$ and sets with stairwell structures.

PROPOSITION 14. *If G is a connected graph, then a set $S\subset G\times(0,1)$ with a stairwell structure of odd height separates $G\times\{0\}$ from $G\times\{1\}$ in $G_\star$.*

Proof. Let $\langle S_1,...,S_k\rangle$ be a stairwell structure for S, where k is odd.

CLAIM 14.1. *For each $x\in G$, the number of integers $i\in\{1,...,k\}$ such that $x\in\pi(S_i)$ is odd.*

Proof. Fix $x\in G$, and define $f:\{0,...,k\}\to\{0,1\}$ by

$$f(i)=\begin{cases}1, & \text{if } i=0,\ i=k, \text{ or } x\in\sigma_{\pi(\beta_i)}(\pi(S_i)),\\ 0, & \text{otherwise.}\end{cases}$$

For each $i=2,...,k$, by property (S3), Proposition 10, and the fact that $\beta_{i-1}=\alpha_i$, we have that $\sigma_{\pi(\beta_{i-1})}(\pi(S_{i-1}))=\sigma_{\pi(\beta_{i-1})}(\pi(S_i))=\sigma_{\pi(\alpha_i)}(\pi(S_i))$. By Proposition 11 and property (F2), it follows that $\sigma_{\pi(\beta_{i-1})}(\pi(S_{i-1}))\cup\sigma_{\pi(\beta_i)}(\pi(S_i))=G$ for each $i=2,...,k-1$. This means that there are no contiguous blocks of more than one integer in $f^{-1}(0)$. Observe that $x\in\pi(S_i)$ if and only if $f(i-1)=f(i)=1$. It follows that if N_1 is the number of integers $i\in\{1,...,k\}$ such that $x\in\pi(S_i)$ and N_2 is the number of integers $i\in\{1,...,k\}$ such that $f(i-1)\neq f(i)$, then $N_1+N_2=k$.

Since $f(0)=f(k)=1$, we have that N_2 is even. By hypothesis, k is odd. Thus N_1 must be odd. □

Given $(x,t)\in G_\star\setminus S$, define $N(x,t)$ as the number of integers $i\in\{1,...,k\}$ such that $(x,s)\in S_i$ for some $s>t$. Let

$$V_1=\{(x,t)\in G_\star\setminus S: N(x,t) \text{ is odd}\} \quad \text{and} \quad V_2=\{(x,t)\in G_\star\setminus S: N(x,t) \text{ is even}\}.$$

From Claim 14.1, we have $G\times\{0\}\subset V_1$, and clearly $G\times\{1\}\subset V_2$ and $V_1\cup V_2=G_\star\setminus S$.

Claim 14.2. *V_1 and V_2 are open in $G_\star\setminus S$.*

Proof. Fix $(x,t)\in V_1$. Let W be a small connected open neighborhood of x in G, and let $\delta>0$ be such that $U=W\times(t-\delta,t+\delta)$ is a neighborhood of (x,t) in $G_\star$ which is disjoint from S.

If $x\notin\pi(\mathcal{E}(S_i))$ for each i, then we may assume that W is small enough so that, for each i, either $W\cap\pi(S_i)=\varnothing$ or $W\subset\pi(S_i)$. It follows easily that, for each $(x',t')\in U$, $N(x',t')=N(x,t)$. Thus $U\subset V_1$.

If $x\in\pi(\mathcal{E}(S_i))$ for some i, say $x\in\pi(\beta_i)$, then by (S5), $x\notin\pi(\mathcal{E}(S_j))$ for each $j\notin\{i,i+1\}$, and so we may assume that W is small enough so that, for each $j\notin\{i,i+1\}$, either $W\cap\pi(S_j)=\varnothing$ or $W\subset\pi(S_j)$. Moreover, we may assume that W is small enough so that $W\cap\pi(S_i)=W\cap\pi(S_{i+1})$.

If there is no $s>t$ such that $(x,s)\in S_i$, then it is easy to see that $N(x',t')=N(x,t)$ for all $(x',t')\in U$. Suppose then that there exists $s>t$ such that $(x,s)\in S_i$ (so that $(x,s)\in\beta_i$). Let $(x',t')\in U$. If $x'\in\pi(S_i)$ then $x'\in\pi(S_{i+1})$ as well, and it is clear that $N(x',t')=N(x,t)$. If $x'\notin\pi(S_i)$, then $x'\notin\pi(S_{i+1})$ as well, and so $N(x',t')=N(x,t)-2$. In any case, we have $(x',t')\in V_1$. Thus $U\subset V_1$.

Therefore V_1 is open. The proof that V_2 is open is identical. □

Thus S separates $G\times\{0\}$ from $G\times\{1\}$ in $G_\star$, and Proposition 14 is proved. □

As a special case, consider a set $S\subset G\times(0,1)$ with a stairwell structure of height 1. In this case, π maps S homeomorphically onto G.

Theorem 15. *Let G be a graph. For any set $M\subseteq G\times(0,1)$ which separates $G\times\{0\}$ from $G\times\{1\}$ in $G_\star$, and any open set $U\subseteq G\times(0,1)$ with $M\subseteq U$, there exists a set $S\subset U$ with a stairwell structure of odd height.*

Proof. Let $M\subset G\times(0,1)$ separate $G\times\{0\}$ from $G\times\{1\}$ in $G_\star$, and fix an open set $U\subseteq G\times(0,1)$ with $M\subseteq U$.

We say that a set $S\subset G\times(0,1)$ *irreducibly separates* $G\times\{0\}$ from $G\times\{1\}$ in $G_\star$ if S separates these two sets, but no proper subset of S does. It is well known (see e.g. [26, Theorems 46.VII.3 and 49.V.3]) that for any set $S\subset G\times(0,1)$ which separates $G\times\{0\}$ from $G\times\{1\}$, there is a closed set $S'\subseteq S$ which irreducibly separates $G\times\{0\}$ from $G\times\{1\}$.

Let Z denote the set of all branch points and endpoints of G. Given a set $L\subset G_\star$ and a point $(x,y)\in L$ such that $x\notin Z$, we say that L *has a side wedge* at (x,y) if there is a closed disk D containing (x,y) in its interior such that $L\cap D=C_1\cup C_2$, where C_1 and C_2 are arcs which both have x as an endpoint but are otherwise disjoint, π is one-to-one on C_1 and on C_2, and $\pi(C_1)=\pi(C_2)$.

CLAIM 15.1. *There exists a set $M'\subset U$ such that*

(1) *M' is a graph*;

(2) *M' irreducibly separates $G\times\{0\}$ from $G\times\{1\}$ in $G_\star$*;

(3) *there is a finite set $T\subset M'$ such that, for all $(x,y)\in M'\setminus T$, there is a neighborhood V of (x,y) such that π maps $M'\cap V$ homeomorphically onto a neighborhood of x in G*;

(4) *for each $(x,y)\in T$, the set M' has a side wedge at (x,y)*;

(5) *$T\cap Z_\star=\varnothing$*;

(6) *if (x_1,y_1) and (x_2,y_2) are two distinct points in T, then $x_1\neq x_2$.*

Proof. We leave it to the reader to show that there exists a set M' having properties (1), (3), (5), and (6), and which separates $G\times\{0\}$ from $G\times\{1\}$. Replacing M' by a subset (which, by abuse of notation, we also denote by M') which irreducibly separates $G\times\{0\}$ from $G\times\{1\}$ in $G_\star$ accomplishes (2). Let $G_\star\setminus M'=R_0\cup R_1$, where R_0 and R_1 are open in $G_\star\setminus M'$, $G\times\{0\}\subset R_0$, and $G\times\{1\}\subset R_1$.

To achieve property (4), consider a point $(x,y)\in T$. Note that (x,y) cannot be an endpoint of M', because x is not an endpoint of G by (5), and $\pi(M'\cap V)$ is open for some neighborhood V of (x,y) by (3). If (x,y) is not a branch point of M', then it is easy to see that M' has a side wedge at (x,y), or else π is one-to-one on M' in a neighborhood of (x,y) in which case we can remove (x,y) from T. Again, by abuse of notation, we denote the resulting set by M'.

Suppose now that (x,y) is a branch point of M'. Let D be a small closed disk, containing (x,y) in its interior, such that $M'\cap D$ is the union of n arcs $C_1,...,C_n$, each having (x,y) as an endpoint, and which are otherwise pairwise disjoint. Because M' is an irreducible separator, the complementary regions of M' in D alternate between R_0 and R_1. It follows that n is even. Now we can modify M' inside D by replacing the arcs $C_1,...,C_n$ with $\frac{1}{2}n$ "wedges", as depicted in Figure 5, and removing (x,y) from T. Some of the resultant wedges may be side wedges, whose "tip" points we add to T. Obviously this can be done without compromising properties (1), (5), and (6), and without leaving U.

Once this is carried out for all the branch points of M' which belong to T, one at a

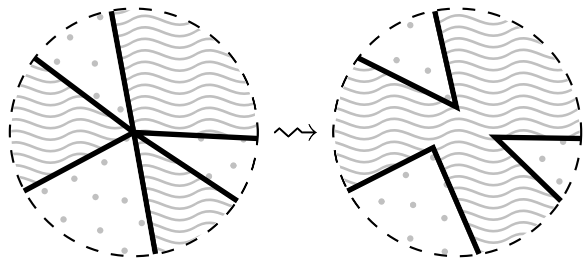

Figure 5. Modifying a graph separator M' in $G_\star$ in a small neighborhood of an unwanted branch point to remove the branch point. The two sides, R_0 and R_1, of $G_\star\setminus M'$ are indicated using wavy lines and dots, respectively.

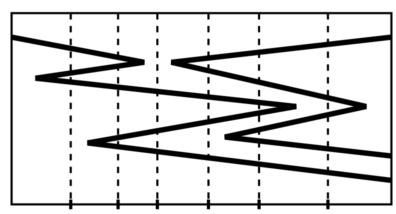

Figure 6. Partitioning the graph into small arcs above each of which the separator M' has at most one side wedge.

time, the resultant set satisfies property (4). It is easy to see that the resultant M' still irreducibly separates $G\times\{0\}$ from $G\times\{1\}$ in $G_\star$. □

We now proceed with the proof of Theorem 15. Given a finite set $B\subset G$, we say that two points $a,b\in B$ are *adjacent* if there is a component of $G\setminus B$ whose closure contains both a and b.

Let M' be a set as described in Claim 15.1. Because of property (6), there exists a finite set $Z'\subset G$ such that

- $Z\subseteq Z'$, $Z'\cap\pi(T)=\varnothing$ and the closure of every component of $G\setminus Z'$ is an arc;
- if $a,b\in Z'$ are adjacent, then there is exactly one component of $G\setminus Z'$ whose closure contains both a and b and we will denote this arc by $[a,b]$;
- if $a,b\in Z'$ are adjacent, then $[a,b]_\star\cap T$ contains at most one point.

Figure 6 illustrates what the set $M'\cap\pi^{-1}(A)$ might look like over some component A of $G\setminus Z'$.

Observe that since $Z'\cap\pi(T)=\varnothing$ and since M' irreducibly separates $G\times\{0\}$ from $G\times\{1\}$, for each point $a\in Z'$, the set $M'\cap\{a\}_\star$ contains an odd number of points. Let k be the maximum cardinality of $M'\cap\{a\}_\star$ among all $a\in Z'$. Then, in particular, k is odd.

Fix two adjacent points $a,b\in Z'$. Let $M'\cap\{a\}_\star=\{(a,y_1),\dots,(a,y_j)\}$, where $j\leqslant k$ is

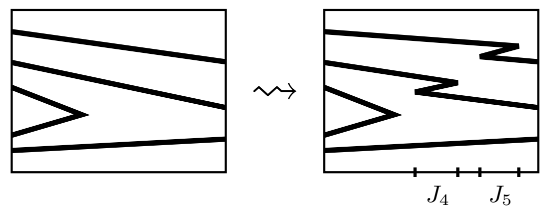

Figure 7. Adding "zig-zags" to two components of $M'\cap[a,b]_\star$, above the arcs J_4 and J_5, in order to obtain a set with a stairwell structure.

odd, and $y_1<y_2<...<y_j$. For each $i=1,...,j$, let C_i be the component of $M'\cap[a,b]_\star$ containing the point (a,y_i).

If there is no side wedge in $M'\cap[a,b]_\star$, then define $S_i^{[a,b]}=C_i$ for each $i=1,...,j$.

On the other hand, suppose that $M'\cap[a,b]_\star$ has a component W which has a side wedge. Assume without loss of generality that $a\in\pi(W)$, so that $b\notin\pi(W)$. Clearly $W\cap\{a\}_\star$ consists of two consecutive points, say (a,y_m) and (a,y_{m+1}) of $M'\cap\{a\}_\star$. Then $C_m=C_{m+1}=W$. Observe that $M'\cap\{b\}_\star$ contains exactly $j-2$ points.

For each $i=m+2,...,j$, let $J_i\subset[a,b]$ be a closed subarc such that $J_i\cap\pi(W)=\varnothing$, and J_{i+1} is between J_i and b for each $i=m+2,...,j-1$. For each $i=m+2,...,j$, in a small neighborhood of C_i in U, define three arcs C_i^1, C_i^2, and C_i^3 such that

- π is one-to-one on C_i^p for each $p=1,2,3$;
- C_i^1 and C_i^2 have a common endpoint, and C_i^2 and C_i^3 have a common endpoint, but these three arcs are otherwise pairwise disjoint;
- $C_i^1\cap\{a\}_\star=C_i\cap\{a\}_\star$ and $C_i^3\cap\{b\}_\star=C_i\cap\{b\}_\star$;
- $\pi(C_i^2)=J_i=\pi(C_i^1)\cap\pi(C_i^3)$.

We call this procedure "adding a zig-zag" to C_i. Refer to Figure 7 for an illustration.

Now for each $i=1,...,m-1$, define $S_i^{[a,b]}=C_i$. For $i=m,...,k$, we define $S_i^{[a,b]}$ by defining the components of these sets in steps, as follows.

Decompose the side wedge W into two arcs W_m and W_{m+1}, where π is one-to-one on each of W_m and W_{m+1}, and W_i contains (a,y_i) for both $i=m,m+1$. We start with $S_i^{[a,b]}=W_i$ for both $i=m,m+1$. Then, for each $i=m+2,...,j$, in order, we start with $S_i^{[a,b]}=C_i^1$, and we add C_i^2 to $S_{i-1}^{[a,b]}$ and add C_i^3 to $S_{i-2}^{[a,b]}$. Finally, for $i=j+1,...,k$, let $S_i^{[a,b]}=\varnothing$.

Define, for each $i=1,...,k$,

$$S_i=\bigcup_{\substack{a,b\in Z'\\ \text{adjacent}}} S_i^{[a,b]},$$

and let $S=\bigcup_{i=1}^k S_i$. Observe that S is in U, and clearly S irreducibly separates $G\times\{0\}$ from $G\times\{1\}$, because M' does.

It is clear that $S_i^{[a,b]}$ is straight for each adjacent pair $a,b\in Z'$ and each $i=1,...,k$. Moreover, if $a\in Z'$ and $S\cap\{a\}_\star=\{(a,y_1),...,(a,y_j)\}$, where $j\leqslant k$ and $y_1<...<y_j$, then from the construction we see that $S_i^{[a,x]}\cap\{a\}_\star=\{(a,y_i)\}$ for any $x\in Z'$ adjacent to a and each $i=1,...,j$ (and $S_i^{[a,x]}\cap\{a\}_\star=\varnothing$ if $i>j$). It follows that S_i is straight for each $i=1,...,k$. Thus property (S1) holds.

Let $\alpha_1=\beta_k=\varnothing$, and for each $i=1,...,k-1$, let $\beta_i=\alpha_{i+1}=S_i\cap S_{i+1}$. The points of the sets $S_i\cap S_{i+1}$ are exactly the tips of side wedges and the zig-zag turning points. Clearly all such points belong to the end sets of the sets S_i, and there are no other points in the end sets of the S_i's because S irreducibly separates $G\times\{0\}$ from $G\times\{1\}$ in $G_\star$. Thus property (S2) holds.

Properties (S3) and (S5) are immediate from the construction.

For property (S4), let C be a component of $G\setminus\pi(S_i)$ for some $i\in\{1,...,k\}$. Note that if $z\in C\cap Z'$, then $|M'\cap\{z\}_*|<i$. If $C\subset[a,b]$ for some adjacent pair $a,b\in Z'$, then it is clear from the construction (refer to the right side of Figure 7) that $\partial C\subset\pi(\alpha_i)$ or $\partial C\subset\pi(\beta_i)$. Suppose, on the other hand, that $x_1,x_2\in\partial C$ do not belong to the same component of $G\setminus Z'$. Let $a_1,...,a_n\in Z'$ be such that a_p and a_{p+1} are adjacent for each $p=1,...,n-1$, $x_1\in[a_1,a_2]$, $x_2\in[a_{n-1},a_n]$, and $[a_p,a_{p+1}]\subset C$ for all $p=2,...,n-2$. For $p=1,...,n$, let j_p be the number of points in $S\cap\{a_p\}_\star$. Then j_p is odd for all $p=1,...,n$. Since $a_i\in C$ for $p=2,...,n-1$, $j_p<i$ for $p=2,...,n-1$ and, as $|j_p-j_{p+1}|\in\{0,2\}$ for each $p=1,...,n-1$, $i\leqslant j_1\leqslant i+1$ and $i\leqslant j_n\leqslant i+1$. Moreover, since $|j_p-j_{p+1}|\in\{0,2\}$ for each $p=1,...,n-1$, we must have that j_1 and j_n have the same parity and, hence, $j_1=j_n$. It is now easy to see that each of x_1 and x_2 corresponds to the tip point of a side wedge or a turning point of a zig-zag joining S_{j_1-1} and S_{j_1}. Hence, if $i=j_1=j_n$, $\{x_1,x_2\}\subset\pi(\alpha_i)$ and, if $i=j_1-1=j_n-1$, then $\{x_1,x_2\}\subset\pi(\beta_i)$ and it follows that (S4) holds.

Since (S1)–(S5) hold $\langle S_1,...,S_k\rangle$ is a stairwell structure of odd height k for S. □

To illustrate that the procedure indicated in Figure 5 may indeed be needed, we offer an example in Figure 8 of a set S in $G_\star$, where G is a *simple triod* with legs T_1, T_2, and T_3; that is, G is the union of three arcs T_1, T_2, and T_3 which have one common endpoint and are otherwise pairwise disjoint. In this case, $G_\star$ is a "3-page book", whose three "pages" are the squares drawn in Figure 8. The left edges of the three squares are identified. We leave it to the reader to observe that this set S irreducibly separates $G\times\{0\}$ from $G\times\{1\}$. The reader may find it informative to remove the unwanted branch point using the procedure indicated in Figure 5 (note that there are two essentially different ways to do this), and then to nudge the set so that all the turning points have distinct projections, and add zig-zags as in Figure 7, to obtain a set with a stairwell structure.

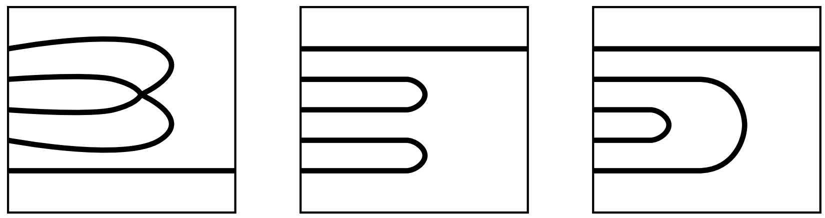

Figure 8. An example of an irreducible separator in $G_\star$, for G a simple triod, with an extra unwanted branch point.

6. Unfolding stairwells

In this technical section, we develop the machinery we need to simplify a set with a stairwell structure by taking its inverse image under a simple fold. As will be seen below, one can reduce the height of a stairwell by taking inverse images under a sequence of simple folds. In the intermediate stages of this process, the resultant sets will not have a stairwell structure; however, they will exhibit a structure very close to it, which is captured by the next definition of a *broken stairwell structure*.

A broken stairwell structure differs from a stairwell structure in that it contains an additional "detour" (which we call a *pit*) at one of the levels. We will observe in Proposition 17 that a set with a stairwell structure of height k can be relabeled so as to have a broken stairwell structure of height $k-2$, with a pit at the first level. We will then prove in Proposition 19 that given a broken stairwell structure, we can take the inverse image under a simple fold to obtain a new set with a broken stairwell structure of the same height in which the pit is at the next level up. This procedure can be repeated to move the pit up to the highest level. Then, once the pit is at the highest level, applying this procedure once more removes the pit altogether, leaving a set with a stairwell structure (not broken). This will be carried out formally in the proof of Theorem 20 in the next section.

Definition 16. Let $S\subset G_\star$. A *broken stairwell structure for S of height k with a pit at level i_0* is a tuple $\langle S_1,...,S_k;P_1,P_2\rangle$ such that

(S1′) $S_1,...,S_k,P_1,P_2$ are non-empty straight subsets of $G_\star$ with $S=S_1\cup...\cup S_k\cup P_1\cup P_2$;

(S2′) property (2) above holds for $S_1,...,S_k$, except that $\mathcal{E}(S_{i_0})$ is decomposed into three disjoint finite sets: $\mathcal{E}(S_{i_0})=\alpha_{i_0}\cup\beta_{i_0}\cup\gamma_{i_0}$. Additionally, $\mathcal{E}(P_2)=\mathcal{E}(P_1)\cup\gamma_{i_0}$, and $\mathcal{E}(P_1)\cap\gamma_{i_0}=\varnothing$;

(S3′) property (3) above holds for $S_1,...,S_k$, and additionally, there is a neighborhood V of $\pi(\mathcal{E}(P_1))$ such that $\pi(P_1)\cap V=\pi(P_2)\cap V$, and a neighborhood W of $\pi(\gamma_{i_0})$

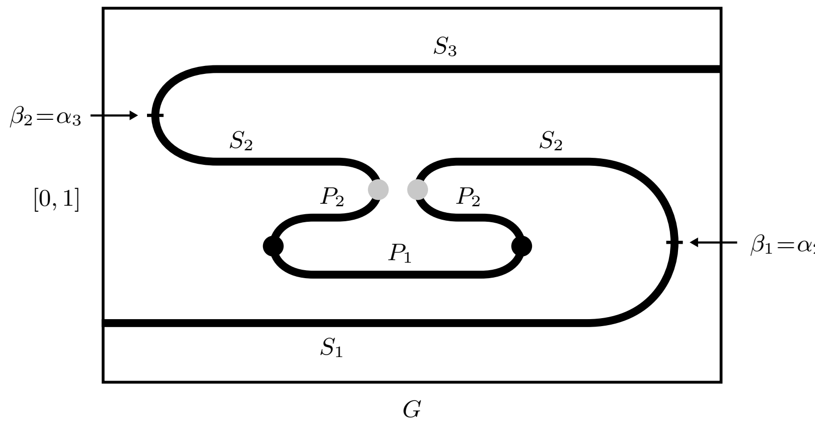

Figure 9. An example of a set with a broken stairwell structure of height 3 with a pit at level 2 in $G_\star$, where G is an arc. The points marked with grey dots comprise the set γ_2, and the points marked with black dots comprise the set $\mathcal{E}(P_1)$.

such that $\pi(P_2)\cap W=\pi(S_{i_0})\cap W$;

(S4′) property (4) above holds for $S_1,...,S_k$, and additionally, $\pi(S_{i_0})$ has consistent complement relative to $\pi(\gamma_{i_0})$, and $\pi(P_2)$ has consistent complement relative to $\pi(\mathcal{E}(P_1))$ and to $\pi(\gamma_{i_0})$;

(S5′) the family $\langle\pi(\alpha_2),...,\pi(\alpha_k),\pi(\mathcal{E}(P_1)),\pi(\gamma_{i_0})\rangle$ (which is the same as the family $\langle\pi(\beta_1),...,\pi(\beta_{k-1}),\pi(\mathcal{E}(P_1)),\pi(\gamma_{i_0})\rangle$) is generic in G;

(S6′) $\pi(\alpha_{i_0})\cap\pi(P_1\cup P_2)=\varnothing$.

See Figure 9 for a simple example of a set with a broken stairwell structure.

Note that even though the sets $S_1,...,S_k$, P_1, and P_2 are all non-empty, we do allow for the possibilities that $\alpha_i=\varnothing$ for some values of $i\in\{2,...,k\}$, that $\mathcal{E}(P_1)=\varnothing$, and that $\gamma_{i_0}=\varnothing$. See also the remarks immediately following Proposition 17 below.

We make the following observation: if $S\subset G_\star$ is a set with a broken stairwell structure with a pit at level i_0, and if C is a component of G such that $S_i\cap C_\star\neq\varnothing$ for each $i=1,...,k$ and $P_j\cap C_\star\neq\varnothing$ for $j=1,2$, then $S\cap C_\star$ has a broken stairwell structure with a pit at level i_0 obtained by intersecting each of the sets S_i, P_1, P_2, α_i, β_i, and γ_{i_0} with $C_\star$.

Proposition 17. *Every set $S\subset G_\star$ which has a stairwell structure of height k has a broken stairwell structure of height $k-2$ with a pit at level 1.*

Proof. Suppose that $S\subset G_\star$ has a stairwell structure $\langle S_1,...,S_k\rangle$ of height k. Let $P_1=S_1$, $P_2=S_2$, and for each $i=1,...,k-2$, let $S'_i=S_{i+2}$. For each $i=2,...,k-2$, let $\alpha'_i=\alpha_{i+2}$ and $\beta'_i=\beta_{i+2}$. Let $\alpha'_1=\varnothing$, $\beta'_1=\beta_3$, and $\gamma'_1=\alpha_3$.

It is now easy to verify that $\langle S'_1,...,S'_{k-2};P_1,P_2\rangle$ is a broken stairwell structure for S of height $k-2$ with a pit at level 1. □

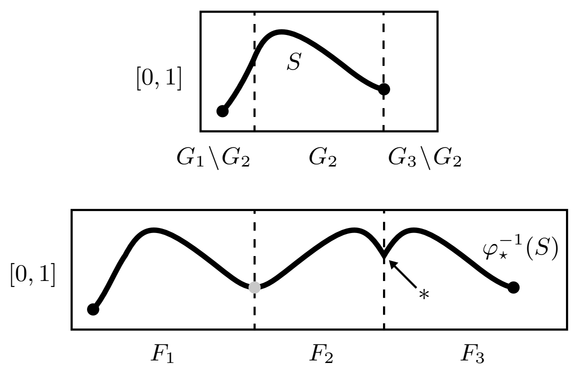

Figure 10. On top, a straight set S in $G_\star$ with end set $\mathcal{E}(S)$ marked with black dots. Underneath, the preimage of S under the map $\varphi_\star$, with the set $\varphi_\star^{-1}(\mathcal{E}(S))$ marked with dots, and only those points in $\mathcal{E}(\varphi_\star^{-1}(S))$ are in black. The point marked with $*$ belongs to the end set of $\varphi_\star^{-1}(S)\cap(F_1\cup F_2)_\star$, even though it does not belong to $\varphi_\star^{-1}(\mathcal{E}(S))$.

We remark that though it may appear at a glance that we could equally well make the pit at level $k-2$ in the above proposition instead of at level 1, property (S6$'$) prevents us from doing so in general.

If $\langle S_1, ..., S_k; P_1, P_2\rangle$ is a broken stairwell structure for $S\subset G_\star$ of height k with a pit at level i_0, and if $\gamma_{i_0}=\varnothing$, then in fact $\langle S_1, ..., S_k\rangle$ is a stairwell structure for $S'=S_1\cup...\cup S_k\subseteq S$. Along the same lines, if $\mathcal{E}(P_1)=\varnothing$ and G is connected, then $\pi(P_1)=G$, and so $\langle P_1\rangle$ is itself a stairwell structure of height 1 for $P_1\subset S$. For these reasons, we will assume in Proposition 19 below that we start with a broken stairwell structure in which $\gamma_{i_0}\neq\varnothing$ and $\mathcal{E}(P_1)\neq\varnothing$.

Our next major task is to prove Proposition 19. Because this is a crucial and delicate part at the heart of the results of this paper, we will treat all the details meticulously. We begin with a lemma to break up and simplify the somewhat involved and tedious proof.

LEMMA 18. *Let $F=F_1\cup F_2\cup F_3$ be a simple fold on a graph G with projection $\varphi: F\to G$, and let $S\subset G_\star$ be straight. Suppose that either $\partial\pi(S)\cap\partial\varphi(F_2)=\varnothing$, or there is a neighborhood V of $\partial\pi(S)\cap\partial\varphi(F_2)$ in G such that $\varphi(F_2)\cap V\subseteq\pi(S)\cap V$. Then, the following properties hold*:

(1) *$S'=\varphi_\star^{-1}(S)$ is straight and $\mathcal{E}(S')=\varphi_\star^{-1}(\mathcal{E}(S))\setminus(\partial F_2)_\star$*;

(2) *$S''=\varphi_\star^{-1}(S)\cap(F_1\cup F_2)_\star$ is straight and*

$$\mathcal{E}(S'')=([\varphi_\star^{-1}(\mathcal{E}(S))\cap(F_1\cup F_2)_\star]\setminus(\partial F_1)_\star)\cup(S''\cap(\partial F_3)_\star).$$

Refer to Figure 10 for an illustration of the situation described in Lemma 18.

Proof. First, we claim that if $x \in \varphi_\star^{-1}(S) \cap (\partial F_2)_\star$, then there is a neighborhood W of $\pi(x)$ such that $\varphi(W) \subset \pi(S)$. To see this, we may assume that $x \in \varphi_\star^{-1}(S) \cap (\partial F_1)_\star$. By hypothesis, there is a neighborhood V of $\varphi(\pi(x))$ such that $\varphi(F_2) \cap V \subseteq \pi(S) \cap V$. We may assume that V is small enough so that if we let $W = \varphi^{-1}(V) \cap (F_1 \cup F_2)$, then W is a neighborhood of $\pi(x)$ and $\varphi(F_1 \cap W) = \varphi(F_2 \cap W) = \varphi(F_2) \cap V$. It follows that $\varphi(W) \subset \pi(S)$. The argument is similar for $x \in \varphi_\star^{-1}(S) \cap (\partial F_3)_\star$.

For (1), note that clearly S' is closed and π is one-to-one on S', since S is closed and π is one-to-one on S. For $x \in S' \setminus (\partial F_2)_\star$, φ is one-to-one in a neighborhood of $\pi(x)$, and so the component of x in S' is non-degenerate since the component of $\varphi_\star(x)$ in S is non-degenerate. For $x \in S' \cap (\partial F_2)_\star$, the component of x in S' is non-degenerate by the above claim. Thus S' is straight.

It is straightforward to see that, for $x \notin (\partial F_2)_\star$, we have that $x \in \mathcal{E}(S')$ if and only if $\varphi_\star(x) \in \mathcal{E}(S)$, since φ is one-to-one on a neighborhood of $\pi(x)$. Moreover, by the above claim, clearly $\mathcal{E}(S') \cap (\partial F_2)_\star = \varnothing$. This establishes (1).

For (2), it can be argued similarly that S'' is a straight. As for the end set of S'', clearly $\mathcal{E}(S'') \subset (F_1 \cap F_2)_\star$ since $S'' \subset (F_1 \cap F_2)_\star$. As in (1), it is straightforward to see that for $x \in S'' \setminus (\partial F_2)_\star$, we have $x \in \mathcal{E}(S'')$ if and only if $\varphi_\star(x) \in \mathcal{E}(S)$, and, by the claim, $\mathcal{E}(S'') \cap (\partial F_1)_\star = \varnothing$. Finally, if $x \in S'' \cap (\partial F_3)_\star$, then clearly any neighborhood of $\pi(x)$ meets both $\pi(S'')$ and the complement of $\pi(S'')$ (since it meets the interior of F_3), and therefore $x \in \mathcal{E}(S'')$. This establishes (2). □

PROPOSITION 19. *Let G be a connected graph, and let $S \subset G_\star$ have a broken stairwell structure $\langle S_1, ..., S_k; P_1, P_2 \rangle$ of height k with a pit at level $i_0 \leqslant k$, in which $\gamma_{i_0} \neq \varnothing$ and $\mathcal{E}(P_1) \neq \varnothing$. Then there exists a simple fold $\varphi: F \to G$ such that F is connected, and $\varphi_\star^{-1}(S)$ contains a set S' with a broken stairwell structure of height k with a pit at level i_0+1 if $i_0 < k$, or simply a stairwell structure of height k if $i_0 = k$.*

The proof of Proposition 19 will occupy the rest of this section. Recall from the comment immediately following Lemma 6 that to uniquely define a simple fold, it suffices to choose three subsets G_1, G_2, and G_3 of G satisfying properties (F1)–(F3). We will define a simple fold in this way, relying on Proposition 11 to verify these properties.

Define the simple fold $F = F_1 \cup F_2 \cup F_3$ by $F_1 \approx G_1 = \pi(P_1)$, $F_2 \approx G_2 = \pi(P_2)$, and $F_3 \approx G_3 = \sigma_{\pi(\gamma_{i_0})}(\pi(S_{i_0}))$, and let $\varphi: F \to G$ be the projection. Hence, F is the union of F_1, F_2 and F_3, with F_1 glued to F_2 along the part corresponding to $\mathcal{E}(P_1)$, and F_2 glued to F_3 along the part corresponding to $\pi(\gamma_{i_0})$. Note that, since G is connected, we have, by the remarks following Definition 9 and by Proposition 10 and (S3′) for S, that

$$\pi(P_1) = \sigma_{\pi(\mathcal{E}(P_1))}(\pi(P_2)) \quad \text{and} \quad \sigma_{\pi(\gamma_{i_0})}(\pi(S_{i_0})) = \sigma_{\pi(\gamma_{i_0})}(\pi(P_2)).$$

So, by (S2′) and Proposition 11, the three sets G_1, G_2, and G_3 do indeed define a simple fold.

We record the following basic observations for reference below:

$$\partial\varphi(F_1)=\varphi(\partial F_1)=\pi(\mathcal{E}(P_1)); \tag{19.1}$$

$$\partial\varphi(F_3)=\varphi(\partial F_3)=\pi(\gamma_{i_0}); \tag{19.2}$$

$$\partial\varphi(F_2)=\partial\varphi(F_1)\cup\partial\varphi(F_3)=\pi(\mathcal{E}(P_1))\cup\pi(\gamma_{i_0}). \tag{19.3}$$

We now describe the set $S'\subseteq\varphi_\star^{-1}(S)$ and its (broken) stairwell structure piece by piece. The reader will find it helpful to refer to Figure 11 when reading the following definitions.

For each $i\notin\{i_0,i_0+1\}$, define

$$S'_i=\varphi_\star^{-1}(S_i),\quad \alpha'_i=\varphi_\star^{-1}(\alpha_i),\quad \beta'_i=\varphi_\star^{-1}(\beta_i).$$

For level i_0, define

$$S'_{i_0}=[\varphi_\star^{-1}(P_1)\cap(F_1)_\star]\cup[\varphi_\star^{-1}(P_2)\cap(F_2)_\star]\cup[\varphi_\star^{-1}(S_{i_0})\cap(F_3)_\star]$$

and

$$\alpha'_{i_0}=\varphi_\star^{-1}(\alpha_{i_0}),\quad \beta'_{i_0}=\varphi_\star^{-1}(\beta_{i_0})\cap(F_3)_\star.$$

If $i_0<k$, then further define

$$\begin{aligned}
S'_{i_0+1}&=\varphi_\star^{-1}(S_{i_0+1}),\\
\alpha'_{i_0+1}&=\varphi_\star^{-1}(\alpha_{i_0+1})\cap(F_3)_\star,\\
\beta'_{i_0+1}&=\varphi_\star^{-1}(\beta_{i_0+1}),\\
\gamma'_{i_0+1}&=\varphi_\star^{-1}(\alpha_{i_0+1})\cap(F_1\cup F_2)_\star,
\end{aligned}$$

as well as

$$P'_1=\varphi_\star^{-1}(P_2)\cap(F_1\cup F_2)_\star \quad\text{and}\quad P'_2=\varphi_\star^{-1}(S_{i_0})\cap(F_1\cup F_2)_\star.$$

We now proceed with confirming that the above sets comprise a (broken) stairwell structure. We begin by showing that the sets $S'_1,\dots,S'_k,P'_1,P'_2$ are all straight, and computing their end sets.

Straightness and end sets

For $i\neq i_0$, we have by (S2′) and (S5′) for S that $\partial\pi(S_i)=\pi(\alpha_i)\cup\pi(\beta_i)$ is disjoint from $\partial\varphi(F_2)=\pi(\mathcal{E}(P_1))\cup\pi(\gamma_{i_0})$ (by (19.3)). Therefore, by Lemma 18, $S'_i=\varphi_\star^{-1}(S_i)$ is straight, and

$$\mathcal{E}(S'_i)=\varphi_\star^{-1}(\mathcal{E}(S_i))\quad\text{for } i\neq i_0. \tag{19.4}$$

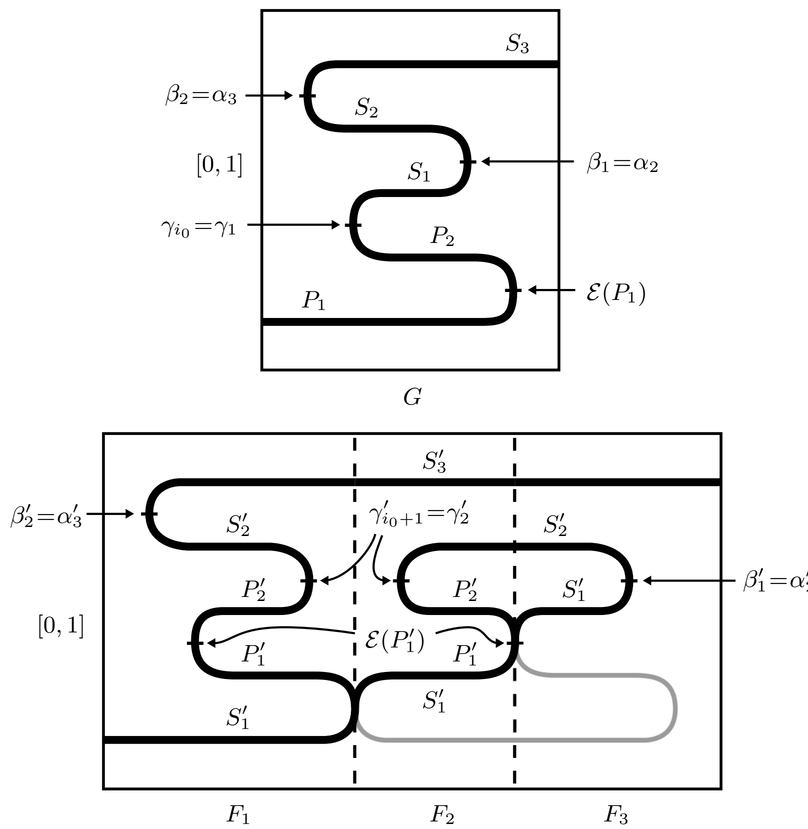

Figure 11. On top, a set S with a broken stairwell structure of height 3 with a pit at level 1. Underneath, the preimage of S under the map $\varphi_\star$, with the subset S' with a broken stairwell structure of height 3 with a pit at level 2 in black. Note that S'_1 and P'_1 overlap in a segment in $(F_2)_\star$.

Observe that $\varphi_\star^{-1}(\mathcal{E}(S_i))$ is disjoint from $(\partial F_2)_\star$.

We now consider S'_{i_0}. Because $\varphi(F_1) = \pi(P_1)$, $\varphi(F_2) = \pi(P_2)$, and

$$\varphi(F_3) = \sigma_{\pi(\gamma_{i_0})}(\pi(S_{i_0})) \supseteq \pi(S_{i_0}),$$

clearly each of $\varphi_\star^{-1}(P_1) \cap (F_1)_\star$, $\varphi_\star^{-1}(P_2) \cap (F_2)_\star$, and $\varphi_\star^{-1}(S_{i_0}) \cap (F_3)_\star$ is straight, as P_1, P_2, and S_{i_0} are straight. From the equalities

$$\varphi_\star^{-1}(P_1) \cap (\partial F_1)_\star = \varphi_\star^{-1}(\mathcal{E}(P_1)) \cap (\partial F_1)_\star = \varphi_\star^{-1}(P_2) \cap (\partial F_1)_\star$$

and

$$\varphi_\star^{-1}(P_2) \cap (\partial F_3)_\star = \varphi_\star^{-1}(\gamma_{i_0}) \cap (\partial F_3)_\star = \varphi_\star^{-1}(S_{i_0}) \cap (\partial F_3)_\star$$

it follows that π is one-to-one on S'_{i_0}. Thus S'_{i_0} is straight.

For the end set of S'_{i_0}, observe that, since $\pi(S'_{i_0})\supset F_1\cup F_2$, one has $\mathcal{E}(S'_{i_0})\subset(F_3)_\star$. Moreover, $\mathcal{E}(S'_{i_0})\cap(\partial F_3)_\star=\varnothing$, because $\pi(S_{i_0})$ agrees with $\varphi(F_3)=\sigma_{\pi(\gamma_{i_0})}(\pi(S_{i_0}))$ near $\pi(\gamma_{i_0})$ in G. Thus

$$F_1\cup F_2\subset\operatorname{int}(\pi(S'_{i_0}))\quad\text{and}\quad\mathcal{E}(S'_{i_0})\subset\operatorname{int}(F_3)_\star. \tag{19.5}$$

By the definition of S'_{i_0}, we have $S'_{i_0}\cap(F_3)_\star=\varphi^{-1}(S_{i_0})\cap(F_3)_\star$, and it follows that

$$\mathcal{E}(S'_{i_0})=\varphi_\star^{-1}(\mathcal{E}(S_{i_0}))\cap\operatorname{int}(F_3)_\star. \tag{19.6}$$

Looking at both cases ($i\neq i_0$ and $i=i_0$) above, we see that

$$\mathcal{E}(S'_i)\cap(\partial F_2)_\star=\varnothing\quad\text{for each } i=1,...,k. \tag{19.7}$$

Next, we consider $P'_1=\varphi_\star^{-1}(P_2)\cap(F_1\cup F_2)_\star$. Observe that $\varphi(F_2)=\pi(P_2)$, so Lemma 18 applies, and we conclude that P'_1 is straight. For the end set of P'_1, we have by Lemma 18 that

$$\mathcal{E}(P'_1)=([\varphi_\star^{-1}(\mathcal{E}(P_2))\cap(F_1\cup F_2)_\star]\setminus(\partial F_1)_\star)\cup(P'_1\cap(\partial F_3)_\star).$$

We simplify this expression using the following straightforward observations:

- $\varphi^{-1}(\pi(\mathcal{E}(P_1)))\cap(F_1\cup F_2)=\partial F_1$, so we can replace $\mathcal{E}(P_2)=\mathcal{E}(P_1)\cup\gamma_{i_0}$ by γ_{i_0} in the above expression;
- $\varphi^{-1}(\pi(\gamma_{i_0}))\subset F_1\cup F_2$ and $\varphi^{-1}(\pi(\gamma_{i_0}))\cap\partial F_1=\varnothing$ (by (19.1) and (S5′) for S), so $[\varphi_\star^{-1}(\gamma_{i_0})\cap(F_1\cup F_2)_\star]\setminus(\partial F_1)_\star=\varphi_\star^{-1}(\gamma_{i_0})$; and
- $P'_1\cap(\partial F_3)_\star=\varphi_\star^{-1}(\gamma_{i_0})$ by (19.2), so $\varphi_\star^{-1}(\gamma_{i_0})\cup(P'_1\cap(\partial F_3)_\star)=\varphi_\star^{-1}(\gamma_{i_0})$.

We thus have

$$\mathcal{E}(P'_1)=\varphi_\star^{-1}(\gamma_{i_0})\subset(F_1\cup F_2)_\star. \tag{19.8}$$

Lastly, we consider $P'_2=\varphi_\star^{-1}(S_{i_0})\cap(F_1\cup F_2)_\star$. By (S5′) for S, we have that

$$\partial\varphi(F_1)\cap\partial\pi(S_{i_0})=\pi(\mathcal{E}(P_1))\cap\partial\pi(S_{i_0})=\varnothing,$$

which means by (19.3) that $\partial\varphi(F_2)\cap\partial\pi(S_{i_0})=\partial\varphi(F_3)=\pi(\gamma_{i_0})$. By (S3′) for S, the sets $\pi(S_{i_0})$ and $\varphi(F_2)=\pi(P_2)$ agree in a neighborhood of $\pi(\gamma_{i_0})$, and hence Lemma 18 applies, and we have that P'_2 is straight.

For the end set of P'_2, we have by Lemma 18 that

$$\mathcal{E}(P'_2)=([\varphi_\star^{-1}(\mathcal{E}(S_{i_0}))\cap(F_1\cup F_2)_\star]\setminus(\partial F_1)_\star)\cup(P'_2\cap(\partial F_3)_\star).$$

We simplify this expression using the following straightforward observations:

- $\partial F_3\subset\varphi^{-1}(\pi(\gamma_{i_0}))\subset F_1\cup F_2$ and $\varphi^{-1}(\pi(\gamma_{i_0}))\cap\partial F_1=\varnothing$ (as above), and hence, since $\mathcal{E}(S_{i_0})=\alpha_{i_0}\cup\beta_{i_0}\cup\gamma_{i_0}$, we obtain

$$\mathcal{E}(P'_2)=([\varphi_\star^{-1}(\alpha_{i_0}\cup\beta_{i_0})\cap(F_1\cup F_2)_\star]\setminus(\partial F_1)_\star)\cup\varphi_\star^{-1}(\gamma_{i_0});$$

- $\varphi^{-1}(\pi(\alpha_{i_0}))\cap(F_1\cup F_2)=\varnothing$ by (S6′) for S, so we can replace $\alpha_{i_0}\cup\beta_{i_0}$ by β_{i_0} in the above expression;
- $\varphi^{-1}(\pi(\beta_{i_0}))\cap\partial F_1=\varnothing$, so

$$[\varphi_\star^{-1}(\beta_{i_0})\cap(F_1\cup F_2)_\star]\setminus(\partial F_1)_\star=\varphi_\star^{-1}(\beta_{i_0})\cap(F_1\cup F_2)_\star=\varphi_\star^{-1}(\alpha_{i_0+1})\cap(F_1\cup F_2)_\star=\gamma'_{i_0+1}.$$

We thus have, by (19.8), that

$$\mathcal{E}(P'_2)=\mathcal{E}(P'_1)\cup\gamma'_{i_0+1}. \tag{19.9}$$

We now continue with the remaining properties to show that the above sets comprise a (broken) stairwell structure.

(S2) / (S2′)

For $i\notin\{i_0,i_0+1\}$, we have $\mathcal{E}(S'_i)=\varphi_\star^{-1}(\alpha_i)\cup\varphi_\star^{-1}(\beta_i)=\alpha'_i\cup\beta'_i$ by (S2′) for S, and clearly $\alpha'_i\cap\beta'_i=\varnothing$ since $\alpha_i\cap\beta_i=\varnothing$. Similarly, if $i_0<k$, then, by (19.4),

$$\begin{aligned}\mathcal{E}(S'_{i_0+1})&=\varphi_\star^{-1}(\mathcal{E}(S_{i_0+1}))\\&=\varphi_\star^{-1}(\alpha_{i_0+1})\cup\varphi_\star^{-1}(\beta_{i_0+1})\\&=[\varphi_\star^{-1}(\alpha_{i_0+1})\cap(F_1\cup F_2)_\star]\cup[\varphi_\star^{-1}(\alpha_{i_0+1})\cap(F_3)_\star]\cup\varphi_\star^{-1}(\beta_{i_0+1})\\&=\gamma'_{i_0+1}\cup\alpha'_{i_0+1}\cup\beta'_{i_0+1}.\end{aligned}$$

We claim that the sets α'_{i_0+1}, β'_{i_0+1}, and γ'_{i_0+1} are pairwise disjoint. Indeed, because $\alpha_{i_0+1}\cap\beta_{i_0+1}=\varnothing$, we immediately have from the definitions of the sets α'_{i_0+1}, β'_{i_0+1}, and γ'_{i_0+1} that $\alpha'_{i_0+1}\cap\beta'_{i_0+1}=\varnothing=\beta'_{i_0+1}\cap\gamma'_{i_0+1}$. Moreover, also from these definitions we see that $\alpha'_{i_0+1}\cap\gamma'_{i_0+1}\subseteq(F_3)_\star\cap(F_1\cup F_2)_\star=(\partial F_3)_\star\subseteq(\partial F_2)_\star$. But also $\alpha'_{i_0+1},\gamma'_{i_0+1}\subseteq\mathcal{E}(S'_{i_0+1})$, and $\mathcal{E}(S'_{i_0+1})\cap(\partial F_2)_\star=\varnothing$ by (19.7). Thus $\alpha'_{i_0+1}\cap\gamma'_{i_0+1}=\varnothing$.

For S'_{i_0}, we have by (19.6) and the fact that $\varphi_\star^{-1}(\gamma_{i_0})\cap\operatorname{int}(F_3)_\star=\varnothing$ that

$$\mathcal{E}(S'_{i_0})=\varphi_\star^{-1}(\mathcal{E}(S_{i_0}))\cap\operatorname{int}(F_3)_\star=[\varphi_\star^{-1}(\alpha_{i_0})\cap(F_3)_\star]\cup[\varphi_\star^{-1}(\beta_{i_0})\cap(F_3)_\star].$$

Moreover, by (S6′) for S and since $\varphi(F_1\cup F_2)=\pi(P_1\cup P_2)$, we have $\varphi_\star^{-1}(\alpha_{i_0})\subset(F_3)_\star$, so that $\varphi_\star^{-1}(\alpha_{i_0})\cap(F_3)_\star=\varphi^{-1}(\alpha_{i_0})=\alpha'_{i_0}$. Therefore we have that $\mathcal{E}(S'_{i_0})=\alpha'_{i_0}\cup\beta'_{i_0}$. Again, clearly $\alpha'_{i_0}\cap\beta'_{i_0}=\varnothing$ since $\alpha_{i_0}\cap\beta_{i_0}=\varnothing$.

It is straightforward to see that $\beta'_i=\alpha'_{i+1}$ for each $i=1,...,k-1$, since $\beta_i=\alpha_{i+1}$ for each $i=1,...,k-1$ by (S2′) for S. The only standout case is when $i=i_0$ (if $i_0<k$), and here we have $\beta'_{i_0}=\varphi_\star^{-1}(\beta_{i_0})\cap(F_3)_\star=\varphi_\star^{-1}(\alpha_{i_0+1})\cap(F_3)_\star=\alpha'_{i_0+1}$. Obviously, $\alpha'_1=\varnothing=\beta'_k$ since $\alpha_1=\varnothing=\beta_k$.

We have already deduced in (19.9) that $\mathcal{E}(P'_2)=\mathcal{E}(P'_1)\cup\gamma'_{i_0+1}$, and clearly the sets $\mathcal{E}(P'_1)=\varphi_\star^{-1}(\gamma_{i_0})$ and $\gamma'_{i_0+1}=\varphi_\star^{-1}(\alpha_{i_0+1})\cap(F_1\cup F_2)_\star$ are disjoint, since, by (S5′) for S, $\gamma_{i_0}\cap\alpha_{i_0+1}=\varnothing$.

(S3) / (S3′)

Since $\pi(\mathcal{E}(S_i'))\cap\partial F_2=\varnothing$ for each $i=1,...,k$ (by (19.7)), we have that φ is one-to-one in a neighborhood of each point of $\pi(\mathcal{E}(S_i'))$. It is then straightforward to see from property (S3′) for S and from the definition of S_i' that there exists a neighborhood V of $\pi(\beta_i')=\pi(\alpha_{i+1}')$ such that $\pi(S_i')\cap V=\pi(S_{i+1}')\cap V$. Again, the only standout case is when $i=i_0$, and here $\pi(\mathcal{E}(S_{i_0}'))\subset\operatorname{int}(F_3)$, and $\pi(S_{i_0}')\cap F_3=\varphi^{-1}(\pi(S_{i_0}))\cap F_3$, so the neighborhood of $\beta_{i_0}=\alpha_{i_0+1}$ in G in which $\pi(S_{i_0})$ and $\pi(S_{i_0+1})$ agree, pulls back under $(\varphi|_{F_3})^{-1}$ to a neighborhood of $\beta_{i_0}'=\alpha_{i_0+1}'$ in which $\pi(S_{i_0}')$ and $\pi(S_{i_0+1}')$ agree.

If $i_0<k$, then by (19.7), we in particular have that $\pi(\gamma_{i_0+1}')\cap\partial F_2=\varnothing$, and so φ is one-to-one in a neighborhood of each point of $\pi(\gamma_{i_0+1}')$. Then as above we have that there is a neighborhood of $\pi(\gamma_{i_0+1}')\subset F_1\cup F_2$ on which $\pi(S_{i_0+1}')=\varphi^{-1}(\pi(S_{i_0+1}))$ and $P_2'=\varphi^{-1}(\pi(S_{i_0}))\cap(F_1\cup F_2)$ agree.

For P_1' and P_2', recall from (19.8) that $\mathcal{E}(P_1')=\varphi_\star^{-1}(\gamma_{i_0})$, which is contained in $(F_1\cup F_2)_\star$. Let $z\in\pi(\mathcal{E}(P_1'))$. Note that $z\notin\partial F_1$ since $\varphi(\partial F_1)=\pi(\mathcal{E}(P_1))$ by (19.1), and $\pi(\mathcal{E}(P_1))\cap\pi(\gamma_{i_0})=\varnothing$ by (S5′) for S. If $z\notin\partial F_3$, then φ is one-to-one in a neighborhood of z, so as above there is a neighborhood of z on which $\pi(P_1')$ and $\pi(P_2')$ agree.

If $z\in\partial F_3$, then by (S5′) for S, $\varphi(z)$ is not a branch point of G, so there is a neighborhood of z in F which is homeomorphic to an open arc J. Note that $J\setminus\{z\}$ is the union of two open arcs J_1 and J_2, where $J_1\subset\operatorname{int}(F_1\cup F_2)$ and $J_2\subset\operatorname{int}(F_3)$. Since P_1' and P_2' are contained in $(F_1\cup F_2)_\star$ and $z\in\pi(\mathcal{E}(P_1'))\subset\pi(\mathcal{E}(P_2'))$ (by (19.9)), there is a neighborhood $W\subset J$ of z in F such that $W\cap(F_1\cup F_2)\subset\pi(P_1')\cap\pi(P_2')$. On the other hand, $W\cap\operatorname{int}(F_3)$ is disjoint from $\pi(P_1')$ and from $\pi(P_2')$. Thus $\pi(P_1')\cap W=\pi(P_2')\cap W$.

(S4) / (S4′)

Given $i\notin\{i_0,i_0+1\}$, let C be a component of $F\setminus\pi(S_i')=F\setminus\varphi^{-1}(\pi(S_i))$. Then $\varphi(C)$ is contained in a component of $G\setminus\pi(S_i)$, and hence $\overline{\varphi(C)}$ meets at most one of $\pi(\alpha_i)$ and $\pi(\beta_i)$. It follows that $\partial C\subseteq\pi(\alpha_i')=\varphi^{-1}(\pi(\alpha_i))$ or $\partial C\subseteq\pi(\beta_i')=\varphi^{-1}(\pi(\beta_i))$.

For level i_0, let C be a component of $F\setminus\pi(S_{i_0}')$. By (19.5), $F_1\cup F_2\subset\operatorname{int}(\pi(S_{i_0}'))$, and hence $\overline{C}\subset\operatorname{int}(F_3)$. Also, we have $C\subset F_3\setminus\varphi^{-1}(\pi(S_{i_0}))$, since $S_{i_0}'\cap(F_3)_\star=\varphi^{-1}(S_{i_0})\cap(F_3)_\star$. Therefore again $\varphi(C)$ is contained in a component of $G\setminus\pi(S_{i_0})$, and hence $\overline{\varphi(C)}$ meets at most one of $\pi(\alpha_{i_0})$, $\pi(\beta_{i_0})$, and $\pi(\gamma_{i_0})$. Note however that $\overline{\varphi(C)}\cap\pi(\gamma_{i_0})=\varnothing$, since $\varphi^{-1}(\pi(\gamma_{i_0}))\cap F_3=\partial F_3$ and $\overline{C}\cap\partial F_3=\varnothing$. As a consequence, we have that $\overline{C}$ meets at most one of $\pi(\alpha_{i_0}')=\varphi^{-1}(\pi(\alpha_{i_0}))$ and $\pi(\beta_{i_0}')=\varphi^{-1}(\pi(\beta_{i_0}))\cap F_3$.

Now suppose that $i_0<k$, and consider the level i_0+1. Let C be a component of $F\setminus\pi(S_{i_0+1}')$. Since $S_{i_0+1}'=\varphi_\star^{-1}(S_{i_0+1})$ and $\beta_{i_0+1}'=\varphi_\star^{-1}(\beta_{i_0+1})$, we have as above that, if $\partial C\cap\pi(\beta_{i_0+1}')\neq\varnothing$, then $\partial C\subset\pi(\beta_{i_0+1}')$.

Suppose, on the other hand, that $\partial C\cap\pi(\alpha'_{i_0+1})\neq\varnothing$ or $\partial C\cap\pi(\gamma'_{i_0+1})\neq\varnothing$. Then $\partial\varphi(C)\cap\pi(\alpha_{i_0+1})\neq\varnothing$. It follows that $\varphi(C)$ is contained in a component $\widetilde{C}$ of $G\setminus\pi(S_{i_0+1})$ whose boundary is contained in $\pi(\alpha_{i_0+1})$. By Proposition 10, $\widetilde{C}$ is also a component of $G\setminus\pi(S_{i_0})$ whose boundary is contained in $\pi(\beta_{i_0})$, because $\pi(S_{i_0+1})$ and $\pi(S_{i_0})$ agree in a neighborhood of $\pi(\alpha_{i_0+1})=\pi(\beta_{i_0})$, and $\pi(S_{i_0})$ has consistent complement relative to $\pi(\beta_{i_0})$.

Observe that $\varphi(\partial F_3)=\pi(\gamma_{i_0})\subset\pi(S_{i_0})$ and $\widetilde{C}\subseteq G\setminus\pi(S_{i_0})$, so $C\cap\partial F_3=\varnothing$. Because C is connected, by Lemma 6 (6) this means that $C\subseteq F_1\cup F_2$ or $C\subseteq F_3$. Therefore, by the definitions of α'_{i_0+1} and γ'_{i_0+1}, either $\partial C\subset\pi(\alpha'_{i_0+1})$ or $\partial C\subset\pi(\gamma'_{i_0+1})$.

Now let D be a component of $F\setminus\pi(P'_2)$. Note that

$$\partial F_3\subset\varphi^{-1}(\pi(\gamma_{i_0}))\cap(F_1\cup F_2)\subset\varphi^{-1}(\pi(S_{i_0}))\cap(F_1\cup F_2)=\pi(P'_2),$$

so $D\cap\partial F_3=\varnothing$. By Lemma 6 (6), this means that $D\subset F_1\cup F_2$ or $D\subset F_3$.

If $D\subset F_3$, then since $F_3\cap\pi(P'_2)=\partial F_3$, we have $\partial D\subset\partial F_3\subset\pi(\mathcal{E}(P'_1))$. If $D\subset F_1\cup F_2$, then since $\pi(P'_2)=\varphi^{-1}(\pi(S_{i_0}))\cap(F_1\cup F_2)$, we have that $\varphi(D)$ is contained in a component $\widetilde{D}$ of $G\setminus\pi(S_{i_0})$. Moreover, $\varphi(D)\subset\pi(P_1\cup P_2)$, so $\partial\widetilde{D}\cap\pi(\alpha_{i_0})=\varnothing$ by (S6$'$) for S. This means that either $\partial\widetilde{D}\subset\pi(\beta_{i_0})$ or $\partial\widetilde{D}\subset\pi(\gamma_{i_0})$. Then because $\mathcal{E}(P'_1)=\varphi_\star^{-1}(\gamma_{i_0})$ and

$$\gamma'_{i_0+1}=\varphi_\star^{-1}(\alpha_{i_0+1})\cap(F_1\cup F_2)_\star=\varphi_\star^{-1}(\beta_{i_0})\cap(F_1\cup F_2)_\star,$$

it follows that $\partial D\subset\pi(\mathcal{E}(P'_1))$ or $\partial D\subset\pi(\gamma'_{i_0+1})$.

(S5) / (S5$'$)

Since $\partial\pi(P_2)$ is disjoint from the set Z of branch points and endpoints of G, we have that the set of branch points and endpoints of F is $\varphi^{-1}(Z)$. It is then trivial to see from the definitions of the sets $\alpha'_2,...,\alpha'_k,\gamma'_{i_0+1},\mathcal{E}(P'_1)$, and from property (S5$'$) for S, that the family $\langle\pi(\alpha'_2),...,\pi(\alpha'_k),\pi(\gamma'_{i_0+1}),\pi(\mathcal{E}(P'_1))\rangle$ (or simply $\langle\pi(\alpha'_2),...,\pi(\alpha'_k)\rangle$ in the case $i_0=k$) is generic.

(S6$'$)

Recall that $\alpha'_{i_0+1}=\beta'_{i_0}\subset\operatorname{int}(F_3)_\star$ by (19.5), which means that $\alpha'_{i_0+1}\cap(F_1\cup F_2)_\star=\varnothing$. Thus $\pi(\alpha'_{i_0+1})\cap\pi(P'_1\cup P'_2)=\varnothing$, since P'_1 and P'_2 are contained in $(F_1\cup F_2)_\star$.

This completes the proof of all the properties required to prove that $\langle S'_1,...,S'_k;P'_1,P'_2\rangle$ is a broken stairwell structure for $S'=S'_1\cup...\cup S'_k\cup P'_1\cup P'_2$ of height k with a pit at level i_0+1, or, in the case that $i_0=k$, that $\langle S'_1,...,S'_k\rangle$ is a stairwell structure for $S'=S'_1\cup...\cup S'_k$.

Finally, to obtain a connected simple fold, we observe that since $\mathcal{E}(P_1)\neq\varnothing$ and $\gamma_{i_0}\neq\varnothing$ (by assumption), and since G is connected and $\pi(P_2)$ has consistent complement relative to $\pi(\mathcal{E}(P_1))$ and to $\pi(\gamma_{i_0})$, there is a component K of $\pi(P_2)=\varphi(F_2)$ such that K meets both $\pi(\mathcal{E}(P_1))=\partial\varphi(F_1)$ and $\pi(\gamma_{i_0})=\partial\varphi(F_3)$.

By Proposition 7, $(\varphi|_{F_2})^{-1}(K)$ is contained in a component C of F such that $\varphi(C)=G$, and $F'=F_1'\cup F_2'\cup F_3'$, where $F_i'=F_i\cap C$ for each $i=1,2,3$, is a connected simple fold. For $i\neq i_0$, since $\varphi(C)=G$ and $S_i'=\varphi_\star^{-1}(S_i)$, we have $S_i'\cap C_\star\neq\varnothing$. Also, all three of S_{i_0}', P_1', and P_2' contain $\varphi_\star^{-1}(\gamma_{i_0})\cap(\partial F_3)_\star$, and clearly C meets ∂F_3 by Lemma 6 (6). Thus $S_{i_0}'\cap C_\star$, $P_1'\cap C_\star$, and $P_2'\cap C_\star$ are all non-empty as well. Therefore, by the remarks following Definitions 13 and 16, the (broken) stairwell structure on $S'\subset F_\star$ yields a (broken) stairwell structure on $S'\cap C_\star$.

This concludes the proof of Proposition 19.

7. Applications

We are now in a position to state and prove our main technical theorem.

THEOREM 20. *A compactum X is hereditarily indecomposable if and only if for any map $f\colon X\to G$ to a graph G, any set $M\subseteq G\times(0,1)$ separating $G\times\{0\}$ from $G\times\{1\}$ in $G\times[0,1]$, any open set $U\subseteq G\times[0,1]$ with $M\subseteq U$, and any $\varepsilon>0$, there exists a map $h\colon X\to U$ such that $d_{\sup}(f,\pi_1\circ h)<\varepsilon$ (where $\pi_1\colon G\times[0,1]\to G$ is the first coordinate projection).*

Proof. Suppose that X is a hereditarily indecomposable compactum. Let $f\colon X\to G$ be a map to a graph G, let $M\subset G\times(0,1)$ separate $G\times\{0\}$ from $G\times\{1\}$ in $G\times[0,1]$, and let U be a neighborhood of M in $G\times[0,1]$. By treating the components of G one at a time, and because the inverse image of any component under f is a hereditarily indecomposable closed and open subset of X, we may assume without loss of generality that G is connected.

By Theorem 15, there is a set $S\subset U$ with a stairwell structure of odd height k_0. We claim that there is a finite sequence $G=F^0,F^1,...,F^n$ of connected graphs such that, for each $i=1,...,n$, F^i is a simple fold on F^{i-1} with projection $\varphi_i\colon F^i\to F^{i-1}$, and the preimage $((\varphi_n)_\star\circ...\circ(\varphi_1)_\star)^{-1}(S)$ contains a set S' with a stairwell structure of height 1. We construct this sequence by induction as follows. Let $F^0=G$.

Step 1. Assume we have a set $S\subset(F^j)_\star$ with a stairwell structure of height k. If $k=1$, then we are done. Otherwise, by Proposition 17, S has a broken stairwell structure of height $k-2$ with a pit at level 1.

Step 2. Assume that $S\subset(F^j)_\star$ and that $\langle S_1,...,S_{k-2},P_1,P_2\rangle$ is a broken stairwell structure on S of height $k-2$ with a pit at level i_0. As per the remarks following Proposition 17, if $\gamma_{i_0}=\varnothing$, then in fact S has a stairwell structure of height $k-2$, and we may return to Step 1 with this stairwell structure. Similarly, if $\mathcal{E}(P_1)=\varnothing$, then in fact $S'=P_1\subseteq S$ itself has a stairwell structure of height 1, and we are done.

Suppose now that $\gamma_{i_0}\neq\varnothing$ and $\mathcal{E}(P_1)\neq\varnothing$. If $i_0<k-2$, then by Proposition 19, there is a simple fold $\varphi_j\colon F^{j+1}\to F^j$, where F^{j+1} is a connected graph, and a set $S'\subseteq\varphi_j^{-1}(S)$ with a broken stairwell structure of height $k-2$ with a pit at level i_0+1, and we may repeat Step 2 for $S'\subset(F^{j+1})_\star$. If $i_0=k-2$, then by Proposition 19, there is a simple fold $\varphi_j\colon F^{j+1}\to F^j$, where F^{j+1} is a connected graph, and a set $S'\subseteq\varphi_j^{-1}(S)$ with a stairwell structure of height $k-2$, and we may repeat the entire process starting at Step 1 for $S'\subset(F^{j+1})_\star$.

In this way, after a sequence of at most $(k_0-1)+(k_0-3)+...+1$ simple folds, we obtain the desired sequence $G=F^0,F^1,...,F^n$ and desired set $S'\subset(F^n)_\star$. Clearly the first coordinate projection $\pi_1\colon F^n\times[0,1]\to F^n$ carries S' one-to-one onto F^n, so there is an inverse $\theta\colon F^n\to S'$.

Let $g_0=f$. By Theorem 8, for each $i=1,...,n$, there is a map $g_i\colon X\to F^i$ such that $d_{\sup}(\varphi_i\circ g_i,g_{i-1})<\varepsilon_i$, where the numbers $\varepsilon_i>0$ are chosen small enough so that if we let $g=\varphi_1\circ...\circ\varphi_n\circ g_n$, then $d_{\sup}(f,g)<\varepsilon$.

Define $h\colon X\to G_\star$ by $h=(\varphi_1)_\star\circ...\circ(\varphi_n)_\star\circ\theta\circ g_n$. We further assume that the numbers ε_i are chosen small enough so that $h(X)\subset U$. Then $\pi_1\circ h=g$, and hence $d_{\sup}(f,\pi_1\circ h)<\varepsilon$.

For the converse, assume that X is compact and that the right side of the "if and only if" statement holds. Let $f\colon X\to[0,1]$ be a map, and let $\varphi\colon F\to[0,1]$ be a simple fold such that F is an arc. Consider a "zig-zag" set $S\subset[0,1]\times(0,1)$ which is the union of three straight sets $S_1,S_2,S_3\subset[0,1]\times(0,1)$ such that $\pi_1(S_i)=\varphi(F_i)$ for $i=1,2,3$, $S_1\cap S_2=\mathcal{E}(S_1)$, $S_2\cap S_3=\mathcal{E}(S_3)$, and $S_1\cap S_3=\varnothing$. Clearly S separates $[0,1]\times\{0\}$ from $[0,1]\times\{1\}$ in the square $[0,1]\times[0,1]$. Note also that there is a homeomorphism $\varrho\colon S\to F$ such that $\varphi\circ\varrho=\pi_1$ on S.

Fix $\varepsilon>0$, and let U be a small neighborhood of S in $[0,1]\times[0,1]$ for which there is a $\frac{1}{2}\varepsilon$-retraction $r\colon U\to S$—in particular, such that on U we have $d_{\sup}(\pi_1\circ r,\pi_1)<\frac{1}{2}\varepsilon$. By hypothesis, there is a map $h\colon X\to U$ such that $d_{\sup}(f,\pi_1\circ h)<\frac{1}{2}\varepsilon$. Let $g=\varrho\circ r\circ h\colon X\to F$. Observe that $\varphi\circ g=\pi_1\circ r\circ h$. Then we have $d_{\sup}(\varphi\circ g,\pi_1\circ h)=d_{\sup}(\pi_1\circ r\circ h,\pi_1\circ h)<\frac{1}{2}\varepsilon$ and $d_{\sup}(f,\pi_1\circ h)<\frac{1}{2}\varepsilon$, and hence $d_{\sup}(f,\varphi\circ g)<\varepsilon$.

Therefore, by Theorem 8, X is hereditarily indecomposable. □

We now recall and prove Theorem 1, from which the classification of homogeneous plane continua (and compacta) follows as detailed in the Introduction above.

Theorem 1. *A continuum X is homeomorphic to the pseudo-arc if and only if X is hereditarily indecomposable and has span zero.*

Proof. The pseudo-arc is hereditarily indecomposable and arc-like, and all arc-like continua have span zero [29], hence the pseudo-arc has span zero.

For the converse, let X be a hereditarily indecomposable continuum in the Hilbert cube $[0,1]^{\mathbb{N}}$ with span zero, and fix $\varepsilon>0$. We will show there is an ε-map from X to an arc.

By Theorem 4, there exists $\delta>0$ small enough so that if $G\subset[0,1]^{\mathbb{N}}$ is a graph and $I\subset[0,1]^{\mathbb{N}}$ is an arc with endpoints p and q, such that the Hausdorff distance from X to each of G and I is less than δ, then the set

$$M=\left\{(x,y)\in G\times(I\setminus\{p,q\}) : d(x,y)<\tfrac{1}{6}\varepsilon\right\}$$

separates $G\times\{p\}$ from $G\times\{q\}$ in $G\times I$. We may assume $\delta\leqslant\frac{1}{6}\varepsilon$.

Let $I\subset[0,1]^{\mathbb{N}}$ be an arc with endpoints p and q such that $d_H(X,I)<\delta$. Since X has span zero, by [31] we have that X is tree-like. Therefore, there exists a tree $T\subset[0,1]^{\mathbb{N}}$ and a map $f\colon X\to T$ such that $d_{\sup}(f,\mathrm{id}_X)<\delta$. It follows that $d_H(X,T)<\delta$. Hence, by choice of δ, the set

$$M=\left\{(x,y)\in T\times(I\setminus\{p,q\}) : d(x,y)<\tfrac{1}{6}\varepsilon\right\}$$

separates $T\times\{p\}$ from $T\times\{q\}$ in $T\times I$.

Let $\pi_1\colon T\times I\to T$ and $\pi_2\colon T\times I\to I$ denote the first and second coordinate projections, respectively. Since M is open, by Theorem 20 there is a map $h\colon X\to M$ such that $d_{\sup}(f,\pi_1\circ h)<\frac{1}{6}\varepsilon$.

We claim that $\pi_2\circ h\colon X\to I$ is such that $d_{\sup}(\pi_2\circ h,\mathrm{id}_X)<\frac{1}{2}\varepsilon$, which means that $\pi_2\circ h$ is an ε-map. Indeed, given $x\in X$, we have

$$d(x,\pi_2\circ h(x))\leqslant d(x,f(x))+d(f(x),\pi_1\circ h(x))+d(\pi_1\circ h(x),\pi_2\circ h(x))<\delta+\tfrac{1}{6}\varepsilon+\tfrac{1}{6}\varepsilon\leqslant\tfrac{1}{2}\varepsilon,$$

where the second inequality follows from $d_{\sup}(f,\mathrm{id}_X)<\delta$, $d_{\sup}(f,\pi_1\circ h)<\frac{\varepsilon}{6}$, and $h(x)\in M$, and the last inequality holds since $\delta\leqslant\frac{1}{6}\varepsilon$.

Therefore X is arc-like. Because X is hereditarily indecomposable and arc-like, it is homeomorphic to the pseudo-arc [3]. □

8. Discussion and questions

A closely related classification problem of significant interest is: What are all the homogeneous hereditarily indecomposable continua? This question was asked by Jones in [17].

It is known, by results of Prajs and Krupski [25], and of Rogers [50], that a homogeneous continuum is hereditarily indecomposable if and only if it is tree-like. Thus far, the pseudo-arc is the only known example of a non-degenerate homogeneous tree-like continuum.

Question 1. If X is a homogeneous tree-like (equivalently, hereditarily indecomposable) continuum, must X be homeomorphic to the pseudo-arc?

By the results of this paper, if there is another such continuum, it would necessarily be non-planar. An affirmative answer to this question would follow if one could prove that every homogeneous tree-like continuum has span zero. The question of whether every homogeneous tree-like continuum has span zero was raised by Ingram in [7, Problem 93].

Theorem 20 can also be applied to the study of hereditarily equivalent spaces. A continuum X is *hereditarily equivalent* if X is homeomorphic to each of its non-degenerate subcontinua. In a forthcoming paper [14], the authors use Theorem 20 to show that the only non-degenerate hereditarily equivalent plane continua are the arc and the pseudo-arc.

Recall from the comments immediately preceding Theorem 4 that a continuum X has *surjective semispan zero* [30] if every subcontinuum $Z \subseteq X \times X$ with $\pi_2(Z) = X$ meets the diagonal $\Delta X = \{(x, x) : x \in X\}$. It is proved in [46] that any continuum with surjective semispan zero is tree-like. Our proof of Theorem 1 in fact establishes the following slightly stronger characterization of the pseudo-arc.

THEOREM 1′. *A continuum X is homeomorphic to the pseudo-arc if and only if X is hereditarily indecomposable and has surjective semispan zero.*

It is clear that every continuum with span zero has surjective semispan zero, but it is not known whether these two properties are equivalent.

Question 2. (Cf. [7, Problem 59]) Does every continuum with surjective semispan zero have span zero?

Besides the property of span zero, another property related to arc-likeness is weak chainability: a continuum is *weakly chainable* if it is the continuous image of an arc-like continuum. This concept was introduced by Lelek [28] who used an equivalent formulation involving weak chain covers. Lelek [28] and Fearnley [9] independently proved the equivalence of these two notions, and observed that every arc-like continuum is the continuous image of the pseudo-arc, which means that a continuum is weakly chainable if and only if it is the continuous image of the pseudo-arc.

It is known that all arc-like continua have span zero [29], and all span zero continua are weakly chainable [46]. Therefore, if the answer to the following question is affirmative, it would yield a still stronger characterization of the pseudo-arc than our Theorem 1.

Question 3. If X is a hereditarily indecomposable and weakly chainable continuum, must X be homeomorphic to the pseudo-arc?

It is known (see e.g. [32] and [40]) that a hereditarily indecomposable and weakly chainable continuum must be tree-like.

It is possible to formulate a version of Theorem 20 without any mention of separators in the product of a graph with an arc, which more directly generalizes Theorem 8. To this end, we give a generalization of the notion of a simple fold (Definition 5), which is inspired by our definition of a stairwell structure (Definition 13).

Definition 21. A *folding map* on G is given by a graph $F=F_1\cup...\cup F_k$ and a function $\varphi: F\to G$, called the *projection*, which satisfy the following properties, where $G_i=\varphi(F_i)$ for $i=1,...,k$:

(FM1) k is odd, and $G_1,...,G_k$ are regular subsets of G;

(FM2) for each $i=1,...,k$, $\partial G_i=A_i\cup B_i$, where A_i and B_i are disjoint finite sets, $A_1=B_k=\varnothing$, and $B_i=A_{i+1}$ for each $i=1,...,k-1$;

(FM3) for each $i=1,...,k-1$ there exists a neighborhood V of $B_i=A_{i+1}$ such that $G_i\cap V=G_{i+1}\cap V$;

(FM4) for each $i=1,...,k$, G_i has consistent complement relative to A_i and to B_i;

(FM5) $\varphi|_{F_i}$ is a homeomorphism $F_i\to G_i$ for each $i=1,...,k$;

(FM6) $\varphi(F_i\cap F_{i+1})=B_i=A_{i+1}$ for $i=1,...,k-1$, and $F_i\cap F_j=\varnothing$ whenever $|i-j|>1$.

It is straightforward to see that given a folding map $\varphi: F\to G$ to a connected graph G, one can construct a set $S\subset G\times(0,1)$ with a stairwell structure corresponding to φ as in the proof of Theorem 20. In this way, one can prove the following result.

THEOREM 22. *A compactum X is hereditarily indecomposable if and only if for any map $f: X\to G$ to a connected graph G, any folding map $\varphi: F\to G$, and any $\varepsilon>0$, there exists a map $g: X\to F$ such that $d_{\sup}(f,\varphi\circ g)<\varepsilon$.*

Observe that the linear ordering of the sets $F_1,...,F_k$, where each of these sets meets only its immediate successor and predecessor, is an essential feature which causes the correspondance between folding maps and sets in $G\times(0,1)$ with stairwell structures (for connected graphs G). However, inspired by the notion of a broken stairwell structure, one could formulate a more general concept of a folding map, in which the adjacency relation on the sets $F_1,...,F_k$ (here we say that F_i and F_j are *adjacent* if $F_i\cap F_j\neq\varnothing$) is a tree (or more generally any graph), instead of an arc (linear order).

Question 4. Can one prove a version of Theorem 8 (and Theorem 22) which pertains to a notion of folding maps $\varphi : F \to G$ for which the subgraphs of F on which φ is one-to-one are allowed to have an adjacency relation which is a tree? More generally, under what conditions on this adjacency relation does there exist, for any map $f : X \to G$ from a hereditarily indecomposable compactum X and any $\varepsilon > 0$, a map $g : X \to F$ such that $d_{\sup}(f, \varphi \circ g) < \varepsilon$?

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LOGAN C. HOEHN
Nipissing University
Department of Computer Science & Mathematics
100 College Drive
P.O. Box 5002
North Bay, Ontario P1B 8L7
Canada
loganh@nipissingu.ca

LEX G. OVERSTEEGEN
University of Alabama at Birmingham
Department of Mathematics
Birmingham, AL 35294
U.S.A.
overstee@uab.edu

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