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Zeros of successive derivatives

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With 7 figures in the text

0. Introduction

We are going to study the distribution of the zeros of the successive derivatives of analytical functions. The main problem is to find the distribution of the limit points of these zeros. The concept of limit point will be used in the following sense. A point x is a limit point of the zeros of the derivatives, if and only if to every neighbourhood of x there is an infinity of derivatives which have zeros in this neighbourhood. Thus e. g. if the analytical function in question can be developed around the origin in a power series with gaps, then the origin is a limit point of the zeros of the derivatives.

0.1. The following notation will be used

$f(z)$ = the given analytical function.

$a_n(x) = \dfrac{f^{(n)}(x)}{\lfloor n}; \quad \because \; f(z) = \sum_{n=0}^{\infty} a_n(x)(z-x)^n.$

$g_{nm}(x) = \sqrt[n]{a_n(x)}$ where the index m indicates the branch of $\sqrt[n]{a_n}$. m is assumed to take one of the values $0; 1; \ldots (n-1)$.

ν denotes a variable which takes the values $1, 2, 3 \ldots$

n_ν denotes a monotonic sequence of natural numbers.

$C(\alpha; \beta)$ denotes *the circumference* of the circle with center α and radius β.

D denotes a bounded simply connected open domain which does not contain any limit point of the zeros of the derivatives of f.

D_- denotes a simply connected closed domain $\subset D$ (D_- is to be thought of as being very close to D).

$x \in D$ will often be used to express that x is not a limit point of zeros. When used with this meaning the relation is to be read thus: x is an element of some D. In an analogous way the relation $x \notin D$ will often mean that x is a limit point, and ought to be read: x is not an element of any D. This meaning of the symbols $x \in D$ and $x \notin D$ is used when D is not specified.

$R(x)$ = distance from x to nearest singular point of $f(z)$.

$r(x)$ = distance from x to nearest limit point of zeros or singular point. Thus $r(x) \leq R(x)$.

A sequence $\sqrt[n_\nu]{|a_{n_\nu}(x)|}$ is said to be maximal at a point x_0 if $\lim \sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ exists and $= \dfrac{1}{R(x_0)}$.

A singular point s is said to be of type A with respect to a point x_0 if x_0 lies within an open sector of a circle, the arc of which touches $C(x_0; R(x_0))$ at s and which does not contain any singular point of f. (Fig. 1.)

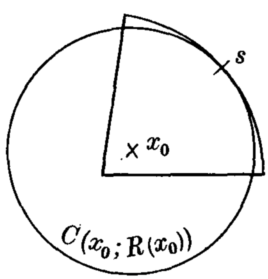


Fig. 1.

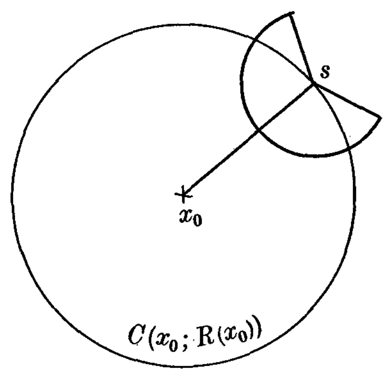


Fig. 2.

A singular point s is said to be of type B with respect to a point x_0 if it lies on $C(x_0; R(x_0))$ and if there exists an open sector of a circle with its cusp at s, the angle of which is bisected by the line from s to x_0 and is greater than π, and which does not contain any singular point of f. (Fig. 2.)

Equivalent limit functions: We are going to study the family $\{g_{nm}\}$. Two limit functions of this family $\varphi_1(x)$ and $\varphi_2(x)$ are said to be equivalent if there is a number η with $|\eta| = 1$ such that $\varphi_1 = \eta\,\varphi_2$.

0.2. Preliminary remarks:

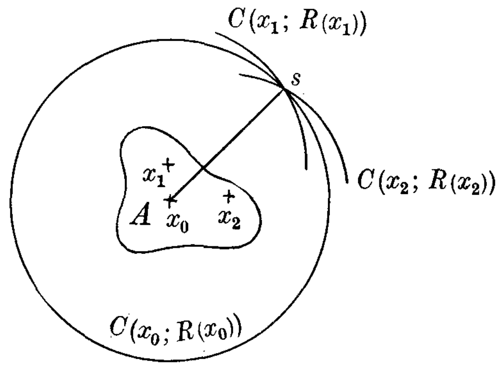
1. The case when $f(z)$ is a polynomial is trivial, for then every point of the plane is a limit point of the zeros of the derivatives. Therefore we shall assume that $f(z)$ is not a polynomial.

2. The following formula follows easily from the definition of a_n: $a_n' = (n+1)\,a_{n+1}$.

3. $R(x)$ and $r(x)$ are continuous functions.

4. The necessary and sufficient condition for $R(x)$ to be the modulus of an analytic function in the neighbourhood of a point x_0 is that there is only one singular point s on $C(x_0; R(x_0))$, and that s is of type B with respect to x_0.

Proof: If s is of type B with respect to x_0 and if s is the only singular point on $C(x_0; R(x_0))$ then there clearly exists a neighbourhood of x_0 all points of which have s as their nearest singular point. Thus $R(x) = |s - x|$. Conversely, assume that $R(x)$ is the modulus of an analytic function in a neighbourhood A of a point x_0. Let s be a singular point on $C(x_0; R(x_0))$. We always have



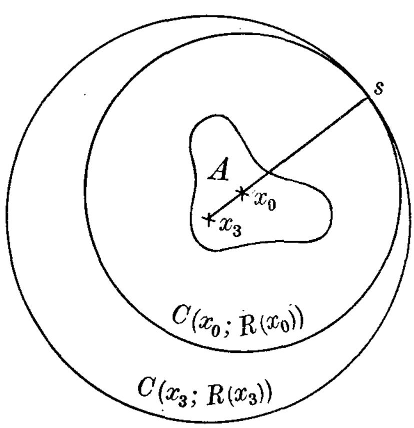


Fig. 3.

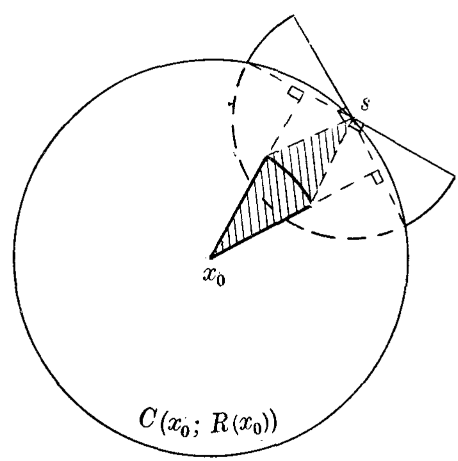


Fig. 4.

$R(x) \leq |s - x|$. On the other hand, for points on the line between x_0 and s the equality sign holds. Thus, as $R(x)$ is the modulus of an analytic function, this sign holds throughout A $\left(\text{maximum modulus theorem for } \dfrac{R(x)}{|s-x|}\right)\cdot$ But this implies that s is of type B, for if we choose two points x_1 and x_2 in A according to Fig. 3, then $C(x_1; R(x_1))$ and $C(x_2; R(x_2))$ pass through s. There are no further singular points on $C(x_0; R(x_0))$, for choose a point $x_3 \in A$ (Fig. 3). Then $C(x_3; R(x_3))$ passes through s.

5. If $s \in C(x_0; R(x_0))$ is of type B with respect to x_0 there exists a sector of a circle with its cusp in x_0 all inner points of which have s as their nearest singular point. Thus for the points of this sector $R(x) = |s - x|$. (Fig. 4.)

(If s is the only singular point on $C(x_0; R\,x_0)$ the same statement holds for a circle around x_0 as was the case in 4.)

6. If $\varphi(x)$ is a limit function of the family $\{g_{nm}(x)\}$ then $|\varphi(x)| \leq \frac{1}{R(x)}$.

7. $R(x) \geq R(x_0) - |x - x_0|$.

8. From 6 and 7 it easily follows that: If $|x - x_0| < R(x_0)$ then $|\varphi(x)| \leq \frac{1}{R(x_0) - |x - x_0|}$.

9. If $\varphi(x)$ is a limit function of the family $\{g_{nm}(x)\}$ then $\eta \cdot \varphi(x)$; $(|\eta| = 1)$ is one.

1. Study of some normal function families

1.1. The problem of finding the limit points of the zeros of the derivatives of f is intimately related to the study of the convergence properties of the functions $g_\nu = \sqrt[\nu]{a_\nu}$. We begin with the following

Theorem: The family $\{g_{nm}(x)\}$ is normal in every bounded simply connected domain which does not contain any singular point of f nor any zero of anyone of the derivatives of f.

Proof: A sufficient condition that a family of functions be normal in a certain domain is that all the functions are holomorphic and the family uniformly bounded in this domain. The function $g_{nm}(x)$ is holomorphic in a certain simply connected domain if f is regular there and if $f^{(n)}$ has no zero within the domain. We now regard a bounded, simply connected, open domain, A, which lies at a positive distance from the set of singular points of f and which does not contain any zero of $f^{(n)}$. Let M denote a number which is greater than $|f(x)|$ within A. It is no restriction to assume that the boundary, C, of A is such that it is possible to define contour integrals along it.

We get

$$\frac{f^{(n)}(x)}{\underline{|n}} = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-x)^{n+1}}.$$

Let L be the length of C, and assume that x lies within a domain $B \subset A$ which has all its points at a distance $> \delta$ from C.

$$|z - x| > \delta; \quad \left|\frac{f^{(n)}(x)}{\underline{|n}}\right| \leq \frac{L}{2\pi}\cdot\frac{M}{\delta^{n+1}}; \quad |g_{nm}(x)| \leq \sqrt[n]{\frac{LM}{2\pi\delta}}\cdot\frac{1}{\delta}.$$

As the right member tends to $\frac{1}{\delta}$ the family $\{g_{nm}\}$ is uniformly bounded in B. Thus the family is normal in B, and as δ can be chosen arbitrarily small, it is normal in A too. Finally we see that the family must be normal in every domain of the type described in the hypothesis because every domain which lies entirely within such a domain is of the same type as A. Thus the theorem is proved.

By the symbol $\{g_{nm}(x)\}_N$ we denote the family consisting of those functions g which have $n > N$. Now, if D is an open domain which does not contain any limit point of the zeros of the derivatives of f, then to every closed domain $D_- \subset D$ it is possible to find a number N such that no function $g_{nm};\ n > N$ has zeros within D_-. If we want to study the limit functions of $\{g_{nm}\}$ we may equally well study those of $\{g_{nm}\}_N$. Thus the limit functions have no connection with isolated zeros of the derivatives of f, but only with their limit points.

We now regard the family $\left\{\frac{1}{g_{nm}(x)}\right\}_N$ where the index, N, has the same meaning as above. This family is normal in a certain domain if $\{g_{nm}\}_N$ is normal in the same domain and if no $g_{nm};\ n > N$ has a zero there. Thus $\left\{\frac{1}{g_{nm}(x)}\right\}_N$ is normal in D_- provided N is sufficiently great. We assume that at a certain point $x_0 \in D$ the numbers $\left|\frac{1}{g_n(x_0)}\right|$ are bounded. Then the family $\left\{\frac{1}{g_{nm}(x)}\right\}_N$ is uniformly bounded within D_-. (MONTEL 2 p. 35.) This fact may as well be expressed in the following way: To every closed $D_- \subset D$ it is possible to find a number N such that $\sqrt[n]{|a_n(x)|} > \alpha > 0$ if $n > N$ (with the same number α for all $x \in D_-$) or shortly: $\varliminf \sqrt[\nu]{|a_\nu(x)|} > 0$ uniformly in D_-.

On the other hand, if $\varliminf \sqrt[\nu]{|a_\nu(x)|} > 0$ uniformly in D_-, then D_- is free from limit points of the zeros of the derivatives of f, for then every $\sqrt[\nu]{|a_\nu(x)|}$ (except possibly for a finite number) is greater than a positive number throughout D_-. Of course the absence of the condition of uniformity would make such a statement impossible, i. e. if we only knew that $\varliminf \sqrt[\nu]{|a_\nu(x)|} > 0$ everywhere in D_-, then there could well be an infinity of points in D_- where $\sqrt[\nu]{|a_\nu(x)|}$ might be equal to zero.

This may be used as a test in the study of the distribution of the limit points. The regular points of f can be divided into three classes:

I: $x_0 \in \mathrm{I}$ if there exists a neighbourhood A of x_0, a number $\alpha > 0$ and a number N such that $\sqrt[n]{|a_n(x)|} > \alpha$ for $n > N$ and $x \in A$.

II: $x_0 \in \mathrm{II}$ if $\varliminf \sqrt[\nu]{|a_\nu(x_0)|} > 0$ and if to every given triple consisting of a neighbourhood A of x_0, a number $\alpha > 0$ and a number N there exist an $x \in A$ and an $n > N$ vith $\sqrt[n]{|a_n(x)|} \leq \alpha$.

III: $x_0 \in \mathrm{III}$ if $\varliminf \sqrt[\nu]{|a_\nu(x_0)|} = 0$.

We have seen that the points belonging to the first class are not limit points and that those belonging to the second class are limit points. As to the points in the third class both cases may occur. PÓLYA (4) has given examples of entire functions where the problem of finding the distribution of the limit points can be completely solved, and the result shows that there exist limit points as well as non-limit points.

1.2. We have seen (Theorem 1.1) that the family $\{g_{nm}\}$ is normal in every domain where the derivatives of $f(z)$ have no zeros. On the other hand, the family is not normal in any domain containing such zeros. As is seen from the proof, this depends entirely upon the fact that not all the functions are holomorphic in such a domain. The upper bound common to all g_{nm} exists independently of the zeros.

The family $\left\{\frac{a_{n+1}}{a_n}\right\}$ consists of functions which may have other singular points than those of f. Such a singularity is a pole. Thus the function $\frac{a_{n+1}}{a_n}$ is meromorphic if f is regular, and its poles are the zeros of $f^{(n)}$.

The family $\{g_{nm}\}$ was treated as a family of holomorphic functions, but we are going to treat $\left\{\frac{a_{n+1}}{a_n}\right\}$ as a family of meromorphic functions. We put the question: In what points of the domain of existence of f is the family $\left\{\frac{a_{n+1}}{a_n}\right\}$ a normal family of meromorphic functions?

First we assume that x_0 is a point which is not a limit point of the zeros of the derivatives of f and that $\underline{\lim} \sqrt[\nu]{|a_\nu(x_0)|} > 0$. We have

$$(n+1)\, a_{n+1} = a_n' = (g_{nm}^n)' = n \cdot g_{nm}^{n-1} \cdot g_{nm}'$$

and

$$a_n = \qquad\qquad = g_{nm}^n$$

As f is not a polynomial, then a_n is not identically zero, and we have

$$\frac{n+1}{n} \cdot \frac{a_{n+1}}{a_n} = \frac{g_{nm}'}{g_{nm}}. \tag{a}$$

In order to prove that $\left\{\frac{a_{n+1}}{a_n}\right\}$ is normal at x_0 we prove that from any given sequence of natural numbers we can choose a subsequence $\{n_\nu\}$ such that $\frac{a_{n_\nu+1}}{a_{n_\nu}}$ converges uniformly in the neighbourhood of x_0. We choose n_ν such that $g_{n_\nu m_\nu}(x)$ with convenient m_ν converges uniformly in the neighbourhood of x_0, which is possible as $\{g_{nm}\}$ is normal at x_0. As $g_{n_\nu m_\nu}(x)$ converges uniformly, $g_{n_\nu m_\nu}'(x)$ does. We have

$$\lim g_{n_\nu m_\nu}(x) \neq 0.$$

Thus

$$\lim \frac{g'_{n_\nu m_\nu}}{g_{n_\nu m_\nu}} \text{ exists and } = \frac{\lim g'_{n_\nu m_\nu}}{\lim g_{n_\nu m_\nu}}$$

viz.

$$\lim \frac{n_\nu + 1}{n_\nu} \cdot \frac{a_{n_\nu+1}}{a_{n_\nu}} = \lim \frac{a_{n_\nu+1}}{a_{n_\nu}} \text{ exists (uniformly).}$$

Secondly we assume that x_0 is a limit point of the zeros of the derivatives of f. If $\left\{\frac{a_{n+1}}{a_n}\right\}$ is normal at x_0, then there exists a sequence of functions $\frac{a_{n_\nu+1}}{a_{n_\nu}}$ all of which have poles in the neighbourhood of x_0, and which converge (spherically) uniformly to a meromorphic function. This function is either the infinite constant or has a pole at x_0. But in both cases $\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}$ tends to infinity.

We have proved that at a point which belongs to class I of § 1.1 $\left\{\frac{a_{n+1}}{a_n}\right\}$ is normal, and that at a point belonging to class II $\left\{\frac{a_{n+1}}{a_n}\right\}$ is not normal provided $\overline{\lim} \frac{a_{\nu+1}(x_0)}{a_\nu(x_0)} < \infty$. For a point x_0 belonging to class III we only know that if x_0 is a limit point and $\overline{\lim} \frac{a_{\nu+1}(x_0)}{a_\nu(x_0)} < \infty$ then $\left\{\frac{a_{n+1}}{a_n}\right\}$ is not normal. Especially we have seen:

If $\underline{\lim} \sqrt[\nu]{|a_\nu(x_0)|} > 0$ *and* $\overline{\lim} \frac{a_{\nu+1}(x_0)}{a_\nu(x_0)} < \infty$, *then* $\left\{\frac{a_{n+1}}{a_n}\right\}$ *is normal (not normal) if* x_0 *is not (is) a limit point of the zeros of the derivatives of* f.

Having stated that $\left\{\frac{a_{n+1}}{a_n}\right\}$ is not normal at a certain point we may use any criterion of normality to draw conclusions concerning the distribution of the values of $\frac{a_{\nu+1}}{a_\nu}$ in the neighbourhood of this point. Thus for example: If $\overline{\lim} \frac{a_{\nu+1}(x_0)}{a_\nu(x_0)} < \infty$ and if x_0 is a limit point of the roots of $a_\nu(x) = 0$ then x_0 is a limit point of the roots of $\frac{a_{\nu+1}(x)}{a_\nu(x)} = C$, except possibly for two values of C.

1.3. 1. We shall prove the following proposition:

If $\underline{\lim} \sqrt[\nu]{|a_\nu(x_0)|} > 0$ for some point $x_0 \in D$, then there exists a number M with $\left|\frac{a_{n+1}}{a_n}\right| < M$ for all $x \in D_-$ and all sufficiently great n.

Let C be a contour within D containing D_-. Let the distance between C and D_- be $\delta > 0$. From the proof of Theorem 1.1 it follows that there exists a number A such that $|g_{nm}(z)| < A$ for all z within and on C and for all n. As g_{nm} is holomorphic within C for all sufficiently great n we have $(x \in D_-)$

$$|g'_{nm}(x)| = \frac{1}{2\pi}\left|\int_C \frac{g_{nm}(z)\,dz}{(z-x)^2}\right| \leq \frac{1}{2\pi}\cdot\frac{A}{\delta^2}\cdot \text{Length of } C.$$

Thus $\{g'_{nm}(x)\}$ is uniformly bounded in D_- for sufficiently great n.

On the other hand, $\left\{\frac{1}{g_{nm}(x)}\right\}$ is uniformly bounded in D_- for sufficiently great n as $\underline{\lim}\sqrt[\nu]{|a_\nu|} > 0$ (§ 1.1). Thus $\left\{\frac{g'_{nm}(x)}{g_{nm}(x)}\right\}$ is uniformly bounded in D_- for sufficiently great n. According to the formula (a) of § 1.2 the same holds for $\left|\frac{a_{n+1}}{a_n}\right|$.

2. But we can prove still more. Even $\sqrt[m]{\left|\left(\frac{a_{n+1}}{a_n}\right)^{(m)}\cdot\frac{1}{\underline{|m}}\right|}$ are bounded in D_- and uniformly with respect to m, n and x, the values of n for which $f^{(n)}$ has a zero within D_- being excluded. There is only a finite number of excluded values of n.

Let C be a contour contained in D and containing D_-. As has just been proved, there exists a number A such that $\left|\frac{a_{n+1}}{a_n}\right| < A$ on and within C except for the values of n for which $f^{(n)}$ has a zero on or within C. We may assume these values to be those for which D_- contains zeros by taking C sufficiently near to D_-. Let the distance between C and D_- be $\delta > 0$. Now $(x \in D_-)$

$$\left|\frac{1}{\underline{|m}}\left(\frac{a_{n+1}(x)}{a_n(x)}\right)^{(m)}\right| = \frac{1}{2\pi}\left|\int_C \frac{\frac{a_{n+1}(z)}{a_n(z)}\,dz}{(z-x)^{m+1}}\right| \leq \frac{1}{2\pi}\cdot\frac{A}{\delta^{m+1}}\cdot \text{Length of } C.$$

In particular this holds for $x = x_0$, which proves the necessity part of the following theorem. The sufficiency part follows easily, considering the fact that the zeros of $f^{(n)}$ are poles of a_{n+1}/a_n.

Theorem: If $\underline{\lim}\sqrt[\nu]{|a_\nu(x_0)|} > 0$ the necessary and sufficient condition that x_0 be not a limit point of the zeros of the derivatives of f is that there exists a number B such that for almost all n and for all m

$$\left|\left(\frac{a_{n+1}}{a_n}\right)^{(m)}(x_0)\right| < \underline{|m}\cdot B^m.$$

1.4. The case $\underline{\lim}\sqrt[\nu]{|a_\nu|} = 0$, which is exceptional as is pointed out in the preceding ought to be studied separately in order to decide for what types of functions it may occur. It will turn out that if $\underline{\lim}\sqrt[\nu]{|a_\nu|} = 0$ at a certain point, then in most cases this point is a limit point of the zeros of the derivatives. Only for a rather limited class of functions f can the contrary occur. In order to determine this class, we assume that x_0 is a point in D and that $\varliminf \sqrt[\nu]{|a_\nu(x_0)|} = 0$, and we try to draw conclusions regarding the behaviour of f from these assumptions. As the family $\{g_{nm}\}$ is normal in D_-, there must exist a limit function φ of the family, holomorphic in D and with $\varphi(x_0) = 0$. Now either $\varphi(x) \equiv 0$ or x_0 is a limit point of zeros of the functions g. (MONTEL, 2, p. 36.) But as $x_0 \in D$, it is not such a limit point. Thus $\varphi(x) \equiv 0$. This means that there exists a sequence $\sqrt[n_\nu]{|a_{n_\nu}(x)|}$ which converges to zero uniformly in D_-.

(We observe that this means that if $\varliminf \sqrt[\nu]{|a_\nu|} = 0$ holds for one single point of D, then it holds for every point of D. This is very often used subsequently.)

We shall say that a power series $\sum_{\mu=0}^{\infty} a_\mu x^\mu$ has great gaps if there is a sequence n_ν of natural numbers and a sequence $\alpha_\nu \to \infty$ of real numbers with $a_\mu = 0$ for $n_\nu \leq \mu < \alpha_\nu n_\nu$. We can now prove the following

Theorem: If $x_0 \in D$ and if there exists a sequence $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ tending to zero, then either f is an entire function, or f is the sum of an entire function and a function the power series (in x_0) of which has great gaps.

Proof: From the function sequence $g_{n_\nu m_\nu}(x)$ where the numbers m_ν are chosen arbitrarily, it is possible to choose a subsequence which is uniformly convergent in D_-. As was proved above, the limit of this subsequence is zero throughout D. (For simplicity we do not change our notation but denote this subsequence instead of the original sequence by $g_{n_\nu m_\nu}(x)$). As the convergence is uniform in D_-, there must exist a circle $C(x_0; \varrho)$ around x_0 with the radius ϱ within and on which $g_{n_\nu m_\nu}(x)$ converges to zero uniformly. Thus, if

$$\operatorname*{Max}_{C(x_0;\,\varrho)} |g_{n_\nu m_\nu}| = \varepsilon_\nu \quad \text{then} \quad \varepsilon_\nu \to 0$$

$$|a_{n_\nu}(x)| \leq \varepsilon_\nu^{n_\nu} \quad \text{for} \quad x \in C(x_0; \varrho) \quad \text{where } \varepsilon_\nu \to 0.$$

It is easy to see that the following formula is correct:

$$\binom{m+n}{m} a_{m+n}(x) = \frac{1}{2\pi i} \int_{C(x_0;\,\varrho)} \frac{a_n(z)\,dz}{(z-x)^{m+1}}$$

where x lies within $C(x_0; \varrho)$. We get:

$$\binom{m+n_\nu}{m} |a_{m+n_\nu}(x_0)| \leq \frac{1}{2\pi} \cdot \frac{\varepsilon_\nu^{n_\nu}}{\varrho^{m+1}} \cdot 2\pi\varrho = \frac{\varepsilon_\nu^{n\nu}}{\varrho^m}$$

$$\text{or} \quad \left(\text{as} \binom{m+n_\nu}{m} \geq 1\right): \ |a_{n_\nu+m}(x_0)| \leq \frac{\varepsilon_\nu^{n_\nu}}{\varrho^m}.$$

Put $m = \mu - n_\nu$:

$$\sqrt[\mu]{|a_\mu(x_0)|} \leq \frac{1}{\varrho} \cdot (\varrho\,\varepsilon_\nu)^{\frac{n_\nu}{\mu}}. \tag{a}$$

Now we choose a sequence a_ν of real numbers such that $a_\nu \to \infty$ and $(\varrho\,\varepsilon_\nu)^{\frac{1}{a_\nu}} \to 0$[1] and divide the non-negative integers μ into two classes

I: μ belongs to class I if for some ν: $n_\nu \leq \mu < a_\nu n_\nu$.

II: μ belongs to class II if for no ν: $n_\nu \leq \mu < a_\nu n_\nu$.

We construct two power series

$$h(x) = \sum_{\nu=0}^{\infty} c_\nu (x - x_0)^\nu \quad \text{and} \quad k(x) = \sum_{\nu=0}^{\infty} d_\nu (x - x_0)^\nu$$

where

$$c_\mu = \begin{cases} a_\mu(x_0) & \text{if } \mu \text{ belongs to I} \\ 0 & \text{if } \mu \text{ belongs to II} \end{cases} \quad \text{and} \quad d_\mu = \begin{cases} 0 & \text{if } \mu \text{ belongs to I} \\ a_\mu & \text{if } \mu \text{ belongs to II} \end{cases}$$

As μ either belongs to I or to II: $f(x) = h(x) + k(x)$. According to (a)

$$\sqrt[\mu]{|c_\mu|} \leq \frac{1}{\varrho} (\varrho\,\varepsilon_\nu)^{\frac{n_\nu}{\mu}}.$$

But then $\dfrac{n_\nu}{\mu} > \dfrac{1}{a_\nu}$

$$\therefore \sqrt[\mu]{|c_\mu|} \leq \frac{1}{\varrho} (\varrho\,\varepsilon_\nu)^{\frac{1}{a_\nu}} \quad \text{provided } \varrho\,\varepsilon_\nu < 1$$

which always holds for sufficiently great ν. Thus

$$\sqrt[\mu]{|c_\mu|} \to 0 \quad \text{if } \mu \in \mathrm{I} \text{ and on the other hand}$$

$$\sqrt[\mu]{|c_\mu|} = 0 \quad \text{if } \mu \in \mathrm{II}$$

Thus h is entire.

Clearly k has great gaps. Thus the theorem is proved.

From a famous theorem of OSTROWSKI's (3 p. 251) we can now draw conclusions concerning the function $f(z)$. Thus, e. g. $f(z)$ has a simply connected domain of existence, which implies that $f(z)$ is uniform and has no isolated singular points in the finite part of the plane.

[1] e. g. choose $a_\nu = -\dfrac{\log(\varrho\,\varepsilon_\nu)}{\log|\log(\varrho\,\varepsilon_\nu)|}$.

2. Cauchy's and d'Alembert's Criteria of Convergence

The application of Cauchy's criterion of convergence to power series leads to the following formula:

$$\overline{\lim}\ \sqrt[\nu]{|a_\nu|} = \frac{1}{R}.$$

However, d'Alembert's criterion does not generally give the radius of convergence exactly but only an inequality:

$$\underline{\lim} \left|\frac{a_{\nu+1}}{a_\nu}\right| \leq \frac{1}{R} \leq \overline{\lim} \left|\frac{a_{\nu+1}}{a_\nu}\right|.$$

It is well known that in the case when d'Alembert's criterion gives the value of R, i. e. when $\lim \left|\frac{a_{\nu+1}}{a_\nu}\right|$ exists, then $\lim \sqrt[\nu]{|a_\nu|}$ exists, but the converse does not necessarily hold. It is clear that for subsequences n_ν of ν the corresponding proposition does not hold, i. e. it is not generally the case that if $\lim \left|\frac{a_{n_\nu+1}}{a_{n_\nu}}\right|$ exists then $\lim \sqrt[n_\nu]{|a_{n_\nu}|}$ exists. Still less does the converse hold. However, we shall see that theorems of this type exist, if the assumptions are completed. In this section some theorems of this type will be proved.

2.1. We begin with the following

Theorem: If the family $\{g_{nm}\}$ is normal in the domain D, and if in this domain the sequence $g_{n_\nu m_\nu}$ converges to a certain limit function $\varphi(x)$ which is not identically zero, then $\frac{a_{n_\nu+1}(x)}{a_{n_\nu}(x)}$ converges to $\frac{\varphi'(x)}{\varphi(x)}$ in D (uniformly in D_-).

Remark: The completion of the assumptions consists in this case primarily of the assumption that the family $\{g_{nm}\}$ is normal. Secondarily we do not only assume the existence of $\lim \sqrt[n_\nu]{|a_{n_\nu}|}$ for a certain value of x, but the existence of $\lim \sqrt[n_\nu]{a_{n_\nu}}$ for all $x \in D$.

Proof: In D_- we have uniformly $\varphi(x) = \lim g_{n_\nu m_\nu}(x)$. Thus, $\lim g'_{n_\nu m_\nu}$ exists and is equal to $\varphi'(x)$. As $\varphi(x)$ is not identically zero we get

$$\frac{\varphi'(x)}{\varphi(x)} = \frac{\lim g'_{n_\nu m_\nu}}{\lim g_{n_\nu m_\nu}} = \lim \frac{g'_{n_\nu m_\nu}}{g_{n_\nu m_\nu}}.$$

Now from the formula (a) of § 1.2 it follows that

$$\frac{\varphi'(x)}{\varphi(x)} = \lim \frac{a_{n_\nu+1}(x)}{a_{n_\nu}(x)}. \qquad \text{Q. E. D.}$$

2.2. As a result of Vitali's theorem it is clearly unnecessary to assume that $\lim \sqrt[n_\nu]{a_{n_\nu}}$ exists for all $x \in D$, as we can accomplish this by assuming the existence of the limit in question at an infinity of points of D_-.

However, instead of requiring convergence of $\sqrt[n_\nu]{a_{n_\nu}}$ at an infinity of points, we can require that $\sqrt[n_\nu]{|a_{n_\nu}|}$ converges at a certain set of points. We get a theorem analogous to the above theorem by assuming this set to be such that any function harmonic in D and zero in the set has to be zero throughout D. Assume that the sequence $\sqrt[n_\nu]{|a_{n_\nu}(x)|}$ converges in such a set M. Let two of the limit functions of the sequence $\sqrt[n_\nu]{a_{n_\nu}}$ be φ_1 and φ_2. These are holomorphic and are assumed to be different from zero in D. Further it is clear that, as neither φ_1 nor φ_2 are $=0$ within D, we can find a function $v(x)$ holomorphic in D and satisfying $\varphi_1(x) = \varphi_2(x) \cdot e^{i\,v(x)}$ in the whole of D. But if $x \in M$ we have $|\varphi_1(x)| = |\varphi_2(x)|$ i. e. $v(x)$ is real. But this means that the function $Im\, v(x)$, harmonic in D, is zero in M. Thus, it is identically zero, and v is a real constant (and we have $|\varphi_1| = |\varphi_2|$ throughout D). Accordingly there is essentially only one limit function φ of the sequence $\sqrt[n_\nu]{a_{n_\nu}}$. All the others are of the form $\eta \cdot \varphi$ where $|\eta| = 1$. Combining this result with the previous theorem we get the following

Theorem: If $\sqrt[n_\nu]{|a_{n_\nu}(x)|}$ converges at a set of points $M \subset D$, if at one point at least it converges to a number different from zero, and if M is such that any function harmonic in D and zero in M is zero identically in D, then all the limit functions of the family $\{\sqrt[n_\nu]{a_{n_\nu}}\}$ are of the form $\eta \cdot \varphi(x)$ where φ is one of them and $|\eta| = 1$. Further $\dfrac{a_{n_\nu+1}}{a_{n_\nu}}$ converges to $\dfrac{\varphi'}{\varphi}$ in D.

2.3. The best theorem of the type discussed here would be the following: If $\sqrt[n_\nu]{|a_{n_\nu}|}$ converges for one single value of x, $x_0 \in D$ then $\dfrac{a_{n_\nu+1}}{a_{n_\nu}}$ converges in D. I have not succeeded in proving either this theorem or any of the theorems with the same hypothesis, only the following less extensive propositions:

$$\text{i. } \frac{a_{n_\nu+1}}{a_{n_\nu}} \text{ converges in } x_0; \quad \text{ii. } \left|\frac{a_{n_\nu+1}}{a_{n_\nu}}\right| \text{ converges in } x_0.$$

On the other hand I have not found any example showing that any of these theorems are false.

However, it is easy to obtain a theorem with the required assumption by imposing further restrictions on the proposition: we can determine an upper bound of $\left|\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right|$ (cf. § 1.31).

Accordingly we assume that $x_0 \in D$ and $\lim \sqrt[n_\nu]{|a_{n_\nu}(x_0)|} = \frac{1}{R_1} \neq 0$. If we knew that all the limit functions of $\{g_{n_\nu m_\nu}\}$ were equivalent to one of them, say $\varphi(x)$, then the best of the theorems mentioned above would be proved. For by Theorem 2.1 we should then have $\lim \frac{a_{n_\nu+1}}{a_{n_\nu}} = \frac{\varphi'}{\varphi}$. However, it is possible that $\{g_{n_\nu m_\nu}\}$ has several non-equivalent limit functions. All of these have nevertheless the modulus $\frac{1}{R_1}$ at the point x_0. To get an upper bound of $\overline{\lim} \left|\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right|$ it is clearly sufficient to find an upper bound of $\left|\frac{\varphi'(x_0)}{\varphi(x_0)}\right|$ where φ passes through the set of all the limit functions which are generated by the sequence $g_{n_\nu m_\nu}(x)$. Now we have (by § 0.28)

$$|\varphi(x)| \leq \frac{1}{R(x_0) - |x - x_0|} \quad \text{for} \quad |x - x_0| < R(x_0).$$

But

$$|\varphi'(x_0)| = \frac{1}{2\pi}\left|\int\limits_{\substack{C(x_0;\,\varrho)\\ \varrho < r(x_0)}} \frac{\varphi(x)\,dx}{(x-x_0)^2}\right| \leq \frac{1}{\varrho(R-\varrho)}.$$

We get the best bound if $\varrho = \frac{R}{2}$, but as ϱ has to be less than r we can accomplish this only when $r > \frac{R}{2}$; in other cases we have to be satisfied with letting ϱ tend to $r(x_0)$.

2.4. By applying the above method to $\varphi^\alpha(x)$ instead of $\varphi(x)$, where α is a real positive number, we can get a better bound. As $\varphi^\alpha(x)$ is holomorphic within $C(x_0; r(x_0))$ we have

$$|\alpha \cdot \varphi^{\alpha-1}(x_0) \cdot \varphi'(x_0)| = \frac{1}{2\pi}\left|\int\limits_{\substack{C(x_0;\,\varrho)\\ \varrho < r(x_0)}} \frac{\varphi^\alpha(x)}{(x-x_0)^2}\,dx\right|.$$

We easily find an upper bound of the integral. After division by $|\alpha \cdot \varphi^\alpha(x_0)|$ we get

$$\left|\frac{\varphi'(x_0)}{\varphi(x_0)}\right| \leq \frac{1}{\alpha\varrho} \cdot \left(\frac{R_1}{R-\varrho}\right)^\alpha.$$

The right membrum has its minimum for $\alpha = \dfrac{1}{\log \dfrac{R_1}{R-\varrho}}$

$$\left|\frac{\varphi'(x_0)}{\varphi(x_0)}\right| \leq \frac{e}{\varrho} \cdot \log \frac{R_1}{R-\varrho}.$$

Here $\varrho < r(x_0)$ but we see that by letting $\varrho \to r$ we can restrict ourselves to requiring only $\varrho \leq r$. Thus

Theorem: If $\lim \sqrt[n_\nu]{|a_{n_\nu}(x_0)|} = \dfrac{1}{R_1} \neq 0$, if $x_0 \in D$ and if $\varrho \leq r(x_0)$, then

$$\overline{\lim} \left|\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right| \leq \frac{e}{\varrho} \log \frac{R_1}{R-\varrho}.$$

Remark: The right membrum has its minimum if $\varrho = Ru$ where u is determined from the equation: $\log \dfrac{R_1}{R} + \log \dfrac{1}{1-u} = \dfrac{u}{1-u}\cdot$ This value can be used if it is smaller than $r(x_0)$. In other cases we get the best result by choosing $\varrho = r(x_0)$. However, the bound we get in that way is not best possible, because the numerical factor e can be replaced by 2, as we shall see later (2.6).

2.5. The bounds given above can be essentially improved for such sequences $\dfrac{a_{n_\nu+1}}{a_{n_\nu}}$ where $R_1 = R$, i. e. for the sequences where the corresponding sequence $\sqrt[n_\nu]{|a_{n_\nu}|}$ is maximal (see 0.1) or converges to $\overline{\lim} \sqrt[\nu]{|a_\nu|} = \dfrac{1}{R}\cdot$ In this case § 2.3 gives the bound $\dfrac{4}{R}$ and Theorem 2.4, which gives the best result for $\varrho \to 0$, gives $\dfrac{e}{R}\cdot$ The following theorem says that these bounds can be replaced by $\dfrac{1}{R}\cdot$

Theorem: If $\lim \sqrt[n_\nu]{|a_{n_\nu}(x_0)|} = \dfrac{1}{R(x_0)}$ and if $x_0 \in D$, then

$$\overline{\lim} \left|\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right| \leq \frac{1}{R(x_0)}.$$

Proof: Let φ be a limit function of $g_{n_\nu m_\nu}(x)$. We investigate the function $h(x) = \log(R(x_0) \cdot \varphi(x))$ which is holomorphic in D. We have according to 0.2: $|R(x_0) \cdot \varphi(x)| \leq \dfrac{R(x_0)}{R(x_0) - |x - x_0|}\cdot$ Now $|R(x_0) \cdot \varphi(x_0)| = 1$. Put $u(r; v) = \log |R(x_0) \cdot \varphi(x)| = Re\, h(x)$ where $r \cdot e^{iv} = x - x_0$

$$\therefore\ u(0; v) = 0 \quad \text{and} \quad u(r; v) \leq \log \frac{R(x_0)}{R(x_0) - |x - x_0|} \leq \frac{K}{R(x_0)} |x - x_0|$$

where K is a constant which can be chosen arbitrarily near to 1 if $|x - x_0|$ is small enough. Let $\left(\frac{\partial u}{\partial r}\right)_v$ be the derivative at the origin in the direction v of the function $u(r; v)$. We have

$$\left(\frac{\partial u}{\partial r}\right)_v = \lim_{r \to 0} \frac{u(r; v) - u(0; v)}{r} \leq \frac{K}{R(x_0)}.$$

As $r \to 0$ we can let $K \to 1$:

$$\left(\frac{\partial u}{\partial r}\right)_v \leq \frac{1}{R(x_0)}.$$

But

$$h'(x_0) = \lim \frac{h(x) - h(x_0)}{x - x_0}.$$

Let $x - x_0$ tend to zero with its argument constant $= v$; we get

$$h'(x_0) = \lim \frac{Re\,[h(x) - h(x_0)]}{|x - x_0| \cdot e^{iv}} + i \frac{Im\,[h(x) - h(x_0)]}{|x - x_0| \cdot e^{iv}}$$

$$\therefore\ h'(x_0) \cdot e^{iv} = \left(\frac{\partial u}{\partial r}\right)_v + i(\cdots); \quad Re\,(h'(x_0) \cdot e^{iv}) = \left(\frac{\partial u}{\partial r}\right)_v \leq \frac{1}{R(x_0)}.$$

But this is valid for all v as the derivative of a holomorphic function is independent of the direction of the differential of the independent variable.

$$\therefore\ |h'(x_0)| \leq \frac{1}{R(x_0)}; \quad \left|\frac{\varphi'(x_0)}{\varphi(x_0)}\right| \leq \frac{1}{R(x_0)}$$

$$\overline{\lim} \left|\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right| \leq \frac{1}{R(x_0)}. \qquad \text{Q. E. D.}$$

2.6. The bound $\frac{e}{\varrho} \log \frac{R_1}{R - \varrho}$ for $\overline{\lim} \left|\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right|$ which has been found in § 2.4 can be improved by use of the following theorem of Landau's (1, p. 620, Theorem 6):

If $|b_1 z + b_2 z^2 + \cdots| < M$ for $|z| \leq \varrho$, then the function $b_1 z + b_2 z^2 + \cdots$ for $|z| \leq \varrho$ takes every value within a circle around the origin with radius $|b_1| \varrho \cdot \tau\left(\frac{M}{|b_1| \varrho}\right)$ where the function $\tau(u)$ is defined by the equations

$$\tau(u) = u \cdot e^{-v}; \quad \frac{\sinh v}{v} = u.$$

$\tau(u)$ is a convex decreasing function of u: $\tau(1) = 1$; $\tau(\infty) = 0$.

We use the theorem in the following form, which is an immediate consequence:

If $|b_1 z + b_2 z^2 + \cdots| < M$ for $|z| \leq \varrho$ and if $h(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ has no zero within $|z| = \varrho$, then $|b_0| \geq |b_1| \varrho \cdot \tau \left(\frac{M}{|b_1| \varrho}\right)$. We define a positive number v by $\frac{\sinh v}{v} = \frac{M}{|b_1| \varrho}$. As $\frac{M}{|b_1| \varrho} > 1$ this number exists, and as $\frac{\sinh v}{v}$ is monotonic it is unique. According to the definition of τ:

$$|b_0| \geq M \cdot e^{-v}; \quad v \geq \log \frac{M}{|b_0|}.$$

We now regard the two cases $|b_0| < M$ and $|b_0| \geq M$. In the first case $\log \frac{M}{|b_0|}$ is a positive number, and as $\frac{\sinh v}{v}$ increases for $v > 0$:

$$\frac{M}{|b_1| \varrho} = \frac{\sinh v}{v} > \frac{\sinh \log \frac{M}{|b_0|}}{\log \frac{M}{|b_0|}} = \frac{\frac{M}{|b_0|} - \frac{|b_0|}{M}}{2 \log \frac{M}{|b_0|}}.$$

In both cases $\frac{M}{|b_1| \varrho} \geq 1$.

We construct a function $\Phi(s)$ in the following way:

$$\Phi(s) = \begin{cases} \dfrac{2 \log \frac{1}{s}}{\frac{1}{s} - s} & \text{for} \quad 0 < s < 1 \\ 1 & \text{for} \quad 1 \leq s \end{cases}$$

Thus we have proved the following

Lemma: If $|b_1 z + b_2 z^2 + \cdots| < M$ for $|z| < \varrho$ and if $h(z) = b_0 + b_1 z + \cdots$ has no zero within $|z| = \varrho$, then

$$\frac{|b_1| \varrho}{M} \leq \Phi\left(\frac{|b_0|}{M}\right).$$

We can now prove the following theorem (cf. § 2.4)

Theorem: If $\lim \sqrt[n_\nu]{|a_{n_\nu}(x_0)|} = \frac{1}{R_1} \neq 0$, if $x_0 \in D$ and if $\varrho \leq r(x_0)$, then

$$\overline{\lim} \left| \frac{a_{n_\nu + 1}(x_0)}{a_{n_\nu}(x_0)} \right| \leq \frac{2}{\varrho} \log \frac{R_1}{R - \varrho}.$$

Proof: First assume $\varrho < r(x_0)$. Then none of the functions $a_n(x)$ has zeros within $C(x_0; \varrho)$ provided n is sufficiently great. Therefore we can apply the lemma to $a_n(x)$. Now

$$a_n(x) = a_n(x_0) + \binom{n+1}{1} a_{n+1}(x_0)(x - x_0) + \cdots;$$

Let $\varrho < \varrho_1 < R(x_0)$ and let the maximum of f on $C(x_0; \varrho_1)$ be A. Then $|a_\nu(x_0)| \leq \frac{A}{\varrho_1^\nu}$, and if $|x - x_0| \leq \varrho$ we get

$$\left|\binom{n+1}{1} a_{n+1}(x_0)(x - x_0) + \binom{n+2}{2} a_{n+2}(x_0)(x - x_0)^2 + \cdots\right| \leq$$

$$\leq \binom{n+1}{1} \frac{A}{\varrho_1^{n+1}} \cdot \varrho + \binom{n+2}{2} \frac{A}{\varrho_1^{n+2}} \varrho^2 + \cdots <$$

$$< \frac{A}{\varrho_1^n} \left(1 + \binom{n+1}{1} \frac{\varrho}{\varrho_1} + \binom{n+2}{2} \left(\frac{\varrho}{\varrho_1}\right)^2 + \cdots\right) = \frac{A}{\varrho_1^n} \cdot \frac{1}{\left(1 - \frac{\varrho}{\varrho_1}\right)^{n+1}} = \frac{A\,\varrho_1}{(\varrho_1 - \varrho)^{n+1}}.$$

Accordingly we put

$$M_n = \frac{A\,\varrho_1}{(\varrho_1 - \varrho)^{n+1}}.$$

The lemma gives

$$\frac{(n+1)\,|a_{n+1}(x_0)|\,\varrho}{M_n} \leq \Phi\left(\frac{|a_n(x_0)|}{M_n}\right).$$

But $\frac{|a_n(x_0)|}{M_n} \leq \frac{A}{\varrho_1^n} \cdot \frac{(\varrho_1 - \varrho)^{n+1}}{A\,\varrho_1}$ tends to zero when $n \to \infty$. Thus for sufficiently great n we can use the expression $\frac{2 \log \frac{1}{s}}{\frac{1}{s} - s}$ for $\Phi(s)$

$$\left|\frac{a_{n+1}(x_0)}{a_n(x_0)}\right| \leq \frac{M_n}{\varrho(n+1)} \cdot \frac{2 \log \frac{M_n}{|a_n(x_0)|}}{\frac{M_n}{|a_n(x_0)|} - \frac{|a_n(x_0)|}{M_n}} \cdot \frac{1}{|a_n(x_0)|}$$

$$\left|\frac{a_{n+1}(x_0)}{a_n(x_0)}\right| \leq \frac{2}{\varrho} \log \sqrt[n+1]{\frac{M_n}{|a_n(x_0)|}} \cdot \frac{1}{1 - \left(\frac{|a_n(x_0)|}{M_n}\right)^2}.$$

Now put n_ν instead of n and let ν tend to infinity

$$\overline{\lim} \left| \frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)} \right| \leq \frac{2}{\varrho} \cdot \log \frac{R_1}{\varrho_1 - \varrho}.$$

Let ϱ_1 tend to $R(x_0)$

$$\overline{\lim} \left| \frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)} \right| \leq \frac{2}{\varrho} \cdot \log \frac{R_1}{R - \varrho}.$$

We have assumed $\varrho < r(x_0)$. But we might equally well assume $\varrho \leq r(x_0)$ which is easily seen by letting ϱ tend to $r(x_0)$.

Thus, the theorem is proved.

3. The Singular Points

In this chapter we are going to study the influence of the singular points on the behaviour of the power series, still under the assumption that the point around which the series is developed does not belong to the set of points where the zeros of the derivatives accumulate.

3.1. We begin with the

Theorem: If $x_0 \in D$, and if s is a singular point on $C(x_0; R(x_0))$ of type B (§ 0.1) with respect to x_0, then $\frac{1}{s-x}$ is a limit function of $\{g_{nm}\}$ in D.

Proof: As s is of type B, there is a sector of a circle within D with its cusp at x_0 such that, if x is a point in its interior, the relation $R(x) = |s - x|$ is satisfied (§ 0.25). Let x_1, be a point within this sector. We know that it is possible to find a sequence of functions $g_{n_\nu m_\nu}(x)$ whose moduli converge to $\frac{1}{R(x_1)}$ at the point x_1. As $R(x) = |s - x|$ then $|g_{n_\nu m_\nu}(x_1)| \to \frac{1}{|s - x_1|}$.

By choosing m_ν conveniently we can make it converge to $\frac{1}{s - x_1}$. Now we can choose a subsequence which converges uniformly in D_-. Then in D it converges to a limit function $\varphi(x)$. We study the function $(s - x) \cdot \varphi(x)$. We have $|(s - x) \cdot \varphi(x)| = |R(x) \cdot \varphi(x)| \leq 1$ within the sector and for $x = x_1$: $|(s - x_1)\, \varphi(x_1)| = 1$. By the maximum modulus theorem, $(s - x) \cdot \varphi(x)$ is then constant. But $(s - x_1)\, \varphi(x_1) = 1$

$$\because \; \varphi(x) = \frac{1}{s - x}. \qquad \text{Q. E. D.}$$

Corollary 1. If s is the only singular point on $C(x_0; R(x_0))$ then every sequence $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ which converges to $\frac{1}{R(x_0)}$ corresponds to a sequence $\sqrt[n_\nu]{a_{n_\nu}(x)}$ which has essentially only one limit function, viz. $\frac{1}{s - x}$. For in this case there must exist a circle that touches $C(x_0; R(x_0))$ at s, and that contains this circle without simultaneously containing any singular point of $f(x)$. The centre of this second circle is the cusp of a sector containing x_0 and to the points of which s is the nearest singular point. The above method of proof can now be used with x_0 instead of x_1 and we find that every limit function of $\sqrt[n_\nu]{a_{n_\nu}}$ is equivalent to $\frac{1}{s-x}$.

Corollary 2. According to theorem 2.1 $\lim \frac{a_{n_\nu+1}}{a_{n_\nu}} = \frac{\varphi'}{\varphi}$ if $\lim g_{n_\nu m_\nu} = \varphi \neq 0$. But if $\varphi = \frac{\eta}{s-x}$; then $\frac{\varphi'}{\varphi} = \frac{1}{s-x}$.

Therefore we can assert: If a singular point s of type B lies on $C(x_0; R(x_0))$ then there exists a sequence n_ν such that $\frac{a_{n_\nu+1}}{a_{n_\nu}}$ converges to $\frac{1}{s-x}$. In particular, if s is the only singular point on $C(x_0; R(x_0))$ and is of type B, and if $\lim \sqrt[n_\nu]{|a_{n_\nu}(x_0)|} = \frac{1}{R(x_0)}$ then

$$\lim \frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)} = \frac{1}{s-x_0}.$$

3.2. The fact pointed out in corollary 2 of the preceding § that it is possible to draw conclusions concerning the sequence $\frac{a_{\nu+1}}{a_\nu}$ from assumptions concerning the singular points, can be used in other ways. If we do not consider the behaviour of the limit function as a whole but only its value at a certain point, we need only assume that the singular point s is of type A with respect to this given point. In order to show this we need a lemma. This lemma is a generalization of the maximum modulus theorem and will serve the same purpose in the proof of the next theorem as did the maximum modulus theorem in the above proof. In order to point out its connection with the maximum modulus theorem, we give the lemma a slightly wider formulation than we actually need.

Lemma: Let $f(x)$ be holomorphic in $C(x_0; \varrho)$ and let for x within the same circle: $|f(x)| \leq |f(x_0)| + K|x-x_0|^n$ where K is a constant and $n \geq 1$. Then $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $|f^{(n)}(x_0)| < e K \cdot \underline{|n}$.

Proof: If $f(x_0) = 0$, the proposition is trivial. Thus we may assume that $f(x_0) \neq 0$. Let m be a positive integer. If $p = 1$, the following formula holds:

$$(f^m)^{(p)}(x_0) = m f^{m-1}(x_0) \cdot f^{(p)}(x_0) \tag{a}$$

and if $p > 1$, it holds, provided we have already proved $f'(x_0) = f''(x_0) = \cdots = f^{(p-1)}(x_0) = 0$. Now suppose, that (a) holds and that $p < n$.

We get

$$|m \cdot f^{m-1}(x_0) \cdot f^{(p)}(x_0)| = \frac{\lfloor p}{2\pi} \left| \int\limits_{\substack{C(x_0;\, \varrho_1) \\ \varrho_1 < \varrho}} \frac{f^m(x)\, dx}{(x-x_0)^{p+1}} \right| \leq \frac{\lfloor p}{\varrho_1^p} (|f(x_0)| + K \varrho_1^n)^m.$$

Divide by $m\,|f^m(x_0)|$

$$\left| \frac{f^{(p)}(x_0)}{f(x_0)} \right| \leq \frac{\lfloor p}{\varrho_1^p\; m} \left(1 + \frac{K}{|f(x_0)|} \varrho_1^n\right)^m.$$

This formula holds for all $\varrho_1 < \varrho$ and all m. We may now let ϱ_1 vary with m: $\varrho_1 = r_m$. We can construct r_m so that it tends to zero with the right member of the inequality as $m \to \infty$. Viz. choose $r_m^n \cdot m = \frac{|f(x_0)|}{K}$;

$$\because \quad m \cdot r_m^p = \frac{|f(x_0)|}{K} \cdot \frac{1}{r_m^{n-p}},.$$

$$\because \quad \left| \frac{f^{(p)}(x_0)}{f(x_0)} \right| \leq r_m^{n-p} \frac{K \lfloor p}{|f(x_0)|} \left(1 + \frac{1}{m}\right)^m \to 0.$$

The last conclusion is no longer correct if $p \geq n$. As (a) holds for $p = 1$ we have $f'(x_0) = 0$. Thus (a) holds for $p = 2$ etc. The induction continues to $p = n - 1$. Finally for $p = n$ we get

$$|f^{(n)}(x_0)| \leq e K \cdot \lfloor n.$$

Thus we have established the lemma. It is clear that if K is zero, we get the maximum modulus theorem because in that case the supposition holds for all n.

It is now possible to prove the following theorem, which is of type i. mentioned in § 2.3, the assumptions having been completed in two directions. On the one hand, we have made an assumption concerning the singular points, and on the other we assume the sequence $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ in question to be maximal.

Theorem: If $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ converges to $\frac{1}{R(x_0)} \neq 0$, if $x_0 \in D$, and if s is the only singular point on $C(x_0; R(x_0))$ and is a singularity of type A, then $\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}$ converges to $\frac{1}{s - x_0}$.

Proof: As s is of type A, it is possible to find a point x_1 on the straight line joining the points s and x_0, x_1 satisfying $R(x_1) = R(x_0) + |x_1 - x_0|$ (Fig. 5).

Put

$$\begin{cases} s - x_1 = R \cdot e^{i v_0} \\ x - x_1 = \varrho \cdot e^{i v} \\ x_0 - x_1 = \varrho_0 \cdot e^{i v_0} \end{cases}$$

Let $\varphi(x)$ be a limit function of $\{g_{n_\nu m_\nu}(x)\}$

$$|\varphi(x_0)| = \frac{1}{R(x_0)}. \tag{a}$$

According to 0.2:

$$|\varphi(x)| \leq \frac{1}{R-\varrho}; \quad \because \quad |(s-x)\cdot\varphi(x)| \leq \frac{|s-x|}{R-\varrho}. \tag{b}$$

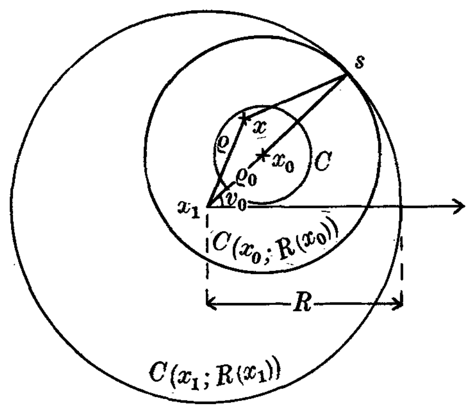


Fig. 5.

For the triangle $(s; x; x_1)$ we get

$$|s-x| = \sqrt{R^2+\varrho^2-2R\varrho\cdot\cos(v-v_0)} =$$

$$= (R-\varrho)\sqrt{1+\frac{2R\varrho}{(R-\varrho)^2}(1-\cos(v-v_0))}.$$

Suppose now that x lies within a sufficiently small circle C around x_0. We can find a constant K_1 such that $1-\cos(v-v_0) \leq K_1(v-v_0)^2$

$$|s-x| \leq (R-\varrho)\sqrt{1+\frac{2R\varrho}{(R-\varrho)^2}K_1(v-v_0)^2}.$$

Within C the formula $|s-x| < (R-\varrho)(1+K_2(v-v_0)^2)$ must hold if K_2 is a new suitable constant. Combine this with (b): $|(s-x)\varphi(x)| < 1+K_2(v-v_0)^2$. Further the formula $|v-v_0| \leq K_3|x-x_0|$ holds with the new constant K_3. Thus, there must exist a constant K such that the formula $|(s-x)\varphi(x)| \leq$ $\leq 1+K|x-x_0|^2$ holds within C. Observing that $R(x_0) = |s-x_0|$ and that (a) holds, we find $|(s-x_0)\varphi(x_0)| = 1$. Thus, $|(s-x)\varphi(x)| \leq |(s-x_0)\varphi(x_0)| +$ $+K|x-x_0|^2$ within C. From the beginning this circle may be chosen so small that it only contains points belonging to D. Thus $\varphi(x)$ and also $(s-x)\cdot\varphi(x)$

are holomorphic within C. It is now possible to use the lemma, and we find $[(s-x)\varphi(x)]' = 0$ for $x = x_0$

$$\therefore \frac{\varphi'(x_0)}{\varphi(x_0)} = \frac{1}{s-x_0}.$$

This holds for every limit function of $\{g_{n_\nu m_\nu}(x)\}$.

Remark: We have proved that $\dfrac{a_{n_\nu+1}(x)}{a_{n_\nu}(x)}$ converges at the point x_0, but we have not proved that it converges in the neighbourhood of x_0 (cf. § 2.3).

3.3. In the last section we have shown (theorem 2.5) that if $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ is maximal, then all the limits of $\dfrac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}$ lie within or on a circle around the origin with radius $\dfrac{1}{R(x_0)}$. This result can be improved. It is possible to show that all the limits lie in the smallest closed convex domain containing all the points $\dfrac{1}{s-x_0}$ where s passes through the set of singular points of $f(x)$ situated on $C(x_0; R(x_0))$.

If all the points of $C(x_0; R(x_0))$ are singular points then the proposition is proved by the theorem just mentioned, because in this case the convex domain coincides with the domain not outside the circle $C\left(0; \dfrac{1}{R(x_0)}\right)$. Thus we may assume that there is on $C(x_0; R(x_0))$ a point a where $f(x)$ is regular. Let $v_0 = \arg(a - x_0)$ and let s_1 and s_2 be the singular points of $f(x)$ situated on $C(x_0; R(x_0))$ nearest to and on either side of a (Fig. 6). Put $v_1 = \arg(s_1 - x_0)$ and $v_2 = \arg(s_2 - x_0)$. We assume $v_1 < v_0 < v_2$ and $0 < v_2 - v_1 \leq 2\pi$. (Equality holds if there is only one singular point on $C(x_0; R(x_0))$, in which case $s_1 = s_2$). Now we choose a positive number $\delta < \mathrm{Min}(v_2 - v_0; v_0 - v_1)$, such that $v_1 + \delta < v_0 < v_2 - \delta$. From x_0 we draw rays with their arguments equal to $v_1 + \delta$ and $v_2 - \delta$. Then it is clear that there exists a number ϱ_1 such that there is no singular point of $f(x)$ in the part of the circular annulus between $C(x_0; R(x_0))$ and $C(x_0; R + \varrho_1)$, which is situated between the rays. We call A the open domain, formed by the part of the circular annulus just mentioned and the interior of $C(x_0; R)$. A contains a but no singular point of $f(x)$. We now examine the set of points x such that the distance from x to a is

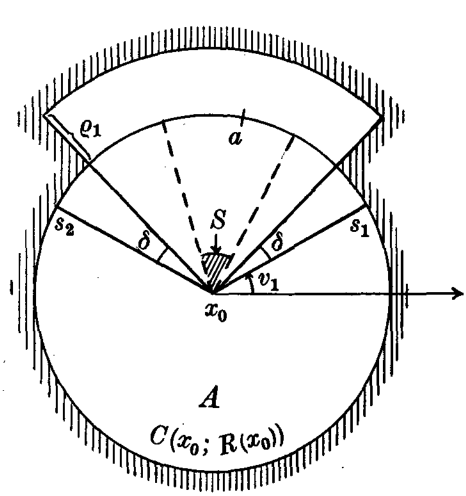


Fig. 6.

smaller than or equal to the distance from x to any point not belonging to A. By a geometrical investigation we can easily see that this set of points contains all the points from a closed circle sector, S, consisting of those points lying within or on $C\left(x_0; \frac{\varrho_1}{2}\right)$, which are seen from x_0 in directions with arguments between $\frac{v_0 + v_1 + \delta}{2}$ and $\frac{v_0 + v_2 - \delta}{2}$. The set contains other points too, a fact however, that we do not need in the following. The cusp of S is x_0. The points of S have the property that their distance from a is $\leq$ their distance to any point outside A. But every singular point is outside A. Thus, if $x \in S$ then $|a - x| \leq R(x)$.

Now we study the function $g(x) = (a - x) \cdot \varphi(x)$ where $\varphi(x)$ is a limit function of the sequence which is supposed to be maximal at $x_0 \in D$. As the sequence is maximal at x_0 then $|\varphi(x_0)| = \frac{1}{R(x_0)}$. As a lies on $C(x_0; R(x_0))$ then $|a - x_0| = R$. Thus $g(x_0) = 1$. But if $x \in S$ then $|a - x| \leq R(x)$ and further we always have $|\varphi(x)| \leq \frac{1}{R(x)}$. Thus, if $x \in S$ we have $|g(x)| \leq 1$. Thus the function $|g|$ is ≤ 1 in the sector S and $= 1$ at its cusp. Now it is possible to use the same method as in the proof of Theorem 2.5.

Put $x = x_0 + \varrho \cdot e^{i\theta}$ and consider the function

$$u(\varrho; \theta) = Re \log g(x) = \log |g(x)|$$

which is harmonic in D. We have $u(0; \theta) = 0$ and $u(\varrho; \theta) \leq 0$ if $\frac{v_0 + v_1 + \delta}{2} \leq$ $\leq \theta \leq \frac{v_0 + v_2 - \delta}{2}$ and $\varrho < \text{Min}\left(r(x_0); \frac{\varrho_1}{2}\right)$.

Now

$$\left(\frac{\partial u}{\partial \varrho}\right)_\theta = \lim_{\varrho \to 0} \frac{u(\varrho; \theta) - u(0; \theta)}{\varrho} \leq 0.$$

But

$$\left(\frac{\partial u}{\partial \varrho}\right)_\theta = Re\left(\frac{g'(x_0)}{g(x_0)} \cdot e^{i\theta}\right).$$

Thus

$$Re\left(\frac{g'(x_0)}{g(x_0)} \cdot e^{i\theta}\right) \leq 0 \quad \text{or} \quad \frac{\pi}{2} \leq \theta + \arg \frac{g'(x_0)}{g(x_0)} \leq \frac{3\pi}{2}$$

for all θ satisfying

$$\frac{v_0 + v_1 + \delta}{2} \leq \theta \leq \frac{v_0 + v_2 - \delta}{2}$$

$$\therefore \frac{\pi}{2} - \frac{v_0 + v_2 - \delta}{2}\left(\leq \frac{\pi}{2} - \theta\right) \leq \arg \frac{g'(x_0)}{g(x_0)}\left(\leq \frac{3\pi}{2} - \theta\right) \leq \frac{3\pi}{2} - \frac{v_0 + v_1 + \delta}{2}.$$

Let $\delta \to 0$;

$$\frac{\pi}{2} - \frac{v_0 + v_2}{2} \leq \arg \frac{g'(x_0)}{g(x_0)} \leq \frac{3\pi}{2} - \frac{v_0 + v_1}{2}.$$

But as $g(x) = (a - x) \cdot \varphi(x)$ we have

$$\frac{g'(x_0)}{g(x_0)} = -\frac{1}{a - x_0} + \frac{\varphi'(x_0)}{\varphi(x_0)}.$$

This means that $\frac{\varphi'(x_0)}{\varphi(x_0)}$ is a point seen from the point $\frac{1}{a - x_0}$ in a direction with argument lying between $\frac{\pi}{2} - \frac{v_0 + v_2}{2}$ and $\frac{3\pi}{2} - \frac{v_0 + v_1}{2}$. According to Theorem 2.5 $\frac{\varphi'(x_0)}{\varphi(x_0)}$ is situated within or on the circle $C\left(0; \frac{1}{R(x_0)}\right)$. Thus we

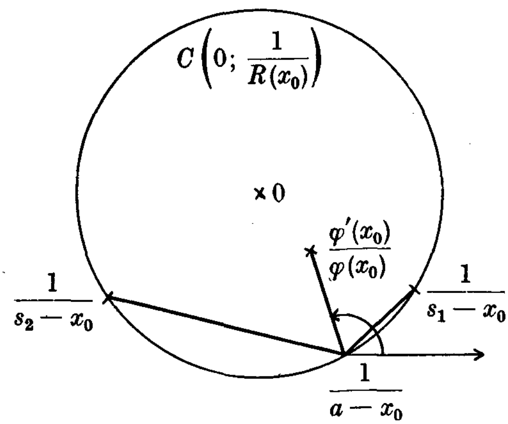


Fig. 7.

have proved that $\frac{\varphi'(x_0)}{\varphi(x_0)}$ lies within a certain peripheral angle of this circle (Fig. 7). This angle can be described simply as the angle which has its cusp at $\frac{1}{a - x_0}$ and the sides of which pass through $\frac{1}{s_1 - x_0}$ and $\frac{1}{s_2 - x_0}$.

Now let a tend to s_1 along $C(x_0; R)$. Then $\frac{1}{a - x_0}$ tends to $\frac{1}{s_1 - x_0}$ along $C\left(0; \frac{1}{R}\right)$, and the side of the peripheral angle which passes through $\frac{1}{s_2 - x_0}$ tends to the chord between $\frac{1}{s_1 - x_0}$ and $\frac{1}{s_2 - x_0}$. Thus $\frac{\varphi'(x_0)}{\varphi(x_0)}$ lies on this chord or on a given side of it, viz. the side opposite to the side where the point $\frac{1}{a - x_0}$ moves.

The same argument is applicable to any pair of adjacent singular points on $C(x_0; R(x_0))$. Thus $\frac{\varphi'(x_0)}{\varphi(x_0)}$ belongs to the closed convex covering of the set of points $\left\{\frac{1}{s-x_0}\right\}$ where s passes through all the singular points of $f(x)$ situated on the circle $C(x_0; R(x_0))$. According to Theorem 2.1 the same holds for the limits of $\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}$.

We have proved the

Theorem: If $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ is maximal $\left(\text{i. e.} \to \frac{1}{R(x_0)} \neq 0\right)$ and if $x_0 \in D$ then all the limits of $\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}$ belong to the closed convex covering of the set of points $\left\{\frac{1}{s-x_0}\right\}$ where s passes through the singular points of $f(x)$ lying on $C(x_0; R(x_0))$.

E. g. if there is only one singular point s on the circle of convergence $C(x_0; R(x_0))$, then $\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}$ converges to $\frac{1}{s-x_0}$. If there are only two singular points s_1 and s_2 on the circle of convergence then all the limits of $\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}$ lie on the chord between $\frac{1}{s_1-x_0}$ and $\frac{1}{s_2-x_0}$.

3.4. From Theorems 3.2 and 3.3 it is possible to deduce a simple method for separating the singular points on $C(x_0; R(x_0))$ from the regular ones. Let $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ be a maximal sequence such that $\left|\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right| \to \frac{1}{R(x_0)}$. Then, according to Theorem 3.3 all the limits of $\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}$ have the form $\frac{1}{s-x_0}$, where s is a singular point on $C(x_0; R(x_0))$. Conversely, if s is a singular point on $C(x_0; R(x_0))$ then from Theorem 3.2 we can deduce that there exists a sequence $\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}$ which converges to $\frac{1}{s-x_0}$ and such that $\left|\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right|$ and $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|} \to \frac{1}{R(x_0)}$.

Viz. let x_ν be a sequence of points $\in D$ and situated on the line between x_0 and s and let $\lim x_\nu = x_0$. Further, let $\{\varepsilon_\nu\}$ be a sequence of positive numbers with $\lim \varepsilon_\nu = 0$. For every μ there exists a sequence $\sqrt[m_{\mu\nu}]{|a_{m_{\mu\nu}}(x)|}$ which is maximal in x_μ. Apparently the hypothesis of Theorem 3.2 holds for such a sequence. Thus $\frac{a_{m_{\mu\nu}+1}(x_\mu)}{a_{m_{\mu\nu}}(x_\mu)}$ converges to $\frac{1}{s-x_\mu}$. It is then possible to choose l_μ so large that

$$\left|\frac{a_{m_\mu l_\mu+1}(x_\mu)}{a_{m_\mu l_\mu}(x_\mu)} - \frac{1}{s-x_\mu}\right| < \varepsilon_\mu$$

and

$$\left|\sqrt[m_\mu l_\mu]{|a_{m_\mu l_\mu}(x_\mu)|} - \frac{1}{R(x_\mu)}\right| < \varepsilon_\mu .$$

Now we designate the numbers $m_\mu l_\mu$ by n_μ. Thus, there exists a sequence n_μ; $\mu = 1, 2, 3, \ldots$ with

$$\left|\frac{a_{n_\mu+1}(x_\mu)}{a_{n_\mu}(x_\mu)} - \frac{1}{s-x_\mu}\right| < \varepsilon_\mu$$

and

$$\left|\sqrt[n_\mu]{|a_{n_\mu}(x_\mu)|} - \frac{1}{R(x_\mu)}\right| < \varepsilon_\mu .$$

An obvious transformation gives (one side in a quadrangle $\leq$ the sum of the three others)

$$\left|\frac{a_{n_\mu+1}(x_0)}{a_{n_\mu}(x_0)} - \frac{1}{s-x_\mu}\right| < \varepsilon_\mu + \left|\frac{1}{s-x_\mu} - \frac{1}{s-x_0}\right| + \left|\frac{a_{n_\mu+1}(x_\mu)}{a_{n_\mu}(x_\mu)} - \frac{a_{n_\mu+1}(x_0)}{a_{n_\mu}(x_0)}\right|$$

and

$$\left|\sqrt[n_\mu]{a_{n_\mu}(x_0)} - \frac{1}{R(x_0)}\right| < \varepsilon_\mu + \left|\frac{1}{R(x_\mu)} - \frac{1}{R(x_0)}\right| + \left|\sqrt[n_\mu]{|a_{n_\mu}(x_\mu)|} - \sqrt[n_\mu]{|a_{n_\mu}(x_0)|}\right| .$$

Let $\mu \to \infty$. Then by hypothesis the two first terms in the right members of the two inequalities tend to zero. The two second terms tend to zero since the functions involved are continuous in x_0. It remains to prove that the two third terms tend to zero. We begin with the first inequality. As the family $\left\{\frac{a_{n+1}}{a_n}\right\}$ is normal in D, its functions are continuous uniformly with respect to n (également continues) (MONTEL 2, p. 28) in D_-. This means that for every value of n (e. g. for $n = n_\mu$)

$$\left|\frac{a_{n+1}(x_\mu)}{a_n(x_\mu)} - \frac{a_{n+1}(x_0)}{a_n(x_0)}\right| < \varepsilon \quad \text{if} \quad |x_\mu - x_0| < \delta_\varepsilon .$$

Thus the term in question tends to zero with $|x_\mu - x_0|$. — In the second inequality we first use the formula $||a|-|b|| \leq |a-b|$ and then the same argument as we used when treating the first inequality, now based on the fact that the family $\{\sqrt[n]{a_n}\}$ is normal in D.

We have now proved that there exists a sequence $\{n_\mu\}$ such that $\dfrac{a_{n_\mu+1}(x_0)}{a_{n_\mu}(x_0)}$ converges to $\dfrac{1}{s-x_0}$ and $\sqrt[n_\mu]{|a_{n_\mu}(x_0)|}$ to $\dfrac{1}{R(x_0)}$. As $\dfrac{1}{|s-x_0|}=\dfrac{1}{R(x_0)}$, our proposition is proved.

Thus it is possible to determine the singular points on the circle of convergence of a power series of the type regarded here by the following method: Find all sequences n_ν such that $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ as well as $\left|\dfrac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right| \to \dfrac{1}{R(x_0)}$. Determine all the limits L of all sequences $\dfrac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}$. Determine all numbers of the form $x_0+\dfrac{1}{L}$. These and no others are the singular points in question.

4. Investigation of some Special Cases

4.1. We have seen (§ 3.1 corollary 1) that if s is the only singular point on $C(x_0;R(x_0))$ and if s is of type B with respect to x_0, then to every maximal sequence $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ corresponds a sequence $g_{n_\nu m_\nu}(x)$ all the limit functions of which are equivalent. As a special case we get: If s is the only singular point on $C(x_0;R(x_0))$, $x_0\in D$, if s is of type B with respect to x_0 and if $\lim \sqrt[\nu]{|a_\nu(x_0)|}$ exists, then all the limit functions of $\{g_{nm}(x)\}$ are equivalent.

From Theorem 2.2 we see that if we do not merely assume $\lim\sqrt[\nu]{|a_\nu(x_0)|}$ to exist but that $\sqrt[\nu]{|a_\nu(x)|}$ converges for all x belonging to D, then the rest of the assumptions in the above proposition becomes superfluous. It is not even necessary to assume convergence throughout D but only at the points of a set $M\subset D$ of the type considered in 2.2. We shall see that the above assumptions concerning s follow from the assumption that $\sqrt[\nu]{|a_\nu(x)|}$ converges at the points of M.

By Theorem 2.2 we conclude that all the limit functions of $\{g_{nm}(x)\}$ in D are equivalent to the same function $\varphi(x)$ and that $\dfrac{a_{\nu+1}(x)}{a_\nu(x)}$ converges to $\dfrac{\varphi'(x)}{\varphi(x)}$. But as $\dfrac{a_{\nu+1}}{a_\nu}$ converges then $\left|\dfrac{a_{\nu+1}}{a_\nu}\right|$ converges and $\sqrt[\nu]{|a_\nu|}$ must converge to the same limit. Thus $|\varphi|=\left|\dfrac{\varphi'}{\varphi}\right|$ or $\dfrac{\varphi'}{\varphi}=\varphi\cdot e^{iv}$, where v is a real constant. This differential equation has the solution $\varphi=\dfrac{e^{-iv}}{s-x}$ where s is the integration constant. s must be the same for all the limit functions of $\{g_{nm}\}$ as they have to be equivalent. As $\sqrt[\nu]{|a_\nu|} \to \frac{1}{R}$ then $|\varphi| = \frac{1}{R}$; i. e. $R(x) = |s - x|$. This means that s lies on $C(x; R(x))$ for every $x \in D$. Let s_1 be a singular point on such a circle $C(x_1; R(x_1))$. Let $x_2 \in D$ lie on the radius from x_1 to s_1. The circle of convergence around x_2, $C(x_2; R(x_2))$ contains only one singular point viz. s_1, and this point is the only one it has common with $C(x_1; R(x_1))$. But as s has to lie on every circle $C(x; R(x))$, s and s_1 must coincide. Thus every singular point on every circle $C(x; R(x))$; $x \in D$ has to coincide with s, viz. s is the nearest singular point to every $x \in D$ and there is no $x \in D$ such that its circle of convergence, $C(x; R(x))$, contains more than one singular point. As $R(x)$ is the modulus of an analytic function s must be of type B with respect to any $x \in D$ (§ 0.24).

Theorem: If $\sqrt[\nu]{|a_\nu(x)|}$ converges for all $x \in D$ to a limit different from zero then there is only one singular point on the circle of convergence $C(x; R(x))$ and this singularity has to be of type B with respect to every $x \in D$.

We can affirm that s is an isolated singular point under the following circumstances: $f(x)$ is uniform in the neighbourhood of s and D describes at s an angle greater than π. For in this case it is possible to find three points in D such that their corresponding circles of convergence cover a domain containing a circle around s.

In the following number we shall see that the sequence $\sqrt[\nu]{|a_\nu|}$ converges at a point which is not a limit point provided it has a sufficiently dense maximal subsequence.

4.2. We shall say that a power series $\sum\limits_{\mu=0}^{\infty} a_\mu x^\mu$ has Ostrowski gaps if there are numbers $\delta > 1$ and $\gamma < \frac{1}{R}$ and a sequence n_ν with $\sqrt[\mu]{|a_\mu|} < \gamma$ for $n_\nu \leq \mu < \delta n_\nu$.

In § 1.4 we noticed that the occurrence of great Ostrowski gaps in the power series around a point in D is a necessary condition that $\underline{\lim} \sqrt[\nu]{|a_\nu|}$ be zero. We shall see that the occurrence of Ostrowski gaps is a necessary condition for $\underline{\lim} \sqrt[\nu]{|a_\nu|}$ to be different from $\overline{\lim} \sqrt[\nu]{|a_\nu|}$.

Theorem: If $x_0 \in D$ and if $\underline{\lim} \sqrt[\nu]{|a_\nu(x_0)|} < \frac{1}{R(x_0)}$ then the power series $\sum\limits_{\mu=0}^{\infty} a_\mu(x_0)(x - x_0)^\mu$ has Ostrowski gaps.

Proof: Put $\underline{\lim} \sqrt[\nu]{|a_\nu(x_0)|} = \alpha$. The theorem has already been proved in the case $\alpha = 0$ (§ 1.4). Therefore suppose $\alpha \neq 0$. Then $\left|\frac{a_{\nu+1}(x_0)}{a_\nu(x_0)}\right|$ is bounded (§ 1.31), say $\left|\frac{a_{\nu+1}(x_0)}{a_\nu(x_0)}\right| < M$. Let β be a number with $\alpha < \beta < \frac{1}{R(x_0)}\cdot$ Thus there exists a sequence n_ν with $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|} < \beta$. Let $\mu \geq n_\nu$

$$|a_\mu(x_0)| = |a_{n_\nu}(x_0)| \cdot \left|\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right| \cdot \left|\frac{a_{n_\nu+2}(x_0)}{a_{n_\nu+1}(x_0)}\right| \cdot \ldots \cdot \left|\frac{a_\mu(x_0)}{a_{\mu-1}(x_0)}\right|;$$

$$|a_\mu(x_0)| < \beta^{n_\nu} \cdot M^{\mu-n_\nu}; \qquad \sqrt[\mu]{|a_\mu(x_0)|} < \beta^{\frac{n_\nu}{\mu}} \cdot M^{1-\frac{n_\nu}{\mu}}.$$

Now, let γ be a number with $\beta < \gamma < \frac{1}{R(x_0)}\cdot$ It is then possible to determine a number $\delta > 1$ such that $\beta^{\frac{1}{\delta}} \cdot M^{1-\frac{1}{\delta}} = \gamma$ viz. $\delta = \frac{\log M - \log \beta}{\log M - \log \gamma}\cdot$ Thus for μ satisfying $n_\nu \leq \mu < \delta n_\nu$ we get

$$\sqrt[\mu]{|a_\mu(x_0)|} \leq \gamma < \frac{1}{R(x_0)}$$

i. e. the power series has Ostrowski gaps. Q. E. D.

Corollary: If $x_0 \in D$ the necessary and sufficient condition that $\lim \sqrt[\nu]{|a_\nu(x_0)|}$ exists is that there exists a maximal sequence $\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}$ with $\lim \frac{n_{\nu+1}}{n_\nu} = 1$.

Proof: The necessity is trivial. The condition is sufficient, for suppose that such a sequence exists, then the series does not present Ostrowski gaps since for sufficiently great ν there would in every such gap lie at least one coefficient a_{n_ν}.

4.3. Let $\left|\frac{a_{n+1}(x)}{a_n(x)}\right| > \delta > 0$ for $x \in D_-$ and $n > N$. Further, let $h(x)$ be a finite linear combination of the derivatives of f: $h(x) = \sum_{\mu=0}^{M} C_\mu \cdot a_\mu(x)$. Then $h^{(n)}(x)$; $n > N_1$, has no zeros within D_-.

Proof: We have

$$h^{(n)}(x) = \sum_{\mu=0}^{M} C_\mu (\mu+1)(\mu+2)\ldots(\mu+n)\, a_{\mu+n}(x).$$

We may obviously assume $C_M \neq 0$. Thus

$$h^{(n)}(x) = C_M \cdot a_{M+n}(x) \cdot (M+1)(M+2)\ldots(M+n) \cdot$$
$$\cdot \left[1 + \sum_{\mu=0}^{M-1} \frac{C_\mu}{C_M} \cdot \frac{a_{n+\mu}}{a_{n+M}} \cdot \frac{(\mu+1)\ldots(\mu+n)}{(M+1)\ldots(M+n)}\right].$$

Substitute R_n for the second term of the [ ] and let $\operatorname{Max}\left|\frac{C_\mu}{C_M}\right| = a$. We have

$$\frac{\mu + i}{M + i} \leq \frac{M - 1 + i}{M + i} = \frac{1}{1 + \frac{1}{M - 1 + i}}$$

and

$$\left|\frac{a_{n+\mu}}{a_{n+M}}\right| = \left|\frac{a_{n+\mu}}{a_{n+\mu+1}}\right| \cdot \ldots \cdot \left|\frac{a_{n+M-1}}{a_{n+M}}\right| < \left(\frac{1}{\delta}\right)^{M-\mu}.$$

Thus:

$$|R_n| < a \frac{\sum_{\mu=0}^{M-1} \left(\frac{1}{\delta}\right)^{M-\mu}}{\prod_{i=1}^{n} \left(1 + \frac{1}{M - 1 + i}\right)}.$$

As the infinite product $\prod_{j=1}^{\infty}\left(1 + \frac{1}{j}\right)$ is divergent, $R_n \to 0$ as $n \to \infty$. Therefore a number N_1 must exist for which $|R_n| < 1$ if $n > N_1$ and then $h^{(n)}(x)$ can have no zero within D_- as $a_{M+n}(x)$ has not and $1 + R_n$ cannot be zero. Q. E. D.

The condition $\left|\frac{a_{n+1}}{a_n}\right| > \delta > 0$ for all $x \in D$ and all $n > N$ (N independent of x); is satisfied if there is one single point x_0 of D where $\underline{\lim}\left|\frac{a_{\nu+1}(x_0)}{a_\nu(x_0)}\right| > 0$. For according to § 1.2 the family $\left\{\frac{a_{n+1}}{a_n}\right\}_N$ is normal in D_-. Thus $\left\{\frac{a_n}{a_{n+1}}\right\}_N$ is normal in D_-. But for N sufficiently great all functions of this family are holomorphic in D_-. Now $\overline{\lim}\left|\frac{a_\nu(x_0)}{a_{\nu+1}(x_0)}\right| < \infty$ i. e. the family is bounded at one point of D_-. Then it is bounded uniformly in every domain interior to D_-. (MONTEL 2, p. 35.)

We get the following theorem

Theorem: If $x_0 \in D_f$, $\underline{\lim}\left|\frac{a_{\nu+1}(x_0)}{a_\nu(x_0)}\right| > 0$ and $h(x) = \sum_{\mu=0}^{M} C_\mu \cdot a_\mu(x)$; ($C_\mu$ constants) then $D_f \subseteq D_h$, and

$$\underline{\lim} \sqrt[\nu]{\left|\frac{h^{(\nu)}(x)}{\underline{|\nu}}\right|} = \underline{\lim} \sqrt[\nu]{|a_\nu(x)|} \quad \text{for} \quad x \in D_f.$$

4.4. In the preceding § we saw that a linear differential transformation of f does not essentially change the distribution of non-limit points of zeros of the derivatives. We shall see that the effect of adding to f a function h does not disturb the distribution of these non-limit points in a domain, the points of which lie sufficiently far from a singular point of h. We have the following theorem:

Theorem: Let $x_0 \in D_f$ and $\varlimsup \sqrt[\nu]{|a_\nu(x_0)|} = \frac{1}{R_1} \neq 0$. Further, let $h(x)$ be a function the singular points of which lie at a distance $> R_1$ from x_0. Then $x_0 \in D_{f+h}$.

Proof: Let ϱ_1 be $> R_1$ but less than the distance from x_0 to the nearest singular point of h. Choose a positive number $\varepsilon < \frac{\varrho_1 - R_1}{2}$. As $\varlimsup \sqrt[\nu]{|a_\nu(x_0)|} = \frac{1}{R_1}$ there exists a number N_1, with $\sqrt[n]{|a_n(x_0)|} > \frac{1}{R_1 + \frac{\varepsilon}{2}}$ for $n > N_1$. Further, choose a number $\varrho < \frac{\varrho_1 - R_1}{2}$ and such that all points within $C(x_0; \varrho)$ belong to D_-. Then there is a number N_2 such that $\{g_{nm}\}_{N_2}$ is normal in $C(x_0; \varrho)$. As $|g_{nm}(x_0)| > \frac{1}{R_1 + \frac{\varepsilon}{2}}$ for $n > N_1$ and $\{g_{nm}(x)\}_{N_2}$ is normal in $C(x_0; \varrho)$ there exists a number N and a number $\varrho_2 < \varrho$ with $|g_{nm}(x)| > \frac{1}{R_1 + \varepsilon}$ for all x within $C(x_0; \varrho_2)$ and all $n > N$

$$\because \quad |a_n(x)| > \left(\frac{1}{R_1 + \varepsilon}\right)^n$$

for all $n > N$ and x within $C(x_0; \varrho_2)$.

On the other hand let $\max\limits_{C(x_0; \varrho_1)} |h|$ be A. Then for x within $C(x_0; \varrho_2)$

$$\left|\frac{h^{(n)}(x)}{\underline{|n}}\right| = \frac{1}{2\pi} \left| \int\limits_{C(x_0; \varrho_1)} \frac{h(z)\,dz}{(z-x)^{n+1}} \right| \leq \frac{A\,\varrho_1}{(\varrho_1 - \varrho_2)^{n+1}}.$$

Now from $\varrho_2 < \frac{\varrho_1 - R_1}{2}$ and $\varepsilon < \frac{\varrho_1 - R_1}{2}$ it follows that $\varrho_1 - \varrho_2 > R_1 + \varepsilon$. Thus for all sufficiently great n:

$$\frac{1}{(R_1 + \varepsilon)^n} > \frac{A\,\varrho_1}{(\varrho_1 - \varrho_2)^{n+1}}$$

i. e.

$$|a_n(x)| > \left|\frac{h^{(n)}(x)}{\underline{|n}}\right| \quad \text{or} \quad |f^{(n)}(x) + h^{(n)}(x)| > 0$$

for all x within $C(x_0; \varrho_2)$. Thus $x_0 \in D_{f+h}$. Q. E. D.

Remark: An immediate consequence is that if $x_0 \notin D_f$ and if

$$\overline{\lim} \sqrt[\nu]{\frac{1}{\underline{|\nu}} \left| f^{(\nu)}(x_0) + h^{\nu}(x_0) \right|} = \frac{1}{R_1} \text{ then } x_0 \notin D_{f+h}.$$

By means of Theorems 4.3 and 4.4 it is possible to solve the problem of finding the distribution of the limit points of the zeros of the derivatives for rather extensive classes of functions. We can for example prove a beautiful theorem of Pòlya concerning meromorphic functions (see PÒLYA 4 or 5 or WHITTAKER 6). Starting from the function $f(x) = \frac{1}{p-x}$ which has the entire plane free from limit points we see from Theorem 4.3 that a meromorphic function which has only one pole has the same property. Then we use Theorem 4.4 in order to prove for an arbitrary meromorphic function, that the points x for which there is only one pole on $C(x;\, R(x))$ are non-limit points.

In the same way we see that if f is a function having (among other singular points) a pole, then in the domain of action of this pole there are no limit points. (The domain of action of a singular point, s, is the set of points to which s is the nearest singular point. This domain is easily seen to be convex.) On the other hand every point on the boundary of this domain is a limit point, for assume this was not the case for a certain point x_0 on the boundary. Then in a neighbourhood A of x_0 the family $\{g_{nm}(x)\}$ would have only limit functions of type $\frac{\eta}{p-x}$, where p is the pole and $|\eta| = 1$, for this is the case in the domain of action of p, a part of which belongs to A. But this means that $\overline{\lim} \sqrt[\nu]{|a_\nu(x)|} = \frac{1}{|p-x|}$ in A, which is impossible as there must be points in A the nearest singular point of which is not p, i. e. for which

$$\overline{\lim} \sqrt[\nu]{|a_\nu(x)|} > \frac{1}{|p-x|}.$$

In this way, the problem of finding the distribution of the zeros for one function being solved, it is automatically solved for an infinity of functions.

5. Further Connections between the Coefficients and the Zeros of the Derivatives

In section 2 we began the study of the influence on the coefficients of a power series of the assumption that the series is developed around a point which has no zeros of the derivatives of the function in its neighbourhood. In this section we are going to continue this investigation.

We begin with two theorems concerning the behaviour of the functions $\sqrt[\nu]{a_\nu}$.

5.1. Theorem: **If $x \in D$ and if $g_{n_\nu m_\nu}$ converges to a non-constant function $\varphi(x)$, it is possible to find a sequence of numbers l_ν such that $g_{n_\nu+1}\, l_\nu(x)$ converges to $\varphi(x)$.**

Proof: We have according to formula (a) of § 1.2:

$$a_{n+1} = \frac{n}{n+1} a_n \cdot \frac{g'_{nm}}{g_{nm}} = \frac{n}{n+1} \cdot g_{nm}^{n+1} \cdot \frac{g'_{nm}}{g_{nm}^2}$$

$$\sqrt[n+1]{|a_{n+1}|} = \sqrt[n+1]{\frac{n}{n+1}} \cdot |g_{nm}| \cdot \left|\left(\frac{1}{g_{nm}}\right)'\right|^{\frac{1}{n+1}}. \tag{a}$$

Since $\varphi \neq 0$ in D, $\frac{1}{\varphi}$ is bounded in D_-. Thus it must be possible to find a domain $E \subset D_-$ where $\left(\frac{1}{\varphi}\right)'$ is bounded. By cutting away small circles around any possible zeros of $\left(\frac{1}{\varphi}\right)'$ it is possible to construct a domain $F \subset E$ within which $\frac{1}{(1/\varphi)'}$ is bounded. $\left(\text{As } \varphi \text{ is not a constant } \left(\frac{1}{\varphi}\right)' \text{ is not identically zero.}\right)$ Thus we can find a positive number $\alpha < 1$ such that for $x \in F$ we have

$$\alpha < \left|\left(\frac{1}{\varphi(x)}\right)'\right| < \frac{1}{\alpha}.$$

On the other hand $g_{n_\nu m_\nu}(x)$ converges uniformly in F to $\varphi(x)$. Thus there exists another number β with:

$$\beta < \left|\left(\frac{1}{g_{n_\nu m_\nu}(x)}\right)'\right| < \frac{1}{\beta} \quad \text{in } F \text{ for all } \nu \text{ (sufficiently great).}$$

By combining this result with formula (a) we find:

$$\sqrt[n_\nu+1]{\beta \frac{n_\nu}{n_\nu+1}}\, |g_{n_\nu m_\nu}(x)| < \sqrt[n_\nu+1]{|a_{n_\nu+1}(x)|} < \sqrt[n_\nu+1]{\frac{1}{\beta} \frac{n_\nu}{n_\nu+1}}\, |g_{n_\nu m_\nu}(x)|.$$

Let $\nu \to \infty$. Then $\lim \sqrt[n_\nu+1]{|a_{n_\nu+1}(x)|}$ exists and is equal to $|\varphi(x)|$. We have proved that if $g_{n_\nu m_\nu}(x) \to \varphi(x)$ in D then $|g_{n_\nu+1}\, l_\nu(x)| \to |\varphi(x)|$ in a certain domain $F \subset D$. Now we choose the number sequence l_ν so that $g_{n_\nu+1}\, l_\nu \to \varphi$ at a certain point $x_1 \in F$, which is always possible. Then every limit function $\psi(x)$ of $g_{n_\nu+1}\, l_\nu(x)$ has the properties:

$$|\psi(x)| = |\varphi(x)|; \quad \psi(x_1) = \varphi(x_1).$$

The first property gives $\psi = \varphi \cdot e^{iv}$ where v is a real constant. From the second property it follows that: $e^{iv} = 1$

$$\psi(x) = \varphi(x) \text{ in } F.$$

Thus $g_{n_\nu+1}\, l_\nu(x) \to \varphi(x)$ in F and on account of Vitali's theorem the same proposition holds in D. Q. E. D.

5.2. Theorem: For every $x \in D$ the sequence $\sqrt[\nu]{|a_\nu(x)|}$ has limits everywhere between $\underline{\lim}\sqrt[\nu]{|a_\nu(x)|}$ and $\overline{\lim}\sqrt[\nu]{|a_\nu(x)|}$, except possibly for those x which have $\underline{\lim}\sqrt[\nu]{|a_\nu(x)|}=0$.

Proof: Assume that the proposition does not hold for a certain value of x, x_0. Then there exist two numbers b and c with $b<c$ situated in an interval that does not contain any limits of $\sqrt[\nu]{|a_\nu(x_0)|}$. Further the two numbers can be chosen so near to each other that none of the values $\sqrt[\nu]{|a_\nu(x_0)|}$ lies between them. The numbers $\sqrt[\nu]{|a_\nu(x_0)|}$ are separated into two classes by the numbers b and c: one class containing the numbers less than b and the other containing those which are greater than c. Both classes contain an infinity of elements, for if they did not, either $\underline{\lim}\sqrt[\nu]{|a_\nu(x_0)|}\geq b$ or $\overline{\lim}\sqrt[\nu]{|a_\nu(x_0)|}\leq c$, which is not in accordance with the assumption that b and c are situated in an interval between these two limits.

Now the class consisting of numbers less than b contains an infinity of elements $\sqrt[n]{|a_n(x_0)|}$ with the property that the following number $\sqrt[n+1]{|a_{n+1}(x_0)|}$ belongs to the other class. Viz. assume that only a finite number of elements from the first class have this property. Then there would exist a greatest n such that $\sqrt[\nu]{|a_n(x_0)|}$ has the property. But as the first class contains an infinity of elements, there must exist a number m greater than n such that $\sqrt[m]{|a_n(x_0)|}$ belongs to the first class. Thus $\sqrt[m+1]{|a_{m+1}(x_0)|}$ and therefore $\sqrt[m+2]{|a_{m+2}(x_0)|}$, and so on, would belong to the first class. But then the second class would not contain an infinity of values. Thus there is in the first class an infinity of values $\sqrt[n]{|a_n(x_0)|}$ such that $\sqrt[n+1]{|a_{n+1}(x_0)|}$ belong to the second class. The values of n in question we call n_ν.

Thus

$$\sqrt[n_\nu]{|a_{n_\nu}(x_0)|}<b \quad \text{and} \quad \sqrt[n_\nu+1]{|a_{n_\nu+1}(x_0)|}>c$$

or

$$|a_{n_\nu}(x_0)|<b^{n_\nu} \quad \text{and} \quad |a_{n_\nu+1}(x_0)>c\cdot c^{n_\nu}$$

$$\left|\frac{a_{n_\nu+1}(x_0)}{a_{n_\nu}(x_0)}\right|>c\cdot\left(\frac{c}{b}\right)^{n_\nu}.$$

But the right member of this inequality tends to infinity with ν. On the other hand, according to § 1.31 the left member is bounded if $\underline{\lim}\sqrt[\nu]{|a_\nu(x_0)|}\neq 0$. Thus we have arrived to a contradiction and the theorem is proved.

5.3. For the functions $\frac{a_{\nu+1}}{a_\nu}$ it is possible to prove theorems analogous to the above two theorems which concern the functions $\sqrt[\nu]{a_\nu}$.

Let $\frac{a_{n_\nu+1}(x)}{a_{n_\nu}(x)}$ be a sequence of functions uniformly convergent in D_- to the limit function $\psi(x)$, holomorphic in D. Further we assume that ψ is different from zero at one point of D. Then it is different from zero throughout D (for if $\psi = 0$ at some point $x_0 \in D$ then either $\psi \equiv 0$ or x_0 is a limit of the zeros of $\frac{a_{n_\nu+1}}{a_{n_\nu}}$. But the last alternative is impossible.)

Now $\psi(x) = \lim \frac{a_{n_\nu+1}(x)}{a_{n_\nu}(x)}$. On account of the uniform convergence

$$\lim \left(\frac{a_{n_\nu+1}(x)}{a_{n_\nu}(x)}\right)'$$

exists and is equal to $\psi'(x)$. Applying the rule $a_n' = (n+1)\,a_{n+1}$ we get

$$\left(\frac{a_{n+1}}{a_n}\right)' = \frac{a_n\,(n+2)\,a_{n+2} - a_{n+1}\,(n+1)\,a_{n+1}}{a_n^2} =$$

$$= \left(\frac{a_{n+1}}{a_n}\right)^2 + (n+2)\left[\frac{a_{n+2}}{a_{n+1}} - \frac{a_{n+1}}{a_n}\right] \cdot \frac{a_{n+1}}{a_n}.$$

Thus

$$\lim \left[\left(\frac{a_{n_\nu+1}}{a_{n_\nu}}\right)^2 + (n_\nu + 2)\left(\frac{a_{n_\nu+2}}{a_{n_\nu+1}} - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right)\frac{a_{n_\nu+1}}{a_{n_\nu}}\right]$$

exists and is equal to ψ'.

$$\because\ \psi' = \psi^2 + \lim \left(\frac{a_{n_\nu+1}}{a_{n_\nu}}\right)(n_\nu + 2)\left(\frac{a_{n_\nu+2}}{a_{n_\nu+1}} - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right)$$

and the limit really exists. As $\lim \frac{a_{n_\nu+1}}{a_{n_\nu}}$ exists and is different from zero, the limit

$$\lim\,(n_\nu + 2)\left(\frac{a_{n_\nu+2}}{a_{n_\nu+1}} - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right)$$

exists. Thus we have proved

Theorem: If $\lim \frac{a_{n_\nu+1}}{a_{n_\nu}}$ exists and is equal to $\psi \neq 0$ within D then

$$\lim n_\nu \left(\frac{a_{n_\nu+2}}{a_{n_\nu+1}} - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right)$$

exists and is equal to $\frac{\psi' - \psi^2}{\psi}$.

This theorem is analogous to Theorem 2.1 which is fundamental in section 2 and in the proofs of Theorems 3.2 and 3.3. Therefore it would probably be possible to deduce theorems for the functions $\frac{a_{\nu+1}}{a_\nu}$ analogous to the theorems for $\sqrt[\nu]{a_\nu}$. However, we shall not follow this program in detail, but only give some of the conclusions which can be drawn as corollaries from Theorem 5.3.

Corollary I. If $\frac{a_{n_\nu+1}}{a_{n_\nu}}$ converges in D to a function $\psi \neq 0$ then

$$\left|\frac{a_{n_\nu+2}}{a_{n_\nu+1}} - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right| = O\left(\frac{1}{n_\nu}\right) \quad \text{(uniformly in } D_-\text{)}.$$

The second corollary is the analogy to Theorem 5.1.

Corollary II. If $\frac{a_{n_\nu+1}}{a_{n_\nu}}$ converges in D to a function $\psi \neq 0$ then

$$\lim \frac{a_{n_\nu+2}}{a_{n_\nu+1}} = \psi \text{ in } D.$$

This follows directly from corollary I.

Corollary III. If $\frac{a_{n_\nu+1}}{a_{n_\nu}}$ converges in D to a function $\psi \neq 0$ then

$$\left|\frac{a_{n_\nu+2}}{a_{n_\nu+1}} - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right| = o\left(\frac{1}{n_\nu}\right)$$

for all $x \in D$ implies $\psi = \frac{1}{s-x}$ for some complex number s, and conversely.

Proof: If $\left|\frac{a_{n_\nu+2}}{a_{n_\nu+1}} - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right| = o\left(\frac{1}{n_\nu}\right)$ then $\lim n_\nu \left(\frac{a_{n_\nu+2}}{a_{n_\nu+1}} - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right) = 0$ for every $x \in D$. Thus $\frac{\psi' - \psi^2}{\psi} = 0$ throughout D. This differential equation has the solution $\psi = \frac{1}{s-x}$ where s is the integration constant. Conversely if $\psi = \frac{1}{s-x}$ then $\frac{\psi' - \psi^2}{\psi} = 0$ and

$$\lim n_\nu \left(\frac{a_{n_\nu+2}}{a_{n_\nu+1}} - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right) = 0.$$

Corollary IV. If $\varliminf \left|\frac{a_{\nu+1}(x_0)}{a_\nu(x_0)}\right| > 0$; $x_0 \in D$ then the set of points

$$\left\{\nu\left(\frac{a_{\nu+2}(x)}{a_{\nu+1}(x)} - \frac{a_{\nu+1}(x)}{a_\nu(x)}\right)\right\}$$

is bounded for every $x \in D$ (uniformly in D_-) (cf. § 1.32).

Proof: The functions $\nu\left(\frac{a_{\nu+2}}{a_{\nu+1}} - \frac{a_{\nu+1}}{a_\nu}\right)$ are holomorphic in D_-. As the family $\frac{a_{\nu+1}}{a_\nu}$ is normal in D_- it is always possible to extract a convergent subsequence from any assigned sequence of functions $\frac{a_{\nu+1}}{a_\nu}$. If $\frac{a_{n_\nu+1}}{a_{n_\nu}}$ is such a uniformly convergent sequence then $n_\nu\left(\frac{a_{n_\nu+2}}{a_{n_\nu+1}} - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right)$ converges uniformly by Theorem 5.3. Thus the family $\left\{\nu\left(\frac{a_{\nu+2}}{a_{\nu+1}} - \frac{a_{\nu+1}}{a_\nu}\right)\right\}$ is normal. It is then only necessary to prove that the family is bounded in one single point in order to have proved that it is uniformly bounded in D_-. Assume that it is not bounded at x_1 ($\in D_-$). Then there exists a sequence n_ν such that $\left|n_\nu\left(\frac{a_{n_\nu+2}(x_1)}{a_{n_\nu+1}(x_1)} - \frac{a_{n_\nu+1}(x_1)}{a_{n_\nu}(x_1)}\right)\right| \to \infty$. It is possible to choose a subsequence m_ν of n_ν such that $\frac{a_{m_\nu+1}}{a_{m_\nu}}$ converges to a function ψ (which is not ∞ by 1.31). Thus $\left|m_\nu\left(\frac{a_{m_\nu+2}}{a_{m_\nu+1}} - \frac{a_{m_\nu+1}}{a_{m_\nu}}\right)(x_1)\right|$ converges to $\frac{\psi'(x_1) - \psi^2(x_1)}{\psi(x_1)}$ what is impossible as the sequence is a subsequence of a sequence which tends to infinity.

The fifth corollary is analogous to Theorem 5.2.

Corollary V. If $\varliminf \left|\frac{a_{\nu+1}(x_0)}{a_\nu(x_0)}\right| > 0$; $x_0 \in D$, then the limits of $\frac{a_{\nu+1}(x_0)}{a_\nu(x_0)}$ form a connected set.

Proof: Viz. assume that the limits of $\frac{a_{\nu+1}(x_0)}{a_\nu(x_0)}$ are elements of two sets at a positive distance from each other. It would then be possible to find two open domains A and B at a positive distance from each other, each covering one of the two separated sets, and such that only a finite number of points $\frac{a_{\nu+1}(x_0)}{a_\nu(x_0)}$ lie outside their sum $A + B$. A and B must contain an infinity of points $\frac{a_{\nu+1}(x_0)}{a_\nu(x_0)}$ for otherwise at least one of the domains would not contain any limit point of these points. Then for an infinity of values n it must happen that $\frac{a_{n+1}(x_0)}{a_n(x_0)} \in A$ and $\frac{a_{n+2}(x_0)}{a_{n+1}(x_0)} \in B$ (cf. § 5.2). But the distance between $\frac{a_{n+1}(x_0)}{a_n(x_0)}$ and $\frac{a_{n+2}(x_0)}{a_{n+1}(x_0)}$ is $O\left(\frac{1}{n}\right)$ (corollary IV) and therefore tends to zero with increasing n, which is not in accordance with the fact that the distance between A and B is positive.

5.4. Most theorems hitherto proved are formulated in the following form: "If $x \in D$ and A, then B", where A and B are various propositions. They might equally well have been formulated in the following equivalent way: "If A and not B, then x is a limit point of the zeros of the derivatives of f." In this form they express sufficient conditions for a point x to be a limit point.

Regarding power series around the origin: $f(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu$ we have thus a multitude of conditions for the coefficients sufficient to guarantee that the origin is such a limit point. But except for what can be concluded from § 1.3, we have no necessary conditions of the same kind. It is then natural to ask whether any of our sufficient conditions are at the same time necessary. This is not the case. Except in §§ 1.3, 4.3 and 5.3 we have only considered sufficient conditions with the following property: The conditions only contain expressions in the numbers a_ν which do not change when the arguments of a_ν are changed but their moduli conserved. If such a condition were at the same time sufficient and necessary, then all the series with the same moduli of coefficients as a given series would have the origin as a limit point of the zeros of their derivatives when and only when this is the case with the given series. But there certainly exist series for which the origin changes its character in this respect if the arguments of the coefficients of the series change. This is clear from the fact that there exist series which satisfy the assumptions of the following theorem.

Theorem: If $f(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu$ is a function with a singular point at a finite distance and if the origin is not a limit point of the zeros of the derivatives of f, then it is possible to choose numbers ω_ν with $|\omega_\nu| = 1$ such that the origin is a limit point of the zeros of the derivatives of $h(x) = \sum_{\nu=0}^{\infty} a_\nu \omega_\nu x^\nu$.

Proof: We use the fact that if α and β are any two complex numbers, then it is possible to choose ω with $|\omega| = 1$ in the way that

$$|\alpha - \beta\omega| \geq |\alpha|.$$

As f is not entire there exists a sequence $\frac{a_{n_\nu+1}}{a_{n_\nu}}$ which tends to a number different from zero. It is no restriction to assume for simplicity that this sequence does not simultaneously contain a number $\frac{a_{n+1}}{a_n}$ and the two next terms

$\frac{a_{n+2}}{a_{n+1}}$ and $\frac{a_{n+3}}{a_{n+2}}$. Now we choose ω_ν in the following way: If m is one of the numbers $n_\nu + 2$ then ω_m is chosen so that

$$\left|\frac{a_{n_\nu+2}}{a_{n_\nu+1}}\omega_m - \frac{a_{n_\nu+1}}{a_{n_\nu}}\right| \geq \left|\frac{a_{n_\nu+1}}{a_{n_\nu}}\right|$$

and if m is not one of the numbers $n_\nu + 2$, then $\omega = 1$. With this convention it is clear that

$$\left|\frac{a_{n_\nu+2}\,\omega_{n_\nu+2}}{a_{n_\nu+1}\,\omega_{n_\nu+1}} - \frac{a_{n_\nu+1}\,\omega_{n_\nu+1}}{a_{n_\nu}\,\omega_{n_\nu}}\right| \geq \left|\frac{a_{n_\nu+1}}{a_{n_\nu}}\right|$$

as

$$\omega_{n_\nu} = \omega_{n_\nu+1} = 1.$$

But if the origin is not a limit point of the zeros of the derivatives of h then the left member of this inequality is $O\left(\frac{1}{n_\nu}\right)$ (5.3 corollary I) which is not in accordance with the fact that the right member converges to a number different from zero.

Remark: The theorem does not necessarily hold for entire functions, which is seen from the example e^x.

BIBLIOGRAPHY. (1) **Landau:** Ueber die Blochsche Konstante und zwei verwandte Weltkonstanten. Mathematische Zeitschrift 30, p. 608 (1929). — (2) **Montel:** Lecons sur les familles normales. Paris 1927. — (3) **Ostrowski:** On representation of analytical functions by power series. Journal London Math. Soc. 1, p. 251 (1926). — (4) **Pòlya:** Ueber die Nullstellen sukzessiver Derivierten. Mathematische Zeitschrift 12, p. 36 (1922). — (5) ——: On the zeros of the derivatives of a function and its analytic character. Bull. Amer. Math. Soc. 49, p. 178 (1943). — (6) **Whittaker:** Interpolatory function theory. Cambridge 1935.

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