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The remainder in Tauberian theorems II

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With 5 figures in the text

Contents

1. Introduction

In an earlier paper [6] the author examined a class of Tauberian relations with exponentially vanishing remainders, i.e. relations of the form

$$\int_{-\infty}^{\infty} \Phi(x-u)\, dF(u) = O(e^{-\gamma x}) \quad \text{as } x \to \infty, \tag{0.1}$$

where $\Phi(x)$ is bounded and $F(x)$ is of bounded variation. Thus, when certain conditions are imposed on the Fourier-Stieltjes transform $f(\xi)$ of $F(x)$, it proves the validity of

$$\Phi(x) = O(e^{-\theta\gamma x}), \quad \text{as } x \to \infty, \tag{0.2}$$

when $\Phi(x)$ satisfies certain Tauberian conditions.

Chapter I of this paper extends these results. First, corresponding results are derived for functions decreasing more slowly than exponentially, the conditions on $f(\xi)$ being the same as before. Second, it is proved possible to weaken the conditions on $f(\xi)$ in theorems 1 and 2 in [6] as was expected ([6] p. 581).

The greater part of the paper investigates conditions necessary to prove the validity of relation (0.2). The principal question concerns the analyticity of $1/f(\xi)$, imposed in all the sufficiency theorems. If $F(x)$ satisfies the supplementary condition

$$\int_{-\infty}^{\infty} |x|^{\frac{1}{2}+\delta} |dF(x)| < \infty \quad \text{for some} \quad \delta > 0, \tag{0.3}$$

then it is necessary for the validity of (0.2) that $1/f(\xi)$ is analytic in a certain strip below the real axis. This result is obtained by reducing the problem to the case where $\Phi(x)$ belongs to L^2.

In order to make the arrangement clear, results concerning the existence of a relation (0.1) are collected in chapter II. In chapter III the validity of (0.2) is presumed, and an analogon of Wiener's result is obtained from this condition. Finally, the above-mentioned results concerning the analyticity of $1/f(\xi)$ are derived. In conclusion an example is set forth proving that the value of the constant θ obtained in Theorems 3 and 4 in [6] cannot be improved.

Chapter I

Improved sufficiency theorems

2. Definitions and notations

V denotes the class of functions of bounded variation and T denotes the class of functions $f(\xi)$ such that

$$f(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}\, dF(x), \qquad F(x) \in V.$$

The total variation of $F(x)$ is denoted by

$$V\{F\} = \int_{-\infty}^{\infty} |dF(x)|.$$

$p(x)$ denotes a weight-function of the kind introduced by Beurling in [3], characterized by the conditions

$$p(x) \geq p(0) = 1, \quad p(x+y) \leq p(x)\,p(y),$$

and

$$p(\varrho x) \geq p(x) \quad \text{for} \quad \varrho > 1.$$

In the sequel, weight-functions of this kind are always considered. Since we need these weight-functions for positive values of x only, we let $p(x) = 1$ for $x < 0$.

If $\Phi(x)$ is real, say that $\Phi(x) \in E_{p(x)} = E_p$ if

$$E_p \begin{cases} \Phi(x) \text{ is bounded and } \Phi(x) + \int\limits_0^x \{p(u)\}^\varepsilon du \text{ is} \\ \text{non-decreasing for every } \varepsilon > 0 \text{ and } x > x_\varepsilon. \end{cases} \tag{1.1}$$

For complex functions $\Phi(x)$ the classes E_p and E are defined as follows:

Definition. *$\Phi(x) \in E_p$ if $\Re\{\Phi(x)\}$ and $\Im\{\Phi(x)\}$ satisfy* (1.1); *$E_p = E$ if $p(x) = e^x$ for $x > 0$.*

We thus have extended class E, defined only for real functions in [6] to contain complex functions. This has no influence on the sufficiency theorems but is more convenient in the proofs of the necessity theorems.

$F(x)$ denotes a real- or complex-valued function, which belongs to V. The notation $\overset{*}{\Phi}(x)$, introduced for bounded functions $\Phi(x)$, is used to denote the function

$$\overset{*}{\Phi}(x) = \int\limits_{-\infty}^{\infty} \Phi(x-u)\,dF(u).$$

For the L^s norm the following notations are used

$$M_s\{f;\,a,b\} = \left\{\int\limits_a^b |f(\xi)|^s d\xi\right\}^{\frac{1}{s}}, \quad M_s\{f\} = M_s\{f;\,-\infty,\infty\},$$

and M is written instead of M_1.

A denotes an absolute positive constant, not necessarily the same one each time it occurs. A constant depending on one or more parameters is usually denoted by C.

3. The sufficiency theorems

The following theorem is a generalization of theorems 1 and 2 in [6].

Theorem 1. *Let $F(x) \in V$ and $f(\xi) = \int\limits_{-\infty}^{\infty} e^{i\xi x}\,dF(x)$, $f(\xi) \neq 0$. Suppose there exists a function $f(\zeta)$, $\zeta = \xi + i\eta$, such that $1/f(\zeta)$ is analytic in the strip $-a \leq \eta < 0$, and*

$$\int\limits_{-\infty}^{\infty} \left(\frac{1}{1+|\xi|}\right)^{2q} \left|\frac{1}{f(\xi+i\eta)}\right|^2 d\xi$$

is bounded for $-a \leq \eta < 0$, *and furthermore*

$$\lim_{\eta \to -0} f(\xi + i\eta) = f(\xi).$$

Let $p(x)$ *be a weight-function,* $p(x) = O(x^{-(\frac{1}{2}+\delta)} e^{ax})$ *as* $x \to \infty$, *for some* $\delta > 0$. *If* $\Phi(x) \in E_p$ *and*

$$\int_{-\infty}^{\infty} \Phi(x-u)\, dF(u) = O\left(\frac{x^{-(\frac{1}{2}+\delta)}}{p(x)}\right) \quad as \quad x \to \infty,$$

then

$$\Phi(x) = O\left\{\left(\frac{1}{p(x)}\right)^{\theta}\right\} \quad as \quad x \to \infty,$$

for every $\theta < 1/(q+1)$.

First, two lemmas will be proved.

Lemma 1. *Let* $F(x) \in V$, $G(x) \in L$ *and* $f(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} dF(x)$,

$$g(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} G(x)\, dx.$$

Suppose that $f(\xi) \neq 0$ *and* $g(\xi)/f(\xi) \in L^2$. *Let* $W(x)$ *be the Fourier transform of* $g(\xi)/f(\xi)$ *and suppose that*

$$p(x)\, W(x) \in L^2, \quad p(x)\, W(x) \in L(0, \infty).$$

If the function $\Phi(x)$ *satisfies*

$$|\Phi(x)| \leq B, \quad p(x) \overset{*}{\Phi}(x) \in L^2(0, \infty),$$

then for $x > 0$ *we have*

$$|\Phi * G(x)| \leq \frac{1}{p(x)} (B\, V\{F\}\, M\{p\, W;\, 0, \infty\} + M_2\{p\, W\}\, M_2\{p \overset{*}{\Phi};\, 0, \infty\}).$$

If we use the relation $p(x) \leq p(x-u)\, p(u)$ and Schwarz' inequality, it is easy to verify that

$$\left.\begin{aligned} |\overset{*}{\Phi}| * |W|(x) &\leq \frac{1}{p(x)} (B\, V\{F\}\, M\{p\, W;\, x, \infty\} + \\ &\qquad + M_2\{p\, W\}\, M_2\{p \overset{*}{\Phi};\, 0, \infty\}), \quad x \geq 0, \\ |\overset{*}{\Phi}| * |W|(x) &= O(1 + |x|^{\frac{1}{2}}), \quad x < 0. \end{aligned}\right\} \tag{1.2}$$

Thus, it is sufficient to prove that

$$\Phi * G(x) = \overset{*}{\Phi} * W(x). \tag{1.3}$$

Let $M(x)$ be a function such that $(1+|x|)^{\frac{1}{2}} M(x)$ belongs to L and such that the Fourier transform, $m(\xi)$, of $M(x)$ vanishes outside a finite interval. The conditions $f(\xi) \in T$ and $f(\xi) \neq 0$ yield $m(\xi)/f(\xi) \in T$ (cf. [3], Theorem III B, p. 13). Let

$$h(\xi) = \frac{g(\xi)\, m(\xi)}{f(\xi)}.$$

Then $h(\xi) \in T$ and $h(\xi)$ is the Fourier transform of $H(x) = M * W(x)$. Thus $H(x) \in L$.

According to (1.2) and the conditions on $M(x)$, the integral $M * \overset{*}{\Phi} * W(x)$ is absolutely convergent and may be inverted, which yields

$$M * \overset{*}{\Phi} * W(x) = \overset{*}{\Phi} * M * W(x) = \overset{*}{\Phi} * H(x).$$

Since $H(x)$ belongs to L, the double integral $\overset{*}{\Phi} * H(x)$ may be inverted to obtain

$$\overset{*}{\Phi} * H(x) = \Phi * \overset{*}{H}(x).$$

Now

$$\overset{*}{H}(x) = M * G(x),$$

since both sides are continuous and have the same Fourier transform, $m(\xi)\, g(\xi)$. Thus, it has been proved that

$$M * \overset{*}{\Phi} * W(x) = \Phi * M * G(x),$$

or, upon another inversion, justified by absolute convergence

$$M * \overset{*}{\Phi} * W(x) = M * \Phi * G(x).$$

If we choose

$$M(x) = M_\lambda(x) = \frac{\sin^2 \frac{1}{2}\lambda x}{\lambda x^2},$$

then $M_\lambda(x)$ satisfies the above conditions for every λ. By letting $\lambda \to \infty$ we find by standard summability theorems

$$\overset{*}{\Phi} * W(x) = \Phi * G(x),$$

since both sides are continuous and $O(1+|x|^{\frac{1}{2}})$. Thus (1.3) is proved and the result of Lemma 1 follows.

Lemma 2. *Let the function $K(x)$ satisfy the following conditions:*

$$K(x) \geq 0; \; K(x) = 0 \text{ outside a finite interval, } 0 < M\{K(x)\} < \infty. \tag{1.4}$$

If $\Phi(x) \in E_p$ and if, for some positive constants C and σ,

$$\left|\Phi * \frac{1}{r} K\left(\frac{x}{r}\right)\right| \leq \frac{C r^{-\sigma}}{p(x)} \tag{1.5}$$

for $x>0$ *and* $0<r\leq 1$, *then as* $x\to\infty$

$$\Phi(x) = O\left\{\left(\frac{1}{p(x)}\right)^{\theta}\right\} \quad \textit{for every} \quad \theta < \frac{1}{1+\sigma}.$$

The proof follows from an obvious modification of the argument on page 579 in [6], and is therefore omitted.

We now turn to the proof of Theorem 1. Let q denote the number introduced in the conditions on $f(\xi)$, in which case $q>0$. Let $K(x)$ be a function which satisfies (1.4), and such that the function

$$k(\zeta) = \int_{-\infty}^{\infty} e^{i\zeta x} K(x)\,dx, \qquad \zeta = \xi + i\eta,$$

fulfils the inequality

$$|k(\xi+i\eta)| \leq \frac{1}{(1+|\xi|)^q} \tag{1.6}$$

in the strip $-a\leq\eta\leq 0$.

The function $K(x)$ may be constructed in the following way. Let $q=n+\frac{1}{2}+\nu$, where n is an integer and $-\frac{1}{2}<\nu\leq\frac{1}{2}$. Let $J_\nu(\zeta)$ denote the Bessel function of order ν, and let

$$k_1(\zeta) = \left(\frac{\sin\zeta}{\zeta}\right)^n \zeta^{-\nu} J_\nu(\zeta).$$

Here $\sqrt{\pi}\,2^\nu\Gamma(\nu+\frac{1}{2})\,\xi^{-\nu}J_\nu(\xi)$ is the Fourier transform of that function, which vanishes outside the interval $(-1,1)$ and equals $(1-x^2)^{\nu-\frac{1}{2}}$ on this interval (cf. [9] p. 178). It follows that $k_1(\xi)$, except for a constant factor, is the Fourier transform of a function $K(x)$ which satisfies the required conditions (cf. [10] Chapter VII).

Let r be real, $0<r\leq 1$. From (1.6) it follows that

$$(1+|\xi|)^q\,|k(r(\xi+i\eta))| \leq r^{-q}, \quad -a\leq\eta\leq 0;$$

therefore, by the conditions on $f(\xi)$,

$$M_2\left\{\frac{k(r(\xi-i\beta))}{f(\xi-i\beta)}\right\} \leq r^{-q} M_2\left\{\left(\frac{1}{1+|\xi|}\right)^q \frac{1}{f(\xi-i\beta)}\right\} < C_1 r^{-q}, \quad 0\leq\beta\leq a. \tag{1.7}$$

Introduce the function

$$w_r(\zeta) = \frac{k(r\zeta)}{f(\zeta)},$$

which is analytic in the strip $-a\leq\eta<0$. Let $W_r(x)$ be the Fourier transform of $w_r(\xi)$. Then $e^{\beta x}W_r(x)$ is the Fourier transform of $w_r(\xi-i\beta)$ if $0\leq\beta\leq a$, and Parseval's relation gives, by the aid of (1.7)

$$\sqrt{2\pi}\, M_2\{e^{\beta x} W_r(x)\} = M_2\{w_r(\xi - i\beta)\} < C_1 r^{-q}, \quad 0 \le \beta \le a.$$

Let

$$p(x) \le C_2 e^{ax}, \; x > 0; \quad M_2\{e^{-ax} p(x);\, 0,\, \infty\} = C_3,$$

then

$$M_2\{p(x)\, W_r(x)\} \le M_2\{W_r(x)\} + C_2 M_2\{e^{ax}\, W_r(x)\} < C_1(1 + C_2)\, r^{-q}$$

and, by Schwarz' inequality,

$$M\{p(x)\, W_r(x);\, 0,\, \infty\} \le M_2\{e^{-ax} p(x);\, 0,\, \infty\}\, M_2\{e^{ax}\, W_r(x)\} < C_1 C_3 r^{-q}.$$

Since $k(r\xi)$ is the Fourier transform of $(1/r)K(x/r)$ the above inequalities imply that the conditions of Lemma 1 are satisfied for the function $G(x) = (1/r)K(x/r)$. An application of Lemma 1 yields, for $x > 0$,

$$\left|\Phi * \frac{1}{r} K\left(\frac{x}{r}\right)\right| \le \frac{C r^{-q}}{p(x)},$$

where C is independent of r, $0 < r \le 1$. Thus Lemma 2 may be applied to find

$$\Phi(x) = O\left\{\left(\frac{1}{p(x)}\right)^{\theta}\right\} \text{ for every } \theta < \frac{1}{q+1},$$

and the theorem is proved.

Incidentally, it may be noticed that if q is an integer, then the function $w_r(\zeta)$, introduced above, is of the same type as the auxiliary function $w_r(\zeta)$, used to prove corresponding theorems in [6].

Theorems 3 and 4 in [6] may be generalized in a similar way to hold for non-exponentially vanishing remainders. If the conditions on $f(\xi)$ in these theorems are unchanged, we find that if

$$\Phi(x) \in E_p, \quad \text{where } p(x) \le e^{ax}, \quad x > 0.$$

and

$$\overset{*}{\Phi}(x) = O\left(\frac{1}{p(x)}\right) \text{ as } x \to \infty,$$

then

$$\Phi(x) = O\left\{\left(\frac{1}{p(x)}\right)^{\theta}\right\} \text{ as } x \to \infty,$$

for the values of θ admitted in these theorems.

Chapter II

The existence of functions $\Phi(x)$ which satisfy a Tauberian relation

4. The problem considered

Let $F(x)$ be of bounded variation and let $f(\xi)$ be the Fourier Stieltjes transform of $F(x)$. Consider a relation of the form

$$\int_{-\infty}^{\infty} \Phi(x-u)\,dF(u) = O(e^{-\gamma x}) \text{ as } x \to \infty, \tag{2.1}$$

where the function $\Phi(x)$ is bounded, and γ denotes a positive number. Our aim is to investigate the necessary conditions under which relation (2.1) implies

$$\Phi(x) = O(e^{-\theta\gamma x}), \quad \text{as } x \to \infty, \tag{2.2}$$

for every $\Phi(x) \in E$ and some $\theta > 0$. Now it immediately follows, by the Wiener argument, that if (2.1) implies $\Phi(x) = o(1)$ as $x \to \infty$ for every $\Phi(x) \in E$, then $f(\xi)$ cannot vanish for any real ξ [cf. [12] p. 26). It is therefore sufficient, in this connection, to consider the case where $f(\xi) \neq 0$.

From the definition of class E (p. 317) it follows that if $\Phi(x) = 0$ a.e. and $\Phi(x) \in E$, then $\Phi(x) = 0$ for $x > x_0$, provided x_0 is large enough. The relation (2.2) is trivial for such a function. We therefore exclude in the sequel the class of functions $\Phi(x)$ vanishing a.e. Let us call a function $\Phi(x)$ *non-trivial* if $\Phi(x) \neq 0$ in a set of positive measure. It is easy to see that the existence of a bounded, non-trivial function, $\Phi(x)$, satisfying the relation (2.1) asserts the existence of a continuous function with the same properties. It should further be noted that if $f(\xi) \neq 0$ and $\overset{*}{\Phi}(x)$ is trivial then $\Phi(x)$ is trivial. (See [4], p. 134.)

In this chapter we investigate the existence of a non-trivial function, $\Phi(x)$, satisfying (2.1), when $F(x)$ is a given function of V such that $f(\xi) \neq 0$. It is clearly no restriction to assume $\Phi(x)$ continuous. In section 6 we examine the case where $\Phi(x)$ is a bounded function of L^2. Then, in section 8, the case is considered where $\Phi(x)$ is merely bounded, and the problem is solved by the aid of the additional condition (0.3).

In this chapter, as in chapter III, we shall make repeated use of bilateral Laplace transforms. For convenience of reference we shall first, in section 5, state some results from the theory of bilateral Laplace transforms in a suitable form.

5. Bilateral Laplace transforms used in the sequel

For the proofs of the results in this section the reader is referred to Widder [11], chapter VI.

Let the function $\Psi(x)$ satisfy

$$e^{\beta x}\Psi(x) \in L^2, \quad \alpha < \beta < \gamma$$

for two numbers α and γ.

Let $\zeta = \xi + i\eta$ and introduce the function

$$\psi(\zeta) = \int_{-\infty}^{\infty} e^{i\zeta x}\Psi(x)\,dx, \quad -\gamma < \eta < -\alpha.$$

By definition, $\psi(i\zeta)$ is the bilateral Laplace transform of $\Psi(x)$. For convenience in notation we shall call $\psi(\zeta)$ the *analytic transform* of $\Psi(x)$.

In the sequel only the case $\gamma > 0$, $\alpha \leq 0$, is considered. Thus it is supposed that

$$e^{\beta x}\Psi(x) \in L^2, \quad 0<\beta<\gamma. \tag{2.3}$$

The function $\psi(\zeta)$ then is analytic in the strip B, $-\gamma < \eta < 0$, bounded in every closed strip inside B, and $\psi(\xi - i\beta)$ is the Fourier transform of $e^{\beta x}\Psi(x)$ if $0<\beta<\gamma$. Parseval's relation gives

$$M_2\{\psi(\xi-i\beta)\} = \sqrt{2\pi}\, M_2\{e^{\beta x}\Psi(x)\} < \infty, \quad 0<\beta<\gamma,$$

and it follows that $|\psi(\xi-i\beta)| \to 0$ when $|\xi| \to \infty$, $0<\beta<\gamma$ (cf. [9], p. 125).

If $\Psi_1(x)$ and $\Psi_2(x)$ are two functions satisfying (2.3), and $\psi_1(\zeta)$ and $\psi_2(\zeta)$ are their analytic transforms, then the function $\Psi_1 * \Psi_2(x)$ still satisfies (2.3) and $\psi_1(\zeta)\psi_2(\zeta)$ is its analytic transform.

Conversely, let the function $\psi(\zeta)$ be analytic in $-\gamma<\eta<0$, and suppose that

$$M_2\{\psi(\xi-i\beta)\} < \infty, \quad 0<\beta<\gamma. \tag{2.4}$$

Let

$$\Psi(x) = \frac{1}{2\pi} \underset{T\to\infty}{\text{l.i.m.}} \int_{-T}^{T} e^{-ix(\xi-i\beta)}\psi(\xi-i\beta)\,d\xi, \quad 0<\beta<\gamma. \tag{2.5}$$

Then $\Psi(x)$ satisfies (2.3) and $\psi(\zeta)$ is the analytic transform of $\Psi(x)$. If, in addition, $M_2\{\psi(\xi-i\beta)\}$ is bounded for $0<\beta<\gamma$, then

$$\lim_{\beta\to 0} M_2\{\psi(\xi-i\beta)-\psi(\xi)\}=0,$$

where $\psi(\xi)$ is the Fourier transform of $\Psi(x)$. Furthermore, for almost all values of ξ it holds that

$$\lim_{\eta\to -0}\psi(\xi+i\eta)=\psi(\xi) \tag{2.6}$$

(cf. [9], Theorem 97, p. 130).

Finally, let $\psi(\zeta)$ be analytic in $-\gamma<\eta<0$, let (2.4) hold, and suppose further that

$$M\{\psi(\xi-i\beta)\} < \infty, \quad 0<\beta<\gamma. \tag{2.7}$$

The function $\Psi(x)$, defined by the integral (2.5) as an ordinary limit, then is continuous and

$$|\Psi(x)| \leq (2\pi)^{-1}e^{-\beta x} M\{\psi(\xi-i\beta)\} < C_\beta e^{-\beta x}, \quad 0<\beta<\gamma.$$

If, in addition, $M\{\psi(\xi-i\beta)\}$ and $M_2\{\psi(\xi-i\beta)\}$ are bounded for $0<\beta<\gamma$, then the function $\psi(\xi)$, defined by (2.6) for almost all values of ξ, belongs to L. Thus its Fourier transform, $\Psi(x)$, is bounded and

$$\Psi(x) = O(e^{-\gamma x}) \quad \text{as } x\to\infty.$$

In the sequel we shall often use the method of constructing the analytic transform of a required function instead of the function itself. Since we always consider a strip of the form $-\gamma < \eta < 0$, the function is uniquely determined whenever its analytic transform, satisfying (2.4) and (2.7) in such a strip, is given.

For further references we quote the simple example

$$g(\zeta) = g_n(\zeta) = (\zeta_1 - \zeta)^{-(n+1)}, \quad \zeta_1 = \alpha_1 - i\beta_1, \quad \beta_1 > 0,$$

where n denotes a non-negative integer. The function $g(\zeta)$ is, in the half-plane $\eta > -\beta_1$, analytic transform of the function $G(x) = G_n(x)$, defined by the relation

$$G(x) = 0, \quad x < 0; \qquad G(x) = \frac{(i)^{n+1}}{n!} x^n e^{-i\zeta_1 x}, \quad x \geq 0.$$

This is easily verified, since $g_0(\xi) = (\zeta_1 - \xi)^{-1}$ is the Fourier transform of $G_0(x) = 0$, $x < 0$; $G_0(x) = i e^{-i\zeta_1 x}$, $x \geq 0$.

6. The function $\Phi(x)$ of L^2

Theorem 2. *Let* $F(x) \in V$, *let* $f(\xi) = \int\limits_{-\infty}^{\infty} e^{i\xi x} dF(x)$, *and* $f(\xi) \neq 0$. *Let* τ *denote a positive number. A necessary and sufficient condition that there should exist a bounded, continuous function* $\Phi(x)$, $\Phi(x) \not\equiv 0$, *such that* $\Phi(x) \in L^2$, $e^{\tau x} \overset{*}{\Phi}(x) \in L^2$, *and* $\overset{*}{\Phi}(x) = O(e^{-\tau x})$, *is that*

$$\int\limits_{-\infty}^{\infty} e^{-\frac{\pi}{\tau}|\xi|} \left|\log|f(\xi)|\right| d\xi < \infty. \tag{2.8}$$

To prove that the condition (2.8) is necessary, suppose that there is a function $\Phi(x)$ which fulfils the conditions of the theorem. We may without loss of generality assume that $F(x)$ is absolutely continuous. For, let $K(x)$ be defined by the relation

$$K(x) = i e^{x}, \quad x < 0; \qquad K(x) = 0, \quad x \geq 0,$$

then $\overset{*}{K}(x)$ belongs to L and

$$\Phi \ast \overset{*}{K}(x) = \overset{*}{\Phi} \ast K(x) = O(e^{-\tau x}),$$

the inversion being justified by absolute convergence. The Fourier transform of $\overset{*}{K}(x)$ is $k(\xi) \equiv f(\xi)/(\xi - i)$. Thus, if the first part of the theorem holds for the function $\int\limits_{-\infty}^{x} \overset{*}{K}(u)\, du$, then it holds for $F(x)$ as well. Obviously, we may also suppose that $V\{F\} < 1$, which implies that $|f(\xi)| < 1$.

Let $\zeta=\xi+i\eta$, and introduce the analytic transform, $\psi(\zeta)$, of $\overset{*}{\Phi}(x)$. The assumption that $\Phi(x)$ is non-trivial yields that $\overset{*}{\Phi}(x)$ is non-trivial; hence the function $\psi(\zeta)$ cannot be equivalent to zero. Also, since $\Phi(x)$ is assumed to belong to L^2, it follows that the function $\overset{*}{\Phi}(x)$ belongs to L^2. Therefore, by the results in section 5, $\psi(\zeta)$ is analytic in the strip $-\tau<\eta<0$, $M_2\{\psi(\xi-i\beta)\}$ is bounded for $0<\beta<\tau$, and $\lim_{\eta\to-0}\psi(\xi+i\eta)=\psi(\xi)$ a.e., where $\psi(\xi)$ is the Fourier transform of $\overset{*}{\Phi}(x)$.

Let us map the strip B, $-\tau<\eta<0$, in the ζ-plane onto the unit circle in the w-plane, and suppose that $\psi(\zeta)$ becomes $\psi_0(w)$ by this transformation. It is easy to verify that $\psi_0(w)$ has a bounded characteristic function in the sense of Nevanlinna ([7], chapter VII). Inverting back to the ζ-plane we thus find

$$\int_{-\infty}^{\infty} e^{-\frac{\pi}{\tau}|\xi|}\left|\log^-|\psi(\xi)|\right|d\xi<\infty.$$

Since we have assumed $F(x)$ absolutely continuous we may write

$$\int_{-\infty}^{\infty}\Phi(x-u)\,F'(u)\,du=\overset{*}{\Phi}(x),$$

where $\Phi(x)$ belongs to L^2 and $F'(x)$ to L. Now, let $\varphi(\xi)$ be the Fourier transform of $\Phi(x)$. Then

$$\varphi(\xi)\,f(\xi)=\psi(\xi)\quad\text{a.e.}$$

(c.f. [9], Theorem 65, p. 90). It follows that

$$\int_{-\infty}^{\infty} e^{-\frac{\pi}{\tau}|\xi|}\left|\log|f(\xi)|\right|d\xi\;<\;\int_{-\infty}^{\infty} e^{-\frac{\pi}{\tau}|\xi|}\{\left|\log^-|\psi(\xi)|\right|+\log^+|\varphi(\xi)|\}\,d\xi<\infty,$$

and the necessity of condition (2.8) is established.

In order to prove the sufficiency part of the theorem we suppose that (2.8) holds, and we wish to construct a function $\Phi(x)$ with the desired properties.

Let us consider the integral equation

$$\int_{-\infty}^{\infty} V(x-u)\,dF(u)=H(x),\tag{2.9}$$

where $H(x)$ is a given continuous function of L^2. Let $h(\xi)$ denote the Fourier transform of $H(x)$. If

$$\frac{h(\xi)}{f(\xi)}\in L^2,\qquad \frac{h(\xi)}{f(\xi)}\in L,\tag{2.10}$$

then the function $V(x)$, defined as the Fourier transform of $h(\xi)/f(\xi)$, is a solution of (2.9) and $V(x)$ belongs to L^2. This is immediate, since $H(x)$ and

$\overset{*}{V}(x)$ are continuous and have the same Fourier transform, $h(\xi)$, which belongs to L. Hence the existence of a function $\Phi(x)$, satisfying the conditions of the theorem, is assured if we can construct a function $H(x)$ such that (2.10) is satisfied, and, in addition,

$$H(x) = O(e^{-\tau x}), \qquad e^{\tau x} H(x) \in L^2.$$

Consider instead the corresponding problem for the analytic transform $h(\zeta)$ of $H(x)$.

If $h(\zeta)$ is analytic in the strip $-\tau < \eta < 0$,

$M_2\{h(\xi - i\beta)\}$ and $M\{h(\xi - i\beta)\}$ are bounded for $0 < \beta < \tau$,

and (2.10) holds for the function $h(\xi)$ defined by

$$\lim_{\eta \to -0} h(\xi + i\eta) = h(\xi) \quad \text{a.e.},$$

then the Fourier transform $H(x)$ of $h(\xi)$ satisfies the required conditions. This is immediate by the results of section 5.

We shall construct a function $h(\zeta)$ with these properties. An analogous problem for a half-plane was treated by Paley-Wiener ([8], Theorem XII, p. 16) and we proceed similarly.

Let B denote the strip $-\tau < \eta < 0$. Then, by means of Green's formula, construct the function $\lambda(\zeta)$ which is harmonic in B and has the boundary values $\log|f(\xi)| - 3\log(1+|\xi|)$ on the real axis, and the boundary values $-2\log(1+|\xi|)$ on the straight line $\eta = -\tau$. The possibility of this construction follows from condition (2.8) and a well-known property of the Green function. Let $\mu(\zeta)$ be the conjugate harmonic function of $\lambda(\zeta)$ and write

$$h(\zeta) = e^{\lambda(\zeta) + i\mu(\zeta)}.$$

The function $h(\zeta)$ is then analytic in B, and $|h(\zeta)|$ has the limiting values $(1+|\xi|)^{-3}|f(\xi)|$ on the real axis and the limiting values $(1+|\xi|)^{-2}$ on the straight line $\eta = -\tau$. Hence the relation (2.10) is fulfilled. Since

$$\log M\{h(\xi - i\beta)\}$$

is a convex function of β, it follows that

$$M\{h(\xi - i\beta)\} \le [M\{h(\xi)\}]^{1-\frac{\beta}{\tau}} [M\{h(\xi - i\tau)\}]^{\frac{\beta}{\tau}} < 2V\{F\}, \quad 0 < \beta < \tau,$$

and, similarly,

$$M_2\{h(\xi - i\beta)\} < V\{F\}, \quad 0 < \beta < \tau.$$

This function $h(\zeta)$ therefore fulfils our conditions, and it results that the function $V(x)$, defined as the Fourier transform of $h(\xi)/f(\xi)$, satisfies the conditions of the theorem. Thus the proof of Theorem 2 is complete.

For further applications we have constructed $h(\zeta)$ in such a way that $\xi h(\xi)/f(\xi) \in L$, which implies that $V'(x)$ is bounded.

In order to examine the existence of bounded functions $\Phi(x)$ satisfying the Tauberian relation (2.1), we will first derive some results concerning the convolution of a bounded function and a function of bounded variation.

7. Some results for harmonic transforms of bounded functions

Let $\Phi(x)$ be bounded, say $|\Phi(x)| \le B$. We write, for $\eta < 0$,

$$U_\Phi(\xi, \eta) = \int_{-\infty}^{\infty} e^{i\xi x + \eta |x|}\, \Phi(x)\, dx,$$

which is harmonic in the half-plane $\eta < 0$. In accordance with a notation used by Beurling in [5] we shall call $U_\Phi(\xi, \eta)$ the harmonic transform of $\Phi(x)$.

Let β denote a positive number. Parseval's relation gives

$$M_2\{U_\Phi(\xi, -\beta)\} = \sqrt{2\pi}\, M_2\{e^{-\beta|x|}\Phi(x)\},$$

hence

$$M_2\{U_\Phi(\xi, -\beta)\} \le \sqrt{2\pi}\, B \beta^{-\frac{1}{2}}. \tag{2.11}$$

Let $F(x) \in V$, $f(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}\, dF(x)$, and introduce $\overset{*}{\Phi}(x)$ and its harmonic transform $U_{\Phi}^{*}(\xi, \eta)$. We will prove that if, for some δ, $0 < \delta \le \frac{1}{2}$,

$$P\{F\} = P_\delta\{F\} = \int_{-\infty}^{\infty} |x|^{\frac{1}{2}+\delta} |dF(x)| < \infty,$$

then

$$I(\beta) \equiv M_2\{U_{\Phi}^{*}(\xi, -\beta) - f(\xi)\, U_\Phi(\xi, -\beta)\} \le 2\sqrt{2\pi}\, P\{F\}\, B\beta^{\delta}, \tag{2.12}$$

and, for an arbitrary set E,

$$\Big\{\int_E |U_{\Phi}^{*}(\xi, -\beta)|^2\, d\xi\Big\}^{\frac{1}{2}} \le B\sqrt{2\pi}\,(2P\{F\}\beta^{\delta} + \beta^{-\frac{1}{2}}\mu\{E\}), \tag{2.13}$$

where

$$\mu\{E\} = \sup_{\xi \in E} |f(\xi)|.$$

The inequality (2.13) is a consequence of (2.11) and (2.12), for, by Minkowski's inequality,

$$\Big\{\int_E |U_{\Phi}^{*}(\xi, -\beta)|^2\, d\xi\Big\}^{\frac{1}{2}} \le I(\beta) + \Big\{\int_E |f(\xi)\, U_\Phi(\xi, -\beta)|^2\, d\xi\Big\}^{\frac{1}{2}}$$
$$\le I(\beta) + \mu\{E\}\, M_2\{U_\Phi(\xi, -\beta)\}.$$

Thus it is sufficient to prove (2.12).

The integral $I(\beta)$ was examined by Beurling in [5] in the case $\delta = 0$, and we proceed similarly. Parseval's relation gives

$$I^2(\beta) = 2\pi M_2^2\Big\{\int_{-\infty}^{\infty} (e^{-\beta|x|} - e^{-\beta|x-u|})\, \Phi(x-u)\, dF(u)\Big\}. \tag{2.14}$$

By Schwarz' inequality,

$$\left| \int_{-\infty}^{\infty} (e^{-\beta|x|} - e^{-\beta|x-u|})\, \Phi(x-u)\, dF(u) \right|^2$$

$$\le P\{F\} B^2 \int_{-\infty}^{\infty} |e^{-\beta|x|} - e^{-\beta|x-u|}|^2 |u|^{-(\frac{1}{2}+\delta)} |dF(u)|.$$

Inserting the above expression in (2.14) and inverting the order of integration, we find

$$I^2(\beta) \le 2\pi P\{F\} B^2 \int_{-\infty}^{\infty} |u|^{\frac{1}{2}+\delta} |dF(u)| \int_{-\infty}^{\infty} |e^{-\beta|x|} - e^{-\beta|x-u|}|^2 |u|^{-(1+2\delta)} dx.$$

The inner integral is dominated by (cf. [5], p. 275)

$$\frac{4\beta|u|^{1-2\delta}}{1+\beta^2u^2} \le 4\beta^{2\delta}.$$

Hence,

$$I^2(\beta) \le 8\pi P^2\{F\} B^2 \beta^{2\delta},$$

which proves (2.12).

8. The function $\Phi(x)$ bounded

The following theorem shows that if we assume that there exists a non-trivial bounded function $\Phi(x)$ satisfying our Tauberian relation, then we have restricted the class of functions $F(x)$ under consideration. The corresponding converse theorem is included in Theorem 2.

Theorem 3. *Let* $F(x) \in V$ *and* $f(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} dF(x)$, $f(\xi) \neq 0$, *and let, for some* $\delta > 0$

$$\int_{-\infty}^{\infty} |x|^{\frac{1}{2}+\delta} |dF(x)| < \infty. \tag{2.15}$$

If there exists a bounded, continuous function $\Phi(x)$, $\Phi(x) \not\equiv 0$, *such that, for some positive number* τ,

$$\overset{*}{\Phi}(x) = O(e^{-\tau x}), \tag{2.16}$$

then

$$\int_{-\infty}^{\infty} e^{-c|\xi|} |\log|f(\xi)|| d\xi < \infty, \quad c > \frac{\pi}{\tau}. \tag{2.17}$$

The proof is based on the analyticity of the analytic transform, $\psi(\zeta)$, of $\overset{*}{\Phi}(x)$ and follows the same idea as the proof of the first part of Theorem 2. This proof is, however, more complicated, due to the fact that, cancelling the hypothesis $\Phi(x) \in L^2$, we cannot use in our argument the strip B, $-\tau < \eta < 0$. Instead of B we shall introduce another domain, D, inside B, which will allow us to make use of the above results for harmonic transforms along a part of the boundary of D. The proof is sketched below, and afterwards worked out in detail.

Consider the curve $\eta = |f(\xi)|^{\varkappa}$, where $\varkappa = \varkappa(\delta)$ is a certain positive constant. This curve is approximated by a step-function, $s(\xi)$, such that $s(\xi)$ is less than $\frac{1}{2}\tau$ and $\log s(\xi) - \varkappa \log |f(\xi)|$ is bounded. Then, by applying (2.13) to the harmonic transform of $\overset{*}{\Phi}(x)$, we can prove that $\psi(\xi - i s(\xi))$ belongs to L^2.

Let S denote the curve obtained in joining the graph of $\eta = -s(\xi)$ by straight lines parallel to the η-axis, and define D as the domain lying between S and a straight line $\eta = -b$ in the ζ-plane. Let the function $w = w(\zeta)$ represent D conformally on the unit circle. Then, if $s(\xi)$ is appropriately constructed, the relation

$$\int_S |\log^- |\psi(\zeta)|| \, |w'(\zeta)| \, |d\zeta| < \infty \tag{2.18}$$

may be derived from the corresponding relation for $\log^+ |\psi(\zeta)|$.

We further introduce a set, H, consisting of small intervals around the points of discontinuity of $s(\xi)$, and denote the complement of H by CH. The set H is constructed such that

$$\int_H e^{-\frac{\pi}{\tau}|\xi|} |\log s(\xi)| \, d\xi < \infty,$$

yet such that

$$|w'(\xi - i s(\xi))| > C_\varepsilon e^{-\left(\frac{\pi}{b}+\varepsilon\right)|\xi|}, \quad \xi \in CH, \tag{2.19}$$

for every $\varepsilon > 0$. The possibility of this construction will follow from condition (2.15).

From the relations (2.18) and (2.19) it can be derived, by the aid of (2.13), that

$$\int_{CH} e^{-c|\xi|} |\log s(\xi)| \, d\xi < \infty, \quad c > \frac{\pi}{b},$$

and the result of the theorem then follows, since b can be chosen arbitrarily close to τ.

Now, proceeding to the detailed proof, let us first normalize our functions in a way convenient for our purpose. Suppose that

$$|\Phi(x)| \le 1, \qquad |\overset{*}{\Phi}(x)| \le e^{-x}, \tag{2.20}$$

$$\int_{-\infty}^{\infty} (1 + |x|)^{\frac{1}{2}+\delta} |dF(x)| \le 2^{-5}, \text{ for some } \delta, \quad 0 < \delta < \tfrac{1}{2}, \tag{2.21}$$

and

$$|f(\xi)| \le 2^{-5}(1 + |\xi|)^{-1}. \tag{2.22}$$

This is no real restriction. For, by the argument used in Theorem 2 (p. 324) we may suppose that $f(\xi)=O\{(1+|\xi|)^{-1}\}$. Also, by changing the variable we obtain from (2.16)

$$\int_{-\infty}^{\infty} \Phi\left(\frac{x-u}{\tau}\right) d F\left(\frac{u}{\tau}\right) = \overset{*}{\Phi}\left(\frac{x}{\tau}\right) = O(e^{-x}).$$

Since $f(\tau\xi)$ is the Fourier Stieltjes transform of $F(x/\tau)$ it is sufficient to consider the case $\tau=1$. And obviously we may always multiply $\Phi(x)$ and $F(x)$ by constants.

In Lemma 3 we derive the inequalities for $\psi(\zeta)$ used in the proof. Then, in Lemma 4, we construct a step-function $s(\xi)$ and a set H with the desired properties. The main proof is given on page 338.

Lemma 3. *Impose the conditions* (2.20) *and* (2.21) *on* $F(x)$ *and* $\Phi(x)$. *Let* $f(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} dF(x)$ *and let* $\psi(\zeta)$ *be the analytic transform of* $\overset{*}{\Phi}(x)$. *If* $t(\xi)$ *is a step-function, taking only the values* e^{-n}, $n=1, 2, \ldots$, *such that for some* $\varrho>0$

$$\varkappa \log|f(\xi)| - \log t(\xi) \le \varrho, \quad \varkappa = \frac{1+\delta}{\frac{1}{2}+\delta}, \tag{2.23}$$

then, for a constant C *depending only on* δ, *we have*

$$M_2\{\psi(\xi - i t(\xi))\} < C e^{\varrho}, \tag{2.24}$$

and, if $\chi(\xi)\in L$, $0\le\chi(\xi)\le 1$,

$$\frac{\delta}{4(1+\delta)} \int_{-\infty}^{\infty} \chi(\xi)|\log t(\xi)| d\xi < \int_{-\infty}^{\infty} \chi(\xi)\,|\log^-|\psi(\xi - i t(\xi))||\, d\xi + (C+\varrho) M\{\chi(\xi)\} + A, \tag{2.25}$$

whenever the right side integral converges.

The proof is an application of the results in the previous section concerning harmonic transforms. C is used to denote a positive constant depending only on δ, but not necessarily the same one each time it occurs. This notation will be used throughout the present section.

Let $U_{\Phi}^{*}(\xi,\eta)$ be the harmonic transform of $\overset{*}{\Phi}(x)$. Parseval's relation gives, for $0<\beta<1$,

$$M_2^2\{\psi(\xi - i\beta) - U_{\Phi}^{*}(\xi, -\beta)\} = 2\pi\int_0^{\infty} (e^{\beta x} - e^{-\beta x})^2 |\overset{*}{\Phi}(x)|^2 dx.$$

Since $|\overset{*}{\Phi}(x)|\le e^{-x}$ we find that

$$M_2\{\psi(\xi - i e^{-n}) - U_{\Phi}^{*}(\xi, -e^{-n})\} < 3\sqrt{2\pi} e^{-n}, \quad n=1, 2, \ldots.$$

Let E_n be the set defined by

$$t(\xi) = e^{-n}, \qquad \xi \in E_n,$$

and let

$$\delta_1 = \frac{1}{\varkappa} - \frac{1}{2} = \frac{\delta}{2(1+\delta)}.$$

Then, by (2.23),

$$|f(\xi)| \le e^{\frac{\varrho}{\varkappa} - \frac{n}{\varkappa}} < e^{\varrho - \frac{n}{\varkappa}}, \qquad \xi \in E_n,$$

and an application of (2.13) gives

$$\Big\{ \int_{E_n} |U_\Phi^*(\xi, -e^{-n})|^2 d\xi \Big\}^{\frac{1}{2}} < \sqrt{2\pi}\,(e^{-\delta n} + e^{\varrho - \delta_1 n}).$$

Using Minkowski's inequality we find that

$$\Big\{ \int_{E_n} |\psi(\xi - i e^{-n})|^2 d\xi \Big\}^{\frac{1}{2}} < \sqrt{2\pi}\,(4 e^{-\delta n} + e^{\varrho - \delta_1 n}) < A e^{\varrho - \delta_1 n}. \tag{2.26}$$

And it follows that

$$M_2^2\{\psi(\xi - i t(\xi))\} = \sum_{n=1}^{\infty} \int_{E_n} |\psi(\xi - i e^{-n})|^2 d\xi < A e^{2\varrho} \sum_{n=1}^{\infty} e^{-2\delta_1 n} < C e^{2\varrho},$$

which proves (2.24).

We now turn to the proof of (2.25). Schwarz' inequality and (2.26) yield

$$\int_{E_n} \chi(\xi) |\psi(\xi - i e^{-n})| d\xi \le \Big\{ \int_{E_n} \chi^2(\xi) d\xi \Big\}^{\frac{1}{2}} A e^{\varrho - \delta_1 n} \tag{2.27}$$

Consider the integers n such that $\int_{E_n} \chi(\xi) d\xi \neq 0$ and let

$$K(n) = \frac{\Big\{ \int_{E_n} \chi^2(\xi) d\xi \Big\}^{\frac{1}{2}}}{\int_{E_n} \chi(\xi) d\xi},$$

then, since $\chi(\xi) \le 1$,

$$\int_{E_n} \chi(\xi) d\xi \le \{K(n)\}^{-2}. \tag{2.28}$$

The inequality (2.27) can be written

$$\frac{1}{\int_{E_n} \chi(\xi) d\xi} \int_{E_n} \chi(\xi) |\psi(\xi - i e^{-n})| d\xi \le K(n) A e^{\varrho - \delta_1 n}.$$

Hence, by the convexity property of the logarithm,

$$\int_{E_n} \chi(\xi) \log |\psi(\xi - i e^{-n})| \, d\xi \leq (A + \varrho + \log K(n) - \delta_1 n) \int_{E_n} \chi(\xi) \, d\xi.$$

Let $\{n\} = M_1 + M_2$, where $n \in M_1$ if $K(n) \leq n^2$, and $n \in M_2$ if $K(n) > n^2$.

If $n \in M_1$ we have

$$\log K(n) - \delta_1 n \leq 2 \log n - \delta_1 n < C - \tfrac{1}{2} \delta_1 n$$

and

$$\sum_{n \in M_1} \int_{E_n} \chi(\xi) \log |\psi(\xi - i e^{-n})| \, d\xi \leq (C + \varrho) M\{\chi(\xi)\} - \tfrac{1}{2} \delta_1 \sum_{n \in M_1} n \int_{E_n} \chi(\xi) \, d\xi.$$

Therefore

$$\int_{-\infty}^{\infty} \chi(\xi) \log^- |\psi(\xi - i t(\xi))| \, d\xi = \sum_{n=1}^{\infty} \int_{E_n} \chi(\xi) \log^- |\psi(\xi - i e^{-n})| \, d\xi$$

$$\leq (C + \varrho) M\{\chi(\xi)\} - \tfrac{1}{2} \delta_1 \sum_{n \in M_1} n \int_{E_n} \chi(\xi) \, d\xi, \qquad (2.29)$$

and the right side sum must converge, as the left side is finite by assumption.
If $n \in M_2$, then $K(n) > n^2$ and, according to (2.28),

$$\int_{E_n} \chi(\xi) \, d\xi < n^{-4}.$$

Since $\log t(\xi) = -n$ in E_n we find

$$\sum_{n \in M_2} \int_{E_n} \chi(\xi) \log t(\xi) \, d\xi = - \sum_{n \in M_2} n \int_{E_n} \chi(\xi) \, d\xi > - \sum_{n \in M_2} n^{-3} > -A,$$

and

$$\int_{-\infty}^{\infty} \chi(\xi) \log t(\xi) \, d\xi = - \sum_{n=1}^{\infty} n \int_{E_n} \chi(\xi) \, d\xi > -A - \sum_{n \in M_1} n \int_{E_n} \chi(\xi) \, d\xi.$$

Taking into account (2.29) we have proved

$$\int_{-\infty}^{\infty} \chi(\xi) \log^- |\psi(\xi - i t(\xi))| \, d\xi < \tfrac{1}{2} \delta_1 \int_{-\infty}^{\infty} \chi(\xi) \log t(\xi) \, d\xi + (C + \varrho) M\{\chi(\xi)\} + \tfrac{1}{2} \delta_1 A,$$

which is equivalent to (2.25). This completes the proof of Lemma 3.

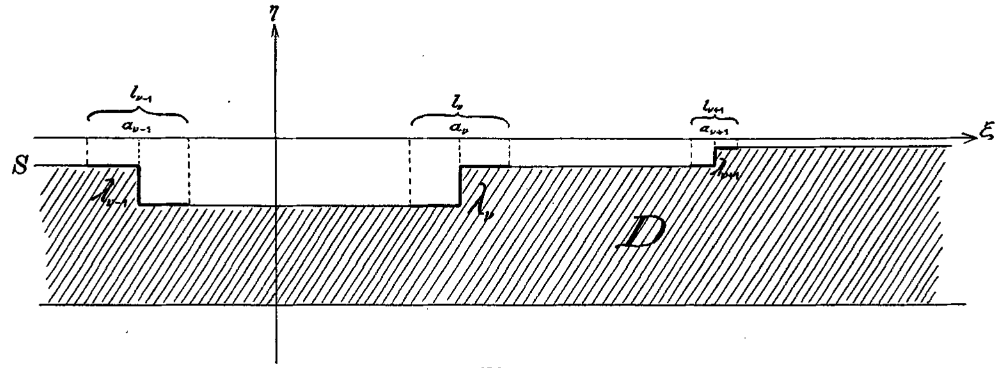

Fig. 1.

We are now going to construct a step-function $\eta = s(\xi)$ of the above type such that the domain D is appropriately represented on the unit circle.

Choose a number b, such that

$$1 - 2^{-4} < b < 1.$$

Let $s(\xi)$ be a step-function such that $0 < s(\xi) \leq e^{-1}$, and introduce the following quantities depending on $s(\xi)$:

S denotes the curve obtained in joining the graph of $\eta = -s(\xi)$ by straight lines parallel to the η-axis at each point of discontinuity of $s(\xi)$.

a_ν ($\nu = \ldots -1, 0, 1, \ldots, a_\nu < a_{\nu+1}$) are the points of discontinuity of $s(\xi)$. l_ν denotes an interval around a_ν of length $|l_\nu|$, and λ_ν is the arc of S whose projection on the real axis is l_ν.

D denotes the domain lying between S and the straight line $\eta = -b$ in the ζ-plane; $\zeta = \xi + i\eta$. The function $w = w(\zeta)$ maps D conformally on the unit circle in the w-plane, $w(\zeta_0) = 0$, $w'(\zeta_0) > 0$; $\zeta_0 = -\frac{1}{2} i$.

$\omega(\zeta, \lambda_\nu, D)$ denotes the harmonic measure of λ_ν with respect to D.

$H = \sum_{\nu=-\infty}^{\infty} l_\nu$, and CH denotes the complement of the set H.

With these notations Lemma 4 can be stated as follows:

Lemma 4. *Impose conditions* (2.21) *and* (2.22) *on* $F(x)$ *and its Fourier Stieltjes transform* $f(\xi)$ *and let* $f(\xi) \neq 0$. *Then we can construct a step-function* $s(\xi)$ *taking only the values* e^{-n}, $n = 1, 2, \ldots$, *and a set* $H = \sum_{\nu=-\infty}^{\infty} l_\nu$ *such that*

$$|\varkappa \log |f(\xi)| - \log s(\xi)| \leq 2, \quad \varkappa = \frac{1+\delta}{\frac{1}{2}+\delta}, \tag{2.30}$$

$$\int_H |\log s(\xi)| \frac{d\xi}{1+\xi^2} < C, \tag{2.31}$$

$$\omega(\zeta_0, \lambda_\nu, D) < C e^{-\frac{\pi}{b}|a_\nu|} |l_\nu|, \tag{2.32}$$

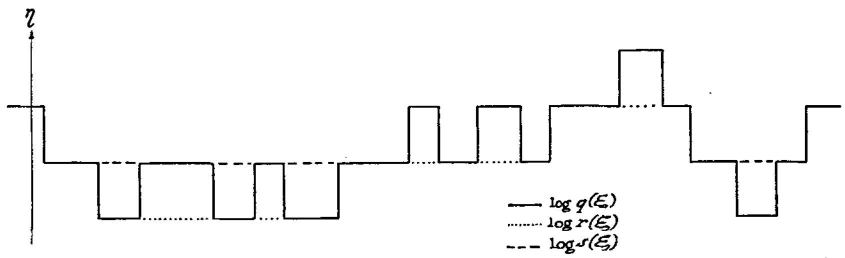

Fig. 2.

and, for every $\varepsilon > 0$,

$$C_\varepsilon e^{-\frac{\pi}{b}(1+\varepsilon)|\xi|} < |w'(\xi - i s(\xi))| < C e^{-\frac{\pi}{b}|\xi|}, \quad \xi \in CH, \tag{2.33}$$

where C *depending only on* δ *and* C_ε *on* δ *and* ε *are positive constants.*

Let us first state the properties of $f(\xi)$ to be used in the proof, viz.

$$0 < |f(\xi)| \leq 2^{-5}(1+|\xi|)^{-1},$$

which is immediate from (2.22), and

$$\big||f(\xi_1)| - |f(\xi_2)|\big| \leq 2^{-4}|\xi_1 - \xi_2|^{\frac{1}{2}+\delta}, \tag{2.34}$$

which is easily verified by use of (2.21). Note further that the assumption $0 < \delta < \frac{1}{2}$ yields $3/2 < \varkappa < 2$.

Now proceeding to the construction of $s(\xi)$, let P_n be the set defined by

$$e^{-n} \leq |f(\xi)|^{\varkappa} < e^{-(n-1)}, \quad \xi \in P_n$$

and introduce the step-function $q(\xi)$, defined by the relation

$$q(\xi) = e^{-n}, \quad \xi \in P_n.$$

To obtain $s(\xi)$ we shall remove, successively, the steps around the intervals in which $q(\xi)$ has a maximum or a minimum, and we proceed as follows: Let i_m, $m = \cdots -1, 0, 1, \ldots$, denote the intervals in which $q(\xi)$ is constant; i_{m+1} lying to the right of i_m. Let Q denote the set consisting of all intervals i_m such that $q(\xi)$ in i_m assumes a value greater than its value in the two adjacent intervals, i_{m-1} and i_{m+1}. Introduce the step-function $r(\xi)$, defined by

$$r(\xi) = \begin{cases} e^{-1} q(\xi) & \text{if } \xi \in Q, \\ q(\xi) & \text{elsewhere.} \end{cases}$$

(The construction is shown in Fig. 2 in logarithmic scale.) Similarly, let j_m, $m = \cdots -1, 0, 1, \ldots$, denote the intervals in which $r(\xi)$ is constant; j_{m+1} lying to the right of j_m. Let R denote the set consisting of all intervals j_m such that $r(\xi)$ in j_m assumes a value smaller than its value in the two adjacent intervals, j_{m-1} and j_{m+1}.

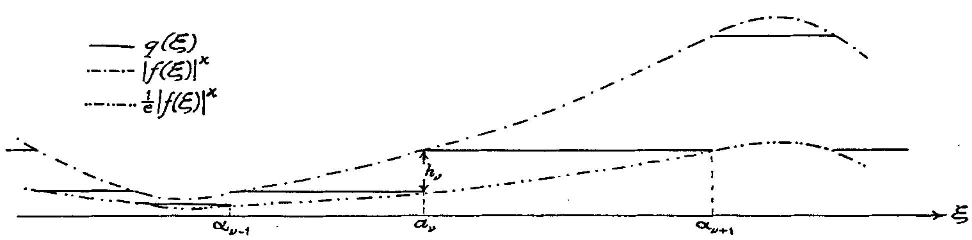

Fig. 3.

We define the step-function $s(\xi)$ as follows

$$s(\xi)=\begin{cases} e\,r(\xi) & \text{if } \xi\in R,\\ r(\xi) & \text{elsewhere.}\end{cases}$$

It is easy to verify that the function $s(\xi)$, thus defined, satisfies condition (2.30) and that $s(\xi)<e^{-2}$.

Now consider the points of discontinuity, a_ν ($a_\nu<a_{\nu+1}$, $\nu=\ldots-1, 0, 1, \ldots$) of $s(\xi)$. Let h_ν denote the step of $s(\xi)$ at a_ν, and put

$$\delta_2=\frac{\delta}{4}, \qquad |l_\nu|=2\,|h_\nu|^{\frac{1}{1+\delta_2}}.$$

We define l_ν as the closed interval of length $|l_\nu|$ lying symmetrically around a_ν.

To prove that the function $s(\xi)$ and the set $H=\sum_{\nu=-\infty}^{\infty} l_\nu$ satisfy the conditions of Lemma 4, we shall first prove the following relations:

$$|l_\nu|<2^{-4}, \tag{2.35}$$

$$s(\xi)>e^{-2}|l_\nu|^{1+\delta_2}, \qquad \xi\in l_\nu, \tag{2.36}$$

$$|l_\nu|^{1-\delta_3}<\tfrac{1}{2}|a_{\nu+\mu}-a_\nu|, \qquad \mu=-1, 1, \tag{2.37}$$

where

$$\delta_3=1-\frac{1+\delta_2}{1+\delta}=\frac{3\,\delta}{4(1+\delta)}.$$

Choose a point a_ν such that a_ν is a point of increase of $s(\xi)$, in which case a_ν is also a point of increase of the functions $q(\xi)$ and $r(\xi)$. Let $\alpha_{\nu+1}$ be the point of discontinuity of $q(\xi)$ immediately following a_ν, and let $\alpha_{\nu-1}$ be the point of discontinuity of $r(\xi)$ immediately preceding a_ν. From the construction (Fig. 3) it may be seen that

$$|h_\nu|=(1-e^{-1})\,|f(a_\nu)|^\varkappa$$

and

$$e^2\,|f(\alpha_{\nu-1})|^\varkappa=e\,|f(a_\nu)|^\varkappa=|f(\alpha_{\nu+1})|^\varkappa.$$

The relation (2.35) follows immediately from these equalities. Furthermore we find

$$|h_\nu| \le K \| f(a_\nu)| - |f(\alpha_{\nu+\mu})\|^{\varkappa}, \quad \mu = -1, 1,$$

where

$$K = (e-1)(e^{\frac{1}{\varkappa}} - 1)^{-\varkappa}.$$

According to (2.34) we have

$$\| f(a_\nu)| - |f(\alpha_{\nu+\mu})\| \le 2^{-4} |a_\nu - \alpha_{\nu+\mu}|^{\frac{1}{2}+\delta}.$$

Since $\varkappa(\frac{1}{2}+\delta) = 1+\delta$, we get

$$|h_\nu| \le K\, 2^{-4\varkappa} |a_\nu - \alpha_{\nu+\mu}|^{1+\delta}, \quad \mu = -1, 1,$$

where, by the construction of $s(\xi)$,

$$|a_\nu - \alpha_{\nu+\mu}| \le |a_\nu - a_{\nu+\mu}|, \quad \mu = -1, 1.$$

As a consequence of the last two inequalities we obtain (2.37). It is easy to see how (2.35) and (2.37) may be derived by a similar argument if a_ν is a point of decrease of $s(\xi)$. From (2.35) and (2.37) it follows that the intervals l_ν are non-overlapping, and through this fact the relation (2.36) is readily verified.

Proceeding to the proof of (2.31), let ω denote an interval of length $|\omega|$. We wish to establish the following relation

$$\int_{\omega \cap H} |\log s(\xi)|\, d\xi < C(1 + |\omega|), \tag{2.38}$$

from which (2.31) follows after a partial integration. To prove (2.38) consider an interval ω. If $\omega \cap H = 0$ there is nothing to prove. If not, let $\omega \cap l_\nu \neq 0$ for $\nu_1 \le \nu \le \nu_2$. Then

$$\int_{\omega \cap H} |\log s(\xi)|\, d\xi \le \sum_{\nu=\nu_1}^{\nu_2} \int_{l_\nu} |\log s(\xi)|\, d\xi.$$

According to (2.36) we have

$$\int_{l_\nu} |\log s(\xi)|\, d\xi < 2|l_\nu| - (1+\delta_2)|l_\nu| \log |l_\nu|,$$

and hence, by aid of (2.37),

$$\int_{l_\nu} |\log s(\xi)|\, d\xi < C |l_\nu|^{1-\delta_3} < C(a_{\nu+1} - a_\nu). \tag{2.39}$$

Therefore,

$$\sum_{\nu=\nu_1}^{\nu_2-1} \int_{l_\nu} |\log s(\xi)|\, d\xi < C(a_{\nu_2} - a_{\nu_1}) < C(|\omega| + \tfrac{1}{2}|l_{\nu_1}| + \tfrac{1}{2}|l_{\nu_2}|).$$

The relation (2.39) yields

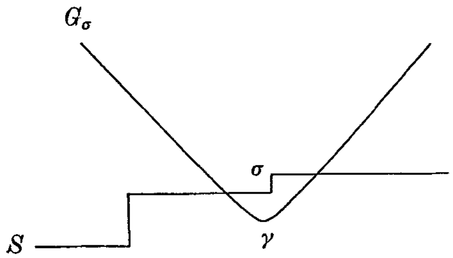

Fig. 4.

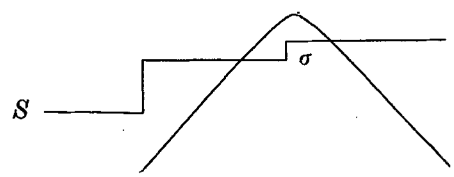

Fig. 5.

$$\int_{l_{\nu_2}} |\log s(\xi)|\, d\xi < C\, |l_{\nu_2}|^{1-\delta_3}$$

and, in view of the fact that $|l_\nu|$ is bounded, the last two inequalities prove (2.38). Thus (2.31) is established.

We now turn our attention to the domain D lying between the straight line $\eta = -b$ and the curve S, $\eta = -s(\xi)$ in the ζ-plane. Let (ξ, ξ') be an interval of the real axis and let σ be the arc of S whose projection on the real axis is (ξ, ξ'). If (ξ, ξ') is contained in CH, then σ is a segment of a straight line paralleling the ξ-axis. With the notations introduced on page 333 we can write

$$|w'(\xi - i s(\xi))| = 2\pi \lim_{\xi' \to \xi} \frac{\omega(\zeta_0, \sigma, D)}{|\xi - \xi'|}, \quad \xi \in CH.$$

It follows that propositions (2.32) and (2.33) of Lemma 4 are verified if we can prove that

$$C_\varepsilon e^{-\frac{\pi}{b}(1+\varepsilon)|\xi|} |\xi - \xi'| < \omega(\zeta_0, \sigma, D) < C e^{-\frac{\pi}{b}|\xi|} |\xi - \xi'|, \quad \varepsilon > 0, \tag{2.40}$$

if

$$(\xi, \xi') = l_\nu \ \text{ for some } \nu, \ \text{ or } \ (\xi, \xi') \subset CH. \tag{2.41}$$

Let $\bar{D}$ denote the image of D in $\eta = -b$ and let $\Omega = \bar{D} + D$. By considering the representation of Ω on a strip and making use of the inequalities of Ahlfors (see [1]) it is easy to verify the following relation

$$K_\varepsilon e^{-\frac{\pi}{b}(1+\varepsilon)|\xi|} < \frac{\omega(\zeta_0, \sigma, D)}{\omega(\xi - ib, \sigma, \Omega)} < A e^{-\frac{\pi}{b}|\xi|}, \quad \varepsilon > 0,$$

where, in view of (2.22), K_ε is a positive constant depending only on ε. Therefore, to establish (2.40) it suffices to show that

$$C|\xi - \xi'| < \omega(\xi - ib, \sigma, \Omega) < C|\xi - \xi'|, \tag{2.42}$$

for each interval (ξ, ξ') of the type described in (2.41).

To prove the right inequality in (2.42) we use the comparison domain U,

$$\eta < |\xi|^{1+\delta_2},$$

where δ_2 denotes the quantity introduced in the definition of l_ν (p. 335) in which case $\delta_2 > 0$. Choose an interval (ξ, ξ') as described in (2.41), let $\zeta = \xi - i\,s(\xi)$ and $\zeta' = \xi' - i\,s(\xi')$. Translate U into a new domain, U_σ, such that the boundary, G_σ, of U_σ passes through the two points ζ and ζ'. From the construction of S and U it follows that the curves S and G_σ intersect only at ζ and ζ'. Let γ denote the arc of G_σ between ζ and ζ' (Fig. 4). Then $\gamma \subset \Omega$, $S - \sigma \subset U_\sigma$, and, by a well-known argument based on the maximum principle,

$$\omega(\xi - i\,b, \sigma, \Omega) < \omega(\xi - i\,b, \gamma, U_\sigma).$$

Now, as is easily verified (cf. [2]),

$$\omega(\xi - i\,b, \gamma, U_\sigma) < C\,|\xi - \xi'|,$$

where C is independent of σ. Hence

$$\omega(\xi - i\,b, \sigma, \Omega) < C\,|\xi - \xi'|.$$

The converse inequality can be derived similarly by considering a comparison domain such as

$$\eta < -|\xi|^{1+\delta_2}, \qquad \xi^2 + \eta^2 < (\tfrac{3}{2})^2$$

(cf. Fig. 5). Thus (2.42) is proved; (2.32) and (2.33) follow. This completes the proof of Lemma 4.

Proof of Theorem 3.

Let $s(\xi)$ denote the step-function and H the set constructed in Lemma 4. Notations used below are defined on page 333.

For each sufficiently large integer k we construct a domain, D_k, as follows: Let the real numbers ξ_k' and ξ_k'' be defined by

$$s(\xi_k') = s(\xi_k'') = e^{-k},$$

$$s(\xi) < e^{-k} \text{ outside the interval } \xi_k' \le \xi \le \xi_k''.$$

Obviously ξ_k' and ξ_k'' exist if $k > k_0$, and $\xi_k' \to -\infty$, $\xi_k'' \to \infty$ as $k \to \infty$. In the sequel we tacitly assume k to be an integer $> k_0$. Let $s_k(\xi)$ be the step-function defined by

$$s_k(\xi) = \begin{cases} s(\xi), & \xi_k' \le \xi \le \xi_k'', \\ e^{-k} & \text{elsewhere,} \end{cases}$$

and let S_k denote the curve obtained in joining the graph of $s_k(\xi)$ by straight lines paralleling the η-axis. D_k is defined as the domain lying between S_k and the straight line $\eta=-b$ in the ζ-plane, and its boundary is denoted by Γ_k. Then $D_k \subset D_{k+1} \subset \cdots \subset D$, and $D_k \to D$ as $k \to \infty$.

Let us map the domain D_k conformally on the unit circle in the w-plane by means of the function $w=w_k(\zeta)$; $w_k(\zeta_0)=0$, $w_k'(\zeta_0)>0$. We wish to prove the following relation

$$\int_{\Gamma_k} \log^+ |\psi(\zeta)|\,|w_k'(\zeta)|\,|d\zeta| = \int_{\eta=-b} + \int_{S_k} < B, \tag{2.43}$$

where B is a constant, only dependent on δ and b. From our normalized conditions (p. 329) it follows that

$$|\psi(\xi - i\beta)| \leq \frac{1}{\beta(1-\beta)}, \qquad 0<\beta<1,$$

and hence, in view of our choice of b,

$$\int_{\eta=-b} \log^+ |\psi(\zeta)|\,|w_k'(\zeta)|\,|d\zeta| < 2\pi \log \frac{2}{1-b}. \tag{2.44}$$

Let H_k be the subset of H consisting of those intervals l_ν which contain some point of discontinuity of $s_k(\xi)$. Let CH_k denote the complement of H_k. If we put

$$m_\nu = \max_{\xi \in l_\nu} \log^+ |\psi(\xi - i s(\xi))|,$$

then we can write

$$\int_{S_k} \log^+ |\psi(\zeta)|\,|w_k'(\zeta)|\,|d\zeta| \leq 2\pi \sum_{l_\nu \subset H_k} \omega(\zeta_0, \lambda_\nu, D_k)\, m_\nu$$
$$+ \int_{CH_k} \log^+ |\psi(\xi - i s_k(\xi))|\,|w_k'(\xi - i s_k(\xi))|\,d\xi = I_1 + I_2.$$

Remembering that the intervals l_ν are non-overlapping we find

$$m_\nu < \log \frac{2e}{s(\xi)} < 2\,|\log s(\xi)|, \qquad \xi \in l_\nu.$$

And since $D_k \subset D$, we obtain by the aid of (2.32)

$$\omega(\zeta_0, \lambda_\nu, D_k) < \omega(\zeta_0, \lambda_\nu, D) < C\,|l_\nu|\,e^{-\frac{\pi}{b}|a_\nu|}, \qquad l_\nu \subset H_k.$$

Observing that $|l_\nu|$ is bounded, we find

$$I_1 < C \sum_{l_\nu \subset H_k} e^{-\frac{\pi}{b}|a_\nu|} \int_{l_\nu} |\log s(\xi)|\,d\xi < C \int_{H_k} |\log s(\xi)| \frac{d\xi}{1+\xi^2}$$

and, by (2.31)

$$I_1 < C \int_H |\log s(\xi)| \frac{d\xi}{1+\xi^2} < C. \tag{2.45}$$

To evaluate the integral I_2, let us notice that the relation

$$|w_k'(\xi - i s_k(\xi))| < C e^{-\frac{\pi}{b}|\xi|}, \quad \xi \in CH_k$$

is a consequence of the construction of D_k and the inequality (2.33). Therefore,

$$I_2 < C \int_{CH_k} \log^+ |\psi(\xi - i s_k(\xi))| \, d\xi < C \int_{CH_k} |\psi(\xi - i s_k(\xi))|^2 \, d\xi.$$

Since $s(\xi)$ is constructed such that $\varkappa \log |f(\xi)| - \log s(\xi) \le 2$, it follows from our choice of ξ_k' and ξ_k'' that $\varkappa \log |f(\xi)| - \log s_k(\xi) \le 2$. An application of (2.24) yields

$$I_2 < C \int_{-\infty}^{\infty} |\psi(\xi - i s_k(\xi))|^2 \, d\xi < C. \tag{2.46}$$

By adding (2.44), (2.45) and (2.46) the relation (2.43) is proved.

Suppose that $\psi(\zeta)$ is transformed to $\psi_k(w)$ by use of $w = w_k(\zeta)$. From the construction of D_k it follows that $\psi_k(w)$ is analytic and bounded in the unit circle for every k. Furthermore, $\psi_k(w)$ cannot be equivalent to zero since $\Phi(x)$ is assumed non-trivial. The inequality (2.43) therefore asserts that the characteristic function of $\psi_k(w)$, $k = k_0, k_0+1, \ldots$, does not exceed B. We may conclude that

$$\int_{\Gamma_k} |\log^- |\psi(\zeta)|| \, |w_k'(\zeta)| \, |d\zeta| < B_1,$$

for some constant B_1 which is independent of k. Accordingly,

$$\int_{\xi_k'}^{\xi_k''} |\log^- |\psi(\xi - i s(\xi))|| \, |w_k'(\xi - i s(\xi))| \, d\xi < B_1.$$

The integrand increases as k increases. Letting $k \to \infty$ we find

$$\int_{-\infty}^{\infty} |\log^- |\psi(\xi - i s(\xi))|| \, |w'(\xi - i s(\xi))| \, d\xi < B_1.$$

Hence, by the aid of (2.33)

$$\int_{CH} e^{-c|\xi|} |\log^- |\psi(\xi - i s(\xi))|| \, d\xi < \infty, \quad c > \frac{\pi}{b}.$$

Choosing $\chi(\xi)=e^{-c|\xi|}$, $\xi \in CH$; $\chi(\xi)=0$, $\xi \in H$ in formula (2.25) and applying (2.25) to the function $s(\xi)$ we find

$$\int_{CH} e^{-c|\xi|}\,|\log s(\xi)|\,d\xi < \infty, \quad c > \frac{\pi}{b}.$$

We already know that

$$\int_{H} |\log s(\xi)|\frac{d\xi}{1+\xi^2} < \infty.$$

The last two inequalities imply

$$\int_{-\infty}^{\infty} e^{-c|\xi|}\,|\log|f(\xi)||\,d\xi < \infty, \quad c > \frac{\pi}{b}.$$

Since b can be chosen arbitrarily close to 1 it follows that

$$\int_{-\infty}^{\infty} e^{-c|\xi|}\,|\log|f(\xi)||\,d\xi < \infty, \quad c > \pi.$$

This was, in fact, the result to be proved. Thus Theorem 3 is established.

Chapter III

The necessity of the conditions

9. Definitions

E denotes the class defined page 317. As before, a function $\Phi(x)$ is called non-trivial if $\Phi(x) \neq 0$ in a set of positive measure (see section 4). Define the class I of functions $F(x)$ as follows:

Definition. *$F(x) \in I$ if $F(x) \in V$ and if two positive constants $\alpha = \alpha_F$ and $\theta = \theta_F$, $0 < \theta \leq 1$, can be found such that the relation*

$$\int_{-\infty}^{\infty} \Phi(x-u)\,dF(u) = O(e^{-\gamma x}) \tag{3.1}$$

implies

$$\Phi(x) = O(e^{-\theta\gamma x})$$

for every function $\Phi(x) \in E$ and every γ, $0 < \gamma < \alpha$; and the class of non-trivial functions $\Phi(x) \in E$, satisfying (3.1) *is non-empty for $\gamma < \alpha$.*

The definition of class I in [6] does not exclude the cases where no non-trivial function $\Phi(x)$ of class E exists such that $\overset{*}{\Phi}(x) = O(e^{-\gamma x})$ as $x \to \infty$. However, we now know, from Theorem 3, that this restriction is essential. In the above definition we therefore exclude cases where the definition is meaningless. Furthermore, in the definition of class I in [6] it is tacitly assumed that $F(x)$ is real. If we restrict ourselves to real functions $F(x)$ then the two definitions, except for the above remark, are equivalent, as is easily seen when separating the real and imaginary parts of $\overset{*}{\Phi}(x)$.

The only property of class E which will be used in the sequel is the following: If $\Phi(x)$ is bounded and $r^{-1}|\Phi(x+r)-\Phi(x)|$ is bounded for $r>0$ and $x>x_0$, then $\Phi(x)$ belongs to E.

10. The Wiener condition

By the argument of section 6 the following property of class I is obvious: *If the function $F(x)$ belongs to class I, then its Fourier Stieltjes transform $f(\xi)$ cannot vanish for any real ξ.* This section demonstrates an analogous theorem, the proof being similar to the Wiener argument.

Let us first note the following result: If $\Phi(x)$ is a bounded function and $G(x)$ belongs to L and V, then the function $\Phi_1(x) = G \ast \Phi(x)$ belongs to E. For, let $|\Phi(x)| \leq B$ and put $Q(x) = \int_{-\infty}^{\infty} \Phi(x-u)\, dG(u)$; then for $r>0$

$$|\Phi_1(x+r)-\Phi_1(x)| = \left|\int_x^{x+r} Q(u)\, du\right| \leq B\, V\{G\}\, r.$$

Theorem 4. *Let $F(x) \in I$ and let $\Phi(x)$ be a non-trivial function of E such that $\overset{*}{\Phi}(x) = O(e^{-\gamma x})$, where $0<\gamma<\alpha$. Let $\varphi(\zeta)$ and $\psi(\zeta)$ denote the analytic transforms of $\Phi(x)$ and $\overset{*}{\Phi}(x)$ respectively. Then the function $\varphi(\zeta)/\psi(\zeta)$ is analytic in the strip $-\theta\gamma < \Im\{\zeta\} < 0$.*

The conditions imply that $\Phi(x) = O(e^{-\theta\gamma x})$, and it follows that, for $\zeta = \xi + i\eta$, $\varphi(\zeta)$ and $\psi(\zeta)$ are analytic in the strip $-\theta\gamma < \eta < 0$ (section 5). Since $\Phi(x)$ is assumed non-trivial, the functions $\varphi(\zeta)$ and $\psi(\zeta)$ cannot vanish identically.

The proof is indirect; thus we suppose that, for some ζ_1,

$$\frac{\psi(\zeta_1)}{\varphi(\zeta_1)} = 0; \quad \zeta_1 = \alpha_1 - i\beta_1, \quad 0<\beta_1<\theta\gamma. \tag{3.2}$$

Choose γ_1 such that $\beta_1/\theta < \gamma_1 < \gamma$. We will construct a function $\Phi_1(x)$ of class E such that, as $x \to \infty$, $\overset{*}{\Phi}_1(x) = O(e^{-\gamma_1 x})$ but $e^{\theta\gamma_1 x}|\Phi_1(x)| \to \infty$, thus obtaining a contradiction. Using the results of section 5 we start with the construction of the analytic transforms of $\Phi_1(x)$ and $\overset{*}{\Phi}_1(x)$.

If $\varphi(\zeta)$ has a zero at the point $\zeta = \zeta_1$, let n denote the order of that zero; if not, let $n=0$. Then, by (3.2), the point ζ_1 is a zero of order $\geq n+1$ of

$\psi(\zeta)$. Put

$$\varphi_1(\zeta) = \frac{\varphi(\zeta)}{(\zeta_1-\zeta)^{n+1}}; \quad \psi_1(\zeta) = \frac{\psi(\zeta)}{(\zeta_1-\zeta)^{n+1}}.$$

In the strip $-\beta_1 < \eta < 0$, $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ are analytic transforms of $G * \Phi(x)$ and $G * \overset{*}{\Phi}(x)$ respectively, where $G(x)$ is the function defined by

$$G(x) = 0, \; x < 0; \quad G(x) = \frac{(i)^{n+1}}{n!} x^n e^{-i\zeta_1 x}, \quad x \geq 0.$$

Writing $\Phi_1(x) = G * \Phi(x)$, we find that since $G(x)$ belongs to L and V,

$$\Phi_1(x) \in E, \tag{3.3}$$

and after an inversion,

$$G * \overset{*}{\Phi}(x) = \overset{*}{\Phi}_1(x).$$

The function $\psi_1(\zeta)$ is analytic in the strip $-\gamma < \eta < 0$, and $\psi_1(\xi - i\beta)$ belongs to L and L^2 if $0 < \beta < \gamma$. Since $\psi_1(\zeta)$ is the analytic transform of $\overset{*}{\Phi}_1(x)$ it follows that

$$\overset{*}{\Phi}_1(x) = O(e^{-\gamma_1 x}). \tag{3.4}$$

The function $\varphi_1(\zeta)$ is analytic in $-\theta\gamma < \eta < 0$, save for a simple pole at $\zeta = \zeta_1$; and $|\varphi_1(\xi - i\beta)| \to 0$ when $|\xi| \to \infty$ if $0 < \beta < \theta\gamma$. For $0 < \beta < \beta_1$ we can write

$$\Phi_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix(\xi - i\beta)} \varphi_1(\xi - i\beta)\, d\xi.$$

Let R_1 be the residue of $\varphi_1(\zeta)$ at $\zeta = \zeta_1$. Then, by Cauchy's theorem,

$$\Phi_1(x) = -iR_1 e^{-i\zeta_1 x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix(\xi - i\theta\gamma_1)} \varphi_1(\xi - i\theta\gamma_1)\, d\xi.$$

Since $\varphi_1(\xi - i\theta\gamma_1) \in L$, this implies

$$|\Phi_1(x)| = |R_1| e^{-\beta_1 x} + O(e^{-\theta\gamma_1 x}) \quad \text{as } x \to \infty,$$

where $\beta_1 < \theta\gamma_1$, due to our choice of β_1. It follows that

$$e^{\theta\gamma_1 x} |\Phi_1(x)| \to \infty \quad \text{as } x \to \infty,$$

and, in view of (3.3) and (3.4), this would imply $F(x) \notin I$. We have obtained a contradiction, and conclude

$$\frac{\psi(\zeta)}{\varphi(\zeta)} \neq 0, \qquad -\theta\gamma < \eta < 0,$$

which proves the theorem.

11. The analyticity of $1/f(\xi)$

Theorem 5. *Let $F(x) \in I$, $f(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} dF(x)$ and suppose that*

$$\int_{-\infty}^{\infty} e^{-c|\xi|} \, |\log |f(\xi)|| \, d\xi < \infty, \tag{3.5}$$

for some real c. Then there exists a function $f(\zeta)$, $\zeta = \xi + i\eta$, such that $1/f(\zeta)$ is analytic in the strip $-\theta\alpha < \eta < 0$,

$$\lim_{\eta \to -0} f(\xi + i\eta) = f(\xi), \tag{3.6}$$

and

$$\frac{\varphi(\zeta)}{\psi(\zeta)} \equiv \frac{1}{f(\zeta)},$$

where $\varphi(\zeta)/\psi(\zeta)$ is the function introduced in Theorem 4. In particular, $\varphi(\zeta)/\psi(\zeta)$ is independent of $\Phi(x)$

We may without restriction suppose that $c > \pi/\alpha$. Writing $c = \pi/\tau$, the above condition (3.5) is precisely condition (2.8) of Theorem 2. Thus we may construct the function $h(\zeta)$ as in the proof of Theorem 2. Let, as before, $H(x)$ denote the Fourier transform of $h(\xi)$ and $V(x)$ that of $h(\xi)/f(\xi)$. Then

$$\overset{*}{V}(x) = H(x) = O(e^{-\tau x}).$$

The function $V(x)$ belongs to E since $V(x)$ and $V'(x)$ are bounded (p. 326). Since $F(x)$ belongs to class I by hypothesis it follows that

$$V(x) = O(e^{-\theta \tau x}).$$

Introduce the analytic transform, $v(\zeta)$, of $V(x)$. Since $V(x)$ and $H(x)$ both belong to L^2 we have

$$\lim_{\eta \to -0} v(\xi + i\eta) = h(\xi)/f(\xi) \quad \text{a.e.}$$

and

$$\lim_{\eta \to -0} h(\xi + i\eta) = h(\xi) \quad \text{a.e.}$$

An immediate result is that, for almost all values of ξ,

$$\lim_{\eta \to -0} \frac{v(\xi + i\eta)}{h(\xi + i\eta)} = \frac{1}{f(\xi)} \tag{3.7}$$

We will first prove that (3.7) holds for all values of ξ.

By their very construction, the functions $h(\zeta)$ and $1/h(\zeta)$ are analytic in $-\tau < \eta < 0$ and

$$\lim_{\eta \to -0} |h(\xi + i\eta)| = \frac{|f(\xi)|}{(1 + |\xi|)^3}.$$

Now $f(\xi) \neq 0$, since $F(x) \in I$. The above results concerning $h(\zeta)$ therefore imply that the function $1/h(\zeta)$ is bounded in every finite sub-domain of the strip $-\frac{1}{2}\tau < \eta < 0$. Furthermore, since $H(x)$ and $V(x)$ both are of L^2, we have

$$\lim_{\beta \to 0} M_2 \left\{ v(\xi - i\beta) - \frac{h(\xi)}{f(\xi)} \right\} = 0$$

and

$$\lim_{\beta \to 0} M_2 \{ h(\xi - i\beta) - h(\xi) \} = 0.$$

Hence, for every finite interval (a, b)

$$\lim_{\beta \to 0} \int_a^b \left| \frac{v(\xi - i\beta)}{h(\xi - i\beta)} - \frac{1}{f(\xi)} \right|^2 d\xi = 0. \tag{3.8}$$

Let ξ_0 be a point on the real axis, and choose ξ_1 and ξ_2 such that $\xi_1 < \xi_0 < \xi_2$ and such that (3.7) holds at ξ_1 and ξ_2. Let R denote the rectangle $\xi_1 < \xi < \xi_2$, $-\frac{1}{2}\theta\tau < \eta < 0$, Γ its boundary and let $g(z, \zeta)$ be Green's function for R. The function $v(\zeta)/h(\zeta)$ is analytic in R. By (3.8), and our choice of ξ_1 and ξ_2, it easily follows that

$$\frac{v(\zeta)}{h(\zeta)} = -\frac{1}{2\pi} \int_\Gamma \frac{\partial g(z, \zeta)}{\partial n} \frac{v(z)}{h(z)} |dz|,$$

$\partial/\partial n$ denoting differentiation along the outward normal. If we observe that on the interval $\xi_1 < x < \xi_2$ we have $v(x)/h(x) = 1/f(x)$ and that $1/f(\xi)$ is continuous, we find from the above representation of $v(\zeta)/h(\zeta)$ that (3.7) holds at $\xi = \xi_0$. Hence (3.7) holds for all values of ξ.

Let us now define

$$\frac{1}{f(\zeta)} \equiv \frac{v(\zeta)}{h(\zeta)}.$$

Then $1/f(\zeta)$ is analytic in $-\theta\tau < \eta < 0$ and

$$\lim_{\eta \to -0} \frac{1}{f(\xi + i\eta)} = \frac{1}{f(\xi)}.$$

It follows that the function $f(\zeta)$ fulfils proposition (3.6) and, for every finite interval (a, b), $f(\zeta)$ is analytic in the rectangle $a < \xi < b$, $-\delta < \eta < 0$, for some $\delta > 0$. Introduce the functions

$$f^+(\zeta) = \int_0^\infty e^{i\zeta x} dF(x), \quad \eta \geq 0;$$

$$f^-(\zeta) = \int_{-\infty}^0 e^{i\zeta x} dF(x), \quad \eta \leq 0,$$

analytic in the upper and lower half-plane respectively. (3.6) can be written

$$\lim_{\eta \to -0} \{f(\xi + i\eta) - f^-(\xi + i\eta)\} = f^+(\xi),$$

which implies that $f^+(\zeta)$ can be analytically continued across every finite interval of the real axis.

Let ω denote an interval. Following Beurling (see [3]) we now introduce the classes $T_p = T_{p(x)}$ and $\underset{\omega}{T_p} = \underset{\omega}{T_{p(x)}}$ defined by

$$f(\xi) \in T_p \quad \text{if} \quad f(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}\, dF(x) \quad \text{where} \quad \int_{-\infty}^{\infty} p(x)\,|dF(x)| < \infty;$$

$$f(\xi) \in \underset{\omega}{T_p} \quad \text{if} \quad f(\xi) = f_1(\xi),\ \xi \in \omega \quad \text{where } f_1(\xi) \in T_p.$$

Let

$$p_1(x) = \begin{cases} (1+x)^{\frac{1}{2}}, & x \geq 0, \\ 1, & x < 0, \end{cases}$$

and let ω be a finite interval. By use of the fact that $f^+(\zeta)$ is analytic in a domain containing ω it is easy to see that

$$f^+(\xi) \in \underset{\omega}{T_{p_1}}.$$

Now $f(\xi) = f^+(\xi) + f^-(\xi)$, where $f^-(\xi) \in T_{p_1}$. Hence

$$f(\xi) \in \underset{\omega}{T_{p_1}} \quad \text{for every finite interval } \omega. \tag{3.9}$$

Let $\Phi(x)$ be a non-trivial function of class E such that $\overset{*}{\Phi}(x) = O(e^{-\gamma x})$, where $0 < \gamma < \alpha$. Let $\varphi(\zeta)$ be the analytic transform of $\Phi(x)$ and $\psi(\zeta)$ be that of $\overset{*}{\Phi}(x)$. We will prove that

$$\frac{1}{f(\zeta)} \equiv \frac{\varphi(\zeta)}{\psi(\zeta)} \tag{3.10}$$

or

$$\varphi(\zeta)\, h(\zeta) \equiv \psi(\zeta)\, v(\zeta)$$

which amounts to showing that

$$\Phi \ast \overset{*}{V}(x) = \overset{*}{\Phi} \ast V(x). \tag{3.11}$$

Now, as $x \to -\infty$,

$$|V| \ast |\Phi|(x) = O\{p_1(-x)\}. \tag{3.12}$$

This is easily verified because $V(x) \in L^2$. Therefore, should $f(\xi)$ belong to T_{p_1}, then the double integral

$$\int_{-\infty}^{\infty} dF(x-u) \int_{-\infty}^{\infty} \Phi(u-v)\, V(v)\, dv$$

would be absolutely convergent, justifying the inversion in (3.11). Here we shall use the method by which Lemma 1 was proved and the aid of the weaker condition (3.9) to prove (3.11).

Let $M(x)$ be a function such that $p_1(x) M(x) \in L$ and such that the Fourier transform, $m(\xi)$, of $M(x)$ vanishes outside a finite interval, ω. Then, by (3.9) $m(\xi) f(\xi) \in T_{p_1}$. The function $m(\xi) f(\xi)$ has the Fourier transform $\overset{*}{M}(x)$, thus $p_1(x) \overset{*}{M}(x) \in L$. Therefore, according to (3.12), the integral $\overset{*}{M} \ast \Phi \ast V(x)$ is absolutely convergent and may be inverted. We thus find

$$\Phi \ast \overset{*}{M} \ast V(x) = V \ast \overset{*}{M} \ast \Phi(x)$$

which, after further inversion, yields

$$\Phi \ast M \ast \overset{*}{V}(x) = V \ast M \ast \overset{*}{\Phi}(x). \tag{3.13}$$

It is easy to verify that $|\Phi| \ast |\overset{*}{V}|(x)$ and $|V| \ast |\overset{*}{\Phi}|(x)$ both are $O\{p_1(-x)\}$ as $x \to -\infty$. Therefore, in view of the conditions on $M(x)$, the integrals in (3.13) are absolutely convergent. Inverting again, we find

$$M \ast \Phi \ast \overset{*}{V}(x) = M \ast V \ast \overset{*}{\Phi}(x).$$

By the argument used in Lemma 1 (p. 319) this yields

$$\Phi \ast \overset{*}{V}(x) = V \ast \overset{*}{\Phi}(x),$$

which proves (3.11). Thus (3.10) is established.

Let us choose another function $\Phi_1(x) \in E$, such that $\Phi_1(x)$ is non-trivial and $\overset{*}{\Phi}_1(x) = O(e^{-\gamma_1 x})$, $0 < \gamma_1 < \alpha$. Denoting the analytic transforms of $\Phi_1(x)$ and $\overset{*}{\Phi}_1(x)$ by $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ we find, since the above argument applies to any function of this kind, $1/f(\zeta) \equiv \varphi_1(\zeta)/\psi_1(\zeta)$. Therefore

$$\frac{\varphi(\zeta)}{\psi(\zeta)} \equiv \frac{\varphi_1(\zeta)}{\psi_1(\zeta)},$$

i.e. $\varphi(\zeta)/\psi(\zeta)$ is independent of the function $\Phi(x)$.

According to Theorem 4, the function $\varphi(\zeta)/\psi(\zeta)$ is analytic in the strip $-\theta\gamma < \eta < 0$. Further, by the definition of class I, we can find, for every $\gamma < \alpha$, a non-trivial function $\Phi(x)$ of class E such that $\overset{*}{\Phi}(x) = O(e^{-\gamma x})$. It follows that $1/f(\zeta)$ is analytic in $-\theta\gamma < \eta < 0$ for every $\gamma < \alpha$. Thus $1/f(\zeta)$ is analytic in $-\theta\alpha < \eta < 0$, and the proof of Theorem 5 is complete.

An immediate consequence of Theorems 3 and 5 is the following

Theorem 6. *Let* $F(x) \in V$, $f(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} dF(x)$ *and let, for some* $\delta > 0$,

$$\int_{-\infty}^{\infty} |x|^{\frac{1}{2}+\delta} |dF(x)| < \infty.$$

If $F(x) \in I$, *then there exists a function* $f(\zeta)$, $\zeta = \xi + i\eta$, *such that* $1/f(\zeta)$ *is analytic in the strip* $-\theta\alpha < \eta < 0$, *and*

$$\lim_{\eta \to -0} f(\xi + i\eta) = f(\xi).$$

12. The constant θ

In the definition of class I (p. 341) we introduced two constants, $\theta = \theta_F$ and $\alpha = \alpha_F$. Let us say that

$$F(x) \in I(\theta, \alpha),$$

if $F(x)$ fulfils the conditions in the definition of class I for the constants θ and α.

Earlier, a theorem has been proved ([6], Theorem 3), which can be restated as follows:

If $F(x) \in V$, $f(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}\, dF(x)$, *and if there exists a function* $f(\zeta)$, $\zeta = \xi + i\eta$, *such that* $1/f(\zeta)$ *is analytic in the strip* $-a \leq \eta < 0$, *and*

$$\left| \frac{d}{d\xi} \frac{1}{f(\xi + i\eta)} \right| < \text{const.}\ (1 + |\xi|)^{r-1} \tag{3.14}$$

for $-a \leq \eta \leq 0$ *and some* $r > 0$, *and furthermore* $\lim_{\eta \to -0} f(\xi + i\eta) = f(\xi)$, *then*

$$F(x) \in I(\theta, a) \quad \text{for} \quad \theta < \frac{1}{r+1}.$$

That in this theorem the boundary $1/(r+1)$ for θ is essential may be seen from the following simple example. Let, for some $r > 0$

$$f(\zeta) = \frac{\Gamma(r)}{(1 + i\zeta)^r}.$$

Then $1/f(\zeta)$ is analytic in $\eta \leq 0$ and satisfies (3.14) in this half-plane. The function $f(\xi)$ is the Fourier Stieltjes transform of an absolutely continuous function $F(x)$ such that

$$F'(x) = |x|^{r-1} e^x, \quad x < 0; \qquad F'(x) = 0, \quad x > 0.$$

Let us choose, for some $c > 0$,

$$\Phi(x) = 0, \quad x < 0; \qquad \Phi(x) = e^{-cx} \sin e^{cx}, \quad x \geq 0.$$

The function $\Phi(x)$ is of class E, since $\Phi'(x)$ is bounded for $x > 0$. Moreover,

$$\overset{*}{\Phi}(x) = \int_{-\infty}^{\infty} F'(x - u)\,\Phi(u)\, du = e^x \int_x^{\infty} e^{-u-cu} (u - x)^{r-1} \sin e^{cu}\, du.$$

It is easy to verify that

$$\overset{*}{\Phi}(x) = O(e^{-(r+1)cx}) \quad \text{as} \quad x \to \infty.$$

It follows from this relation and the order of magnitude of $\Phi(x)$ that the function $F(x)$, introduced above, cannot belong to $\mathrm{I}(\theta, \alpha)$ if $\theta > 1/(r+1)$ for any α.

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