ON THE STEADY-STATE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS, III

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Contents

Introduction

In this work we study the relations connecting a solution of the Navier-Stokes equations

$$\mu \triangle \mathbf{w} - \varrho \mathbf{w} \cdot \nabla \mathbf{w} - \nabla p = 0 \tag{1}$$
$$\nabla \cdot \mathbf{w} = 0,$$

(1) This investigation was supported by the Office of Naval Research.

with the values achieved by the solution on the boundary of the region of definition, and with the magnitudes of certain energy integrals which are associated naturally with the solutions of (1).

The notation in (1) is the usual one of vector analysis. Each of the quantities which appears admits a simple physical interpretation. The solution $\mathbf{w}(\mathbf{x})$ can be interpreted as the velocity field of an incompressible fluid motion, and $p(\mathbf{x})$ is then the associated pressure. The constant μ is the viscosity coefficient of the fluid, and the term $\mu \triangle \mathbf{w}$ denotes accordingly the shearing force on a unit volume due to relative motion at a fluid interface. ϱ denotes the density of the fluid, $\varrho \mathbf{w} \cdot \nabla \mathbf{w}$ the inertial reaction of a unit volume, and ∇p is the force per unit volume acting normal to a fluid interface. The first equation expresses the equilibrium of these forces at points of the flow; the second expresses the assumption that ϱ is constant in the motion.

Because of the difficulty in integrating (1) in a general case, it is natural to consider the linear equations satisfied by the perturbations of a particular solution. The simplest of these are the Stokes equations

$$\begin{gathered} \mu \triangle \mathbf{w} - \nabla p = 0 \\ \nabla \cdot \mathbf{w} = 0 \end{gathered} \tag{2}$$

which correspond to the identically vanishing solution of (1). A major task of the present work will be to examine the connection between the solutions of (2), and the solutions of (1) which correspond to small boundary data.

The system (2) has been studied in considerable detail by Odqvist [11], who proved the existence of a Green's Tensor for an arbitrary region. Odqvist used this tensor to obtain an integral equation for the solutions of (1), and this equation led in turn to a proof of existence of a solution of (1) in a finite region $\mathcal{G}$, corresponding to prescribed data $\mathbf{w}^*$ on the boundary Σ which satisfy the (necessary) condition

$$\oint_{\Sigma} \mathbf{w}^* \cdot \mathbf{n} \, dS = 0 \tag{3}$$

($\mathbf{n}$ = unit exterior direct normal), provided only that $|\mathbf{w}^*|$ is everywhere sufficiently small ([11], see also [10]).

The first general study of (1) for arbitrary prescribed data is due to Leray [8]. Leray derived general a priori estimates on the solutions of (1), depending only on Σ and on the boundary data. He applied these estimates, using a device which is now classical but which was at that time not yet clearly formulated, to prove the existence of at least one solution of (1) in $\mathcal{G}$ corresponding to arbitrary (sufficiently smooth) data $\mathbf{w}^*$ on Σ. The solution is obtained by a continuous deformation in function space, starting with the solution of (2) given by Odqvist, and essential use is made of Odqvist's integral equation and of his estimates on the Green's Tensor.

From a physical point of view the problem just discussed has little meaning, since the natural boundary condition is $\mathbf{w}^* \equiv 0$, and in this case one sees easily that the only solution of (1) in $\mathcal{G}$ is $\mathbf{w} \equiv 0$. Of more interest is the exterior problem, in which a solution is sought which assumes data $\mathbf{w}^*$ on Σ and which tends to a given constant vector $\mathbf{w}_0$ at infinity. In this case, however, new difficulties arise. Experimental evidence indicates that, at least for large prescribed data, the solution either does not exist or is unstable. For the linear system (2), it is known that no solution exists in two dimensions.(1) In three dimensions, the solution exists but is known to violate, in a neighborhood of infinity, the assumptions under which the equations were derived (see, e,g. [12, p. 165]). Also for the strict equations (1) there is evidence that solutions may exhibit pathological behavior at infinity. An example in two dimensions is discussed in § 6 of this paper. Nevertheless, Leray succeeded in constructing, for arbitrary prescribed data in three dimensions, a solution of (1) in the exterior $\mathcal{E}$ of Σ which equals $\mathbf{w}^*$ on Σ, for which the Dirichlet Integral is finite, and which tends to $\mathbf{w}_0$ in the sense of an integral norm. Leray also obtained a priori estimates on the solution which are valid in any compact subregion of $\mathcal{E}$.

The behavior of the solution at infinity has been discussed in some detail in [1] and in [2]. In [1] we have proved that the solution of Leray (more generally any solution with finite Dirichlet Integral) necessarily tends to a limit in the strict sense as $\mathbf{x} \to \infty$, and a representation of the solution by means of an integral equation is obtained. In [2] we discuss solutions which need not have finite Dirichlet Integral. We show there that whenever $\mathbf{w} \to \mathbf{w}_0$ at infinity, then necessarily all first order derivatives of $\mathbf{w}$ tend to zero. If, in addition, $|\mathbf{w}-\mathbf{w}_0| < Cr^{-\frac{1}{4}-\varepsilon}$ for some $\varepsilon > 0$, then $\mathbf{w}(\mathbf{x})$ has the same asymptotic structure at infinity as the corresponding solution of the system obtained by linearizing (1) about the solution $\mathbf{w} \equiv \mathbf{w}_0$. In particular, $|\mathbf{w}-\mathbf{w}_0| < Cr^{-1}$ and there exists a paraboloidal "wake" region outside of which $|\mathbf{w}-\mathbf{w}_0| < Cr^{-2}$. It is not known however whether there exist solutions which exhibit the assumed rate of decay to $\mathbf{w}_0$ at infinity.

(1) See, e.g., [4]. An improved discussion of this phenomenon will appear in a forthcoming work of I. D. Chang and the author.

The crucial step in the method of Leray consists in obtaining an a priori bound for the Dirichlet Integral [1] of any possible solution, depending only on boundary data. Leray proved the existence of such a bound, and also gave an independent demonstration which yielded an explicit estimate [2]. In § 2 of this paper we obtain a bound for the Dirichlet Integral by a method which derives conceptually from that of Leray. Our result is a slight improvement on that of Leray, in the sense that we do not insist that the outflow integral (3) vanish, but merely require it to be sufficiently small. The demonstration we give uses a technical device due to E. Hopf [5] which, we believe, simplifies and clarifies the reasoning considerably.

In § 3, we apply the bounds on Dirichlet Integral in order to obtain a-priori estimates on any possible solution and on its first derivatives, depending only on prescribed data. In the case of a finite region $\mathcal{G}$, these bounds are essentially those of Leray. For the region $\mathcal{E}$ exterior to Σ, we improve the results of Leray by giving estimates which are uniformly valid throughout the flow region.

We show in § 4 that solutions with finite Dirichlet Integral are necessarily continuous at infinity. We present here a proof which is more elementary than the one we have given in [1]. In § 5 we prove the existence of a solution corresponding to prescribed boundary data. Again the result is essentially that of Leray when the region is finite. The new features in the other case are that the solution is shown to attach continuously to the prescribed value at infinity, that some outflux is permitted, and that uniform bounds are available for the solution and its derivatives.

The principal new results of this paper are presented in § 7. Here we study the manner in which the solutions of (1) transform into those of (2) as the prescribed data tend to zero. Precisely, we consider data of the form $\lambda \mathrm{w}^*$, $\lambda \mathrm{w}_0$, $0<\lambda<1$. [3]

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(1) This integral can be interpreted physically as half the sum of the rate at which energy is converted into heat by the fluid, and the total vorticity in the flow.

(2) Another proof of the existence of a bound, based on an inequality of Sobolev, has been given by O. A. Ladyzhenskaia [7]. The method of Leray, besides yielding an explicit estimate, is intrinsically simpler and more elementary.

(3) Equivalently, we could keep the boundary data fixed and let $\mu \to \infty$ or $\varrho \to 0$. In the former case we would find $|w(x;\mu) - W_0(x)| < C/\mu$ in a bounded region, and

$$|w(x;\mu) - W_0(x)| < C(\mu^{-\frac{1}{2}} r^{-1} + \mu^{-1})$$

in an exterior region. The estimate for a bounded region can be obtained also from the work of Odqvist [11]. The emphasis in the present paper is on the behavior of the solution in a neighborhood of infinity, to which the methods of Odqvist do not seem to apply.

We prove that if $\mathbf{w}(\mathbf{x};\lambda)$ is a solution of (1) with these data and if $\mathbf{W}_0(\mathbf{x})$ is the solution of (2) with data $\mathbf{w}^*$, $\mathbf{w}_0$ then $|\lambda^{-1}\mathbf{w}(\mathbf{x};\lambda)-\mathbf{W}_0(\mathbf{x})|=O(\lambda)$ when the region is finite, and $|\lambda^{-1}\mathbf{w}(\mathbf{x};\lambda)-\mathbf{W}_0(\mathbf{x})|=O(\sqrt{\lambda}\,r^{-1}+\lambda)$ in the case of an infinite region.(1) Analogous estimates for the derivatives are also given. Thus, we see that the solutions of (2) in an exterior region are uniformly close to the corresponding solutions of the Navier-Stokes equations (1), even though the perturbation from the latter solutions to the former is singular in this region. These considerations are applied to a discussion of the force exerted on Σ by the fluid. and an estimate is given for the error incurred by using the solution of (2) to calculate the force. The demonstrations are straightforward, but lean heavily on the developments in the earlier sections of the paper.

In the final section we improve the classical uniqueness theorem for (sufficiently small) solutions in a finite region $\mathcal{G}$ by showing that this result can be given in an a priori formulation, depending only on boundary data. (The classical result assumes a knowledge of one solution in the entire region, see, e.g. [16].) We obtain this theorem as a special case of a more general result, that the difference of two sufficiently small solutions $\mathbf{w}_1(\mathbf{x})$ and $\mathbf{w}_2(\mathbf{x})$ can be bounded uniformly in $\mathcal{G}$ in terms of the solution of the linear equations (2) with boundary data equal to $\mathbf{w}_1(\mathbf{x})-\mathbf{w}_2(\mathbf{x})$. To our knowledge, this is the first result on continuous dependence of the solutions of (1) on boundary data to be published.

The chief concern of this paper is with solutions of the system (1) in three dimensions. Those of our results which pertain to solutions in a finite region are presumably valid also in the corresponding two dimensional case, but a rigorous proof requires certain general estimates which are not yet available. The behavior of a two dimensional solution at infinity appears to present difficulties of a more profound nature, and a precise discussion must probably await the development of new methods.

1. Notation and definitions; preliminary estimates; the representation formula

We consider a vector field $\mathbf{w}(\mathbf{x})$, $\mathbf{w}=(w_1, w_2, w_3)$, which is defined in a region $\mathcal{G}$ of three dimensional Euclidean space, $\mathbf{x}=(x_1, x_2, x_3)$. Such a field is said to be a *solution of the Navier-Stokes equations*,

(1) The origin of coordinates is assumed interior to Σ. The result implies, in particular, the uniform inequality $|\mathbf{W}(\mathbf{x};\lambda)-\mathbf{W}_0(\mathbf{x})|<C\sqrt{\lambda}$ in $\mathcal{E}+\Sigma$.

$$\mu \triangle \mathbf{w} - \varrho \mathbf{w} \cdot \nabla \mathbf{w} - \nabla p = 0 \qquad (1)$$
$$\nabla \cdot \mathbf{w} = 0$$

in $\mathcal{G}$ whenever there exists a scalar field $p(x)$ in $\mathcal{G}$, such that (1) is satisfied throughout $\mathcal{G}$ by the pair $(\mathbf{w}, p)$. It is assumed that $\mathbf{w}(\mathbf{x})$ and $p(\mathbf{x})$ are sufficiently smooth that all quantities entering in (1) are defined and continuous throughout $\mathcal{G}$. The vector field $\mathbf{w}(\mathbf{x})$ and scalar $p(\mathbf{x})$ have the physical significance of velocity and pressure, respectively. We have found these interpretations helpful in providing motivation and suggesting methods, but they are of course unnecessary for the formal mathematical developments. In order to simpliy notation we shall normalize (1) so that $\mu = \varrho = 1$. This can always be achieved by multiplication of $\mathbf{w}$ and of p by appropriate constant factors. Equations (1) then take the form

$$\triangle \mathbf{w} - \mathbf{w} \cdot \nabla \mathbf{w} - \nabla p = 0 \qquad (4)$$
$$\nabla \cdot \mathbf{w} = 0.$$

Since most of our results are valid for every value of the Reynold's number, (1) this normalization entails no loss of generality. In § 7 we shall permit the Reynold's number to vary, but we shall effect this by varying the boundary values of the velocity field and keeping all other parameters constant.

A particular solution of (4) is the uniform flow, $\mathbf{w} \equiv \mathbf{w}_0 = \text{const.}$ The perturbations of this solution are solutions of the linear system,

$$\triangle \mathbf{w} - \mathbf{w}_0 \cdot \nabla \mathbf{w} - \nabla p = 0 \qquad (5)$$
$$\nabla \cdot \mathbf{w} = 0,$$

the *equations of Oseen*. In the case $\mathbf{w}_0 = 0$, we obtain the *equations of Stokes*,

$$\triangle \mathbf{w} - \nabla p = 0 \qquad (6)$$
$$\nabla \cdot \mathbf{w} = 0.$$

We shall need a *fundamental solution tensor* $\boldsymbol{\chi}(\mathbf{x}, \mathbf{y})$ associated with (5). Such a tensor has been determined explicitly by Oseen [12, p. 34]. It can be obtained from the relations,

(1) The Reynold's number is defined by the relation $R = \varrho UL/\mu$, where U and L denote a characteristic speed and length in the flow. For a discussion of the role played by this quantity in the theory of (1) and in experimental observation, see, e.g. [6].

$$\chi_{ij} = \delta_{ij} \triangle \Phi - \frac{\partial^2 \Phi}{\partial x_i \partial x_j}, \qquad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{7}$$

$$\psi_j = -\frac{\partial}{\partial x_j} [\triangle \Phi + \mathbf{w}_0 \cdot \nabla \Phi]$$

$$\Phi = -\frac{1}{8\pi\sigma} \int_0^{\sigma s} \frac{1 - e^{-\alpha}}{\alpha} d\alpha$$

$$\sigma = \frac{|\mathbf{w}_0|}{2}, \quad s = r_{\mathbf{xy}} + \frac{\mathbf{w}_0 \cdot (\mathbf{y} - \mathbf{x})}{|\mathbf{w}_0|}.$$

The tensor $\boldsymbol{\chi} = (\chi_{ij})$ and vector $\boldsymbol{\psi} = (\psi_j)$ become singular at $\mathbf{x} = \mathbf{y}$ in such a way that

$$\delta_{ik} = \lim_{r \to 0} \int_{\Sigma_r} \left\{ \psi_k \delta_{ij} - \left(\frac{\partial \chi_{ik}}{\partial x_j} + \frac{\partial \chi_{jk}}{\partial x_i} \right) \right\} n_j \, dS_y, \tag{8}$$

where Σ_r denotes the surface of a sphere of radius r about $\mathbf{x}$ as center and $\mathbf{n} = (n_j)$ is the unit exterior directed normal on Σ_r. For fixed j, the column vectors χ_{ij} define, as function of $\mathbf{x}$, a solution of (5) with ψ_j as corresponding pressure. As function of $\mathbf{y}$, χ_{ij} defines a solution of the adjoint system,

$$\begin{gathered} \triangle \mathbf{w} + \mathbf{w}_0 \cdot \nabla \mathbf{w} - \nabla p = 0 \\ \nabla \cdot \mathbf{w} = 0. \end{gathered} \tag{9}$$

In the case $\mathbf{w}_0 = 0$, the tensor (χ_{ij}) takes a particularly simple form. We then have

$$\chi_{ij} = \frac{-1}{8\pi} \left\{ \frac{\delta_{ij}}{r_{xy}} + \frac{(x_i - y_i)(x_j - y_j)}{r_{xy}^3} \right\}, \qquad \psi_j = -\frac{x_j - y_j}{4\pi r_{xy}^3}. \tag{10}$$

We define the *stress tensor* $T\mathbf{w}$ by the relation

$$(T\mathbf{w})_{ij} = -p\,\delta_{ij} + \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right). \tag{11}$$

Formal integration by parts leads to the relations, valid for any divergence-free vector fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ and associated scalars $p(\mathbf{x}), q(\mathbf{x})$ defined in a region $\mathcal{G}$ with boundary Σ,

$$\int_{\mathcal{G}} \mathbf{u} \cdot (\triangle \mathbf{u} - \nabla p) \, dV + 2 \int_{\mathcal{G}} (\operatorname{def} \mathbf{u})^2 \, dV = \oint_{\Sigma} \mathbf{u} \cdot T\mathbf{u} \, dS \tag{12}$$

$$\int_{\mathcal{G}} [\mathbf{u} \cdot (\triangle \mathbf{v} - \nabla q) - \mathbf{v} \cdot (\triangle \mathbf{u} - \nabla p)] \, dV = \oint_{\Sigma} (\mathbf{u} \cdot T\mathbf{v} - \mathbf{v} \cdot T\mathbf{u}) \, dS,$$

where $\operatorname{def} \mathbf{u} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is the *deformation tensor* associated with the motion.(1) These identities are to be understood in the sense that $\mathbf{u}\cdot T\mathbf{v} = u_i (Tv)_{ij} n_j$ and summation is extended over repeated indices.

From (8) and (12) we obtain the representation, valid for any solution of (4) in $\mathcal{G}$,

$$\mathbf{w}(\mathbf{x}) = \oint_{\Sigma} \{\mathbf{w}\cdot T\boldsymbol{\chi} - \boldsymbol{\chi}\cdot T\mathbf{w} + (\boldsymbol{\chi}\cdot\mathbf{w})(\mathbf{w}_0\cdot\mathbf{n})\}\, dS_{\mathbf{y}} + \int_{\mathcal{G}} \boldsymbol{\chi}\cdot(\mathbf{w}-\mathbf{w}_0)\cdot\nabla\mathbf{w}\, dV_{\mathbf{y}}, \quad (13)$$

where $T\boldsymbol{\chi}$ is formed by interpreting the components of $\boldsymbol{\psi}$ as pressures. One sees easily that, conversely, any vector field $\mathbf{w}(\mathbf{x})$ which satisfies (13) is a solution of the Navier-Stokes equations (4). We find similarly a relation for the pressure,

$$p(\mathbf{x}) = \oint_{\Sigma} \{\mathbf{w}\cdot T\boldsymbol{\psi} - \boldsymbol{\psi}\cdot T\mathbf{w} + (\boldsymbol{\psi}\cdot\mathbf{w})(\mathbf{w}_0\cdot\mathbf{n})\}\, dS_{\mathbf{y}} + \int_{\mathcal{G}} \boldsymbol{\psi}\cdot(\mathbf{w}-\mathbf{w}_0)\cdot\nabla\mathbf{w}\, dV_{\mathbf{y}}, \quad (14)$$

where we have introduced a "pressure" $\psi^* = \left[\mathbf{w}_0\cdot\nabla\left(\frac{1}{r_{\mathbf{xy}}}\right)\right]$ corresponding to the vector $\boldsymbol{\psi}(\mathbf{x}, \mathbf{y})$.

We collect here some elementary properties of the tensor $\boldsymbol{\chi}(\mathbf{x}, \mathbf{y})$ for later reference. In what follows, we assume (without loss of generality) that the vector $\mathbf{w}_0$ is directed along the positive x_1-axis. We denote by φ the (polar) angle made by a ray which starts from the point $\mathbf{x}$, with the positively directed x_1-axis, and by r the distance from $\mathbf{x}$ to a point $\mathbf{y}$ of this ray. We present the estimates for $\boldsymbol{\chi}(\mathbf{x}, \mathbf{y})$ as function of $\mathbf{y}$ for fixed $\mathbf{x}$. Considered as function of $\mathbf{x}$, all estimates for $\boldsymbol{\chi}(\mathbf{x}, \mathbf{y})$ remain true if φ is replaced by $(\pi - \varphi)$. Since $\boldsymbol{\chi}$ is a function only of $(\mathbf{y}-\mathbf{x})$, we may assume that $\mathbf{x}$ is the origin of coordinates. Letting $|\boldsymbol{\chi}|$ denote an upper bound for the magnitudes of all components of $\boldsymbol{\chi}$, we then have for some positive constant C,

i) as $r \to \infty$, $|\boldsymbol{\chi}| < C\dfrac{1}{r}\dfrac{1-e^{-\sigma s}}{\sigma s}$,

$$|\nabla\boldsymbol{\chi}| < C\frac{1-e^{-\sigma s} - \sigma s e^{-\sigma s}}{(\sigma s)^{\frac{3}{2}}}\frac{\sigma^{\frac{1}{2}}}{r^{\frac{3}{2}}},$$

where $s = r + y_1 = r(1+\cos\varphi)$, and $\sigma = \frac{1}{2}|\mathbf{w}_0|$,

ii) for any integer $N > 0$, the Nth derivative $\boldsymbol{\chi}^{(N)}$ of $\boldsymbol{\chi}$ in any direction satisfies the inequality $|\boldsymbol{\chi}^{(N)}| < C r^{-\frac{1}{2}(N+2)}$ uniformly in φ for sufficiently large r,

(1) Throughout this paper, we denote volume integrals by $\int \ldots dV$, and integrals over closed surfaces by $\oint \ldots dS$.

iii) for a spherical surface Σ_R of radius R and center at $\mathbf{x}$, there holds

$$\oint_{\Sigma_R} |\chi|\,dS < C, \quad \oint_{\Sigma_R} |T\chi|\,dS < C,$$

$$\oint_{\Sigma_R} |\nabla\chi|\,dS < CR^{-\frac{1}{2}}, \quad \oint_{\Sigma_R} |\chi|^2\,dS < CR^{-1},$$

iv) in a neighborhood of the singular point $\mathbf{x}=\mathbf{y}$, $|\chi| < Cr^{-1}$, $|\nabla\chi| < Cr^{-2}$.

Property iv) follows immediately from the definition (7) of $\chi(\mathbf{x}, \mathbf{y})$. Property ii) is obtained from a generalization of property i) to higher derivatives. Property iii), except for the estimate on $T\chi$, follows easily from property i). Property i) is obtained by tedious but formal computation, starting from (7). We omit details. The analogous estimates for $\psi(\mathbf{x}, \mathbf{y})$ are obvious. We have, in fact, $\psi(x, y) = \nabla(r^{-1})$. The estimate in iii) for $\int_{\Sigma_R} |T\chi|\,dS$ follows from this and from the estimate for $\int_{\Sigma_R} |\nabla\chi|\,dS$. Some further estimates on $\chi(\mathbf{x}, \mathbf{y})$ are included in [2]. We have given here only those which are necessary in the present context.

We shall deal with solutions $\mathbf{w}(\mathbf{x})$ of (4) which are defined in a bounded region $\mathcal{G}$, and with solutions defined in a region $\mathcal{E}$ which contains a neighborhood of infinity. In either case we denote the boundary of the region by Σ. Σ is to consist of a finite number of closed, connected component surfaces. By a *smooth surface* Σ we shall mean a surface which admits in a neighborhood of each of its points a parametric representation by means of functions which have continuous derivatives of all orders entering in the context. Although in most cases slightly weaker assumptions will suffice, we will be safe to assume that these functions are of class $\mathfrak{C}^{(3)}$. Correspondingly, we will usually assume that the boundary values $\mathbf{w}^*$ of $\mathbf{w}$ are of class $\mathfrak{C}^{(3)}$ when considered as functions of the same parameters.

Throughout this paper the symbol C will be used to denote a positive constant, the value of which may however change even within a given context. Thus, from the relation $\beta < C(1+\alpha^2)$ we may conclude $\beta < C\alpha^2$ for all $\alpha > 1$.

2. A priori estimation of the Dirichlet integral

In this section we derive the bounds on the Dirichlet integral of solutions of (4), which are basic to the subsequent developments. We prove first a preliminary result concerning the possibility of constructing solenoidal extensions of prescribed vector fields. (1)

Lemma 2.1. *Let* $\mathbf{w}^*$ *be prescribed data on a smooth surface* Σ, *which are of class* $C^{(3)}$ *on* Σ *and which satisfy the outflow condition* $\oint_{\Sigma_i} (\mathbf{w}^* \cdot \mathbf{n})\, dS = 0$ *for each component* Σ_i *of* Σ. *Then there exists an infinity of vector fields* $\boldsymbol{\psi}(\mathbf{x})$ *which are defined throughout space, which vanish outside a neighborhood of* Σ, *and are such that* curl $\boldsymbol{\psi}(\mathbf{x}) = \mathbf{w}^*$ *on* Σ. *The field* $\boldsymbol{\psi}(\mathbf{x})$ *can be chosen to be bounded together with all its partial derivatives up to third order, the bounds depending only on the corresponding derivatives of* $\mathbf{w}^*$ *with respect to suitable surface parameters and on the smoothness of* Σ.

The vector field $\mathbf{v}(\mathbf{x}) = \text{curl}\, \boldsymbol{\psi}$ then provides a solenoidal extension of the given data $\mathbf{w}^*$.

Lemma 2.1 is true in any number of dimensions. We present here a proof for three dimensional case. We consider first a representative component Σ_0 of Σ, and introduce a partition of unity over Σ_0. That is, we cover Σ_0 by a finite number of neighborhoods $\Sigma_0^{(k)}$, $k = 1, \ldots, N$, and define non-negative functions $f^{(k)}$ of class C^∞ on Σ_0 such that, a) each $f^{(k)}$ vanishes outside $\Sigma_0^{(k)}$ and, b) $\sum_{k=1}^{N} f^{(k)} = 1$ at all points of Σ_0. (For details of the construction see, e.g., De Rham [15].) We may assume that the $\Sigma_0^{(k)}$ and $f^{(k)}$ are chosen in such a way that each such neighborhood admits a representation of class $C^{(3)}$ onto the interior of a plane unit disc $\Gamma^{(k)}$ and that each $f^{(k)} = 0$ in the annular region consisting of all points in the disc whose distance from the origin exceeds $\frac{1}{2}$. Finally, we extend each representation to a mapping from a (thin) cylinder $Z^{(k)}$ of which $\Gamma^{(k)}$ is the mid-section, onto a neighborhood of $\Sigma_0^{(k)}$, by mapping the normals to the disc isometrically onto the normals to Σ_0 at corresponding points, and we extend the partition functions by constancy along the normals.

At each point of Σ_0 we have $\mathbf{w}^* = \sum_{k=1}^{N} f^{(k)} \mathbf{w}^*$. Denote by α_1, α_2, the rectilinear coordinates of the disc $\Gamma^{(k)}$ and by α_3 the distance along the normal to the disc. Let $A_1^{(k)}, A_2^{(k)}, A_3^{(k)}$ be the components of $f^{(k)} \mathbf{w}^*$, and let

(1) This lemma has been used by several authors, but we know of no proof in the literature previous to a demonstration we have given in [2]. The proof presented here is due to Professor C. Loewner (oral communication). It is more elementary than the one in [2], and it has the advantage that estimates on the extension field can easily be found from a knowledge of the corresponding estimates for w^*.

$$P_1^{(k)} = A_1^{(k)} \begin{pmatrix} x_2 & x_3 \\ \alpha_2 & \alpha_3 \end{pmatrix} + A_2^{(k)} \begin{pmatrix} x_3 & x_1 \\ \alpha_2 & \alpha_3 \end{pmatrix} + A_3^{(k)} \begin{pmatrix} x_1 & x_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}$$

$$P_2^{(k)} = A_1^{(k)} \begin{pmatrix} x_2 & x_3 \\ \alpha_3 & \alpha_1 \end{pmatrix} + A_2^{(k)} \begin{pmatrix} x_3 & x_1 \\ \alpha_3 & \alpha_1 \end{pmatrix} + A_3^{(k)} \begin{pmatrix} x_1 & x_2 \\ \alpha_3 & \alpha_1 \end{pmatrix}$$

$$P_3^{(k)} = A_1^{(k)} \begin{pmatrix} x_2 & x_3 \\ \alpha_1 & \alpha_2 \end{pmatrix} + A_2^{(k)} \begin{pmatrix} x_3 & x_1 \\ \alpha_1 & \alpha_2 \end{pmatrix} + A_3^{(k)} \begin{pmatrix} x_1 & x_2 \\ \alpha_1 & \alpha_2 \end{pmatrix},$$

where the quantities in parentheses denote Jacobians. Denote by $P^{(k)}$ the vector, $P^{(k)} = (P_1^{(k)}, P_2^{(k)}, P_3^{(k)})$. $P^{(k)}$ is defined on $\Gamma^{(k)}$ and vanishes outside a circle of radius $\frac{1}{2}$.

We now seek a vector field $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ defined in $Z^{(k)}$, such that curl $\boldsymbol{\omega} = P^{(k)}$ on $\Gamma^{(k)}$ and $\boldsymbol{\omega} = 0$ whenever $\alpha_1^2 + \alpha_2^2 \geqq \frac{1}{4}$. In general there is no such solution, for one sees easily that a necessary condition is $\int_{\Gamma^{(k)}} P_3^{(k)} \, d\alpha_1 \, d\alpha_2 = 0$. We therefore modify $P_3^{(k)}$ in order to achieve this condition. To do this, observe first that each point where $0 < f^{(k)} < 1$ in $\Sigma_0^{(k)}$ is interior to one of the other covering neighborhoods, say $\Sigma_0^{(j)}$, and $0 < f^{(j)} < 1$ at this point. We select a neighborhood N of such a point which lies interior to both neighborhoods, and modify $P_3^{(k)}$ in N so that $\int_{\Gamma^{(k)}} P_3^{(k)} \, d\alpha_1 \, d\alpha_2 = 0$. Simultaneously, we modify the corresponding term in $\Sigma_0^{(j)}$ so that $\mathbf{w}^* = \sum_{k=1}^{N} f^{(k)} \mathbf{w}^*$ remains unchanged. With this new function $P_3^{(k)}$ we determine a vector field $\bar{\boldsymbol{\omega}} = (\bar{\omega}_1, \bar{\omega}_2, 0)$ so that curl $\bar{\boldsymbol{\omega}} = P_3^{(k)}$ on $\Gamma^{(k)}$. We begin by defining this field on $\Gamma^{(k)}$. We may, for example, set $\bar{\omega}_2 = 0$ in $\Gamma^{(k)}$, $\bar{\omega}_1 = 0$ on the semicircle $\alpha_1^2 + \alpha_2^2 = 1$, $\alpha_2 < 0$, and determine $\bar{\omega}_1$ in $\Gamma^{(k)}$ by the condition $\partial \bar{\omega}_1 / \partial \alpha_2 = -P_3^{(k)}$. Then in the annular region where $P_3^{(k)} = 0$ we have $\partial \bar{\omega}_1 / \partial \alpha_2 - \partial \bar{\omega}_2 / \partial \alpha_1 = 0$, hence there exists a function $\varphi(\alpha_1, \alpha_2)$ such that $\bar{\boldsymbol{\omega}} = \nabla \varphi$. (The condition $\int_{\Gamma^{(k)}} P_3^{(k)} \, d\alpha_1 \, d\alpha_2 = 0$ shows that φ is necessarily single valued.) We may now extend $\varphi(\alpha_1, \alpha_2)$ to the entire disc $\Gamma^{(k)}$, and define

$$\boldsymbol{\omega} = \bar{\boldsymbol{\omega}} - \nabla \varphi$$

on $\Gamma^{(k)}$. Finally, we extend $\boldsymbol{\omega}$ to the entire cylinder $Z^{(k)}$ by setting $\omega_3 \equiv 0$ and extending ω_1, ω_2 so that $\partial \omega_2 / \partial \alpha_3 = -P_1^{(k)}$ and $\partial \omega_1 / \partial \alpha_2 = P_2^{(k)}$ on $\Gamma^{(k)}$. We may clearly also arrange that $\boldsymbol{\omega} = 0$ on the boundary surface of $Z^{(k)}$. The resulting field $\boldsymbol{\omega}(\alpha_1, \alpha_2, \alpha_3)$ then satisfies the desired relation curl $\boldsymbol{\omega} = P^{(k)}$ on $\Gamma^{(k)}$.

Since Σ_0 is by assumption connected, any covering of the sort described has the property that for any two of the neighborhoods $\Sigma_0^{(j)}$ and $\Sigma_0^{(k)}$, there is a chain of neighborhoods $\Sigma_0^{(j)}, \Sigma_0^{(l)}, \ldots, \Sigma_0^{(k)}$, such that each adjacent pair intersects at points where neither partition function vanishes. Thus, *starting from any given neighborhood* $\Sigma_0^{(1)}$, it is possible by repetition of the above argument to construct, in a finite number of steps, corresponding fields $\boldsymbol{\omega}$ for each of the $\Sigma_0^{(j)}$ with the possible exception of one last one, which we denote by $\Sigma_0^{(N)}$, in which we can not modify the function $P_3^{(N)}$ without affecting the previous construction. But by assumption,

$$\oint_{\Sigma_0}\left(\sum_{j=1}^{N} f^{(j)}\mathbf{w}^*\right)\cdot\mathbf{n}\,dS=\oint_{\Sigma_0}(\mathbf{w}^*\cdot\mathbf{n})\,dS=0,$$

while by the nature of the construction,

$$\oint_{\Sigma_0}\left(\sum_{j=1}^{N-1} f^{(j)}\mathbf{w}^*\right)\cdot\mathbf{n}\,dS=0,$$

hence

$$\oint_{\Sigma_0}(f^{(N)}\mathbf{w}^*)\cdot\mathbf{n}\,dS=0,$$

and we see that no modification of this last function is necessary. Thus a vector field $\boldsymbol{\omega}^{(j)}$ can be defined in a neigborhood of each of the images of the $\Sigma_0^{(j)}$ such that curl $\boldsymbol{\omega}^{(j)}=P^{(j)}$ in $\Gamma^{(j)}$, and $\boldsymbol{\omega}^{(j)}$ vanishes on the boundary of $Z^{(j)}$.

We now transform these fields into a neighborhood of the original surface Σ_0. To do this, set

$$2\,\psi_k^{(j)}=\omega_i^{(j)}\frac{\partial\alpha_i}{\partial x_k},\qquad \boldsymbol{\psi}^{(j)}=(\psi_1^{(j)},\,\psi_2^{(j)},\,\psi_3^{(j)})$$

in a neighborhood of $\Sigma_0^{(j)}$, with summation extended over repeated indices, and set

$$\boldsymbol{\psi}=\sum_{j=1}^{N}\boldsymbol{\psi}^{(j)}.$$

A simple calculation then shows that $\boldsymbol{\psi}$ is a field of the type sought, i.e., curl $\boldsymbol{\psi}=\boldsymbol{\omega}^*$ on Σ_0. Repeating the entire procedure for each component of Σ completes the construction of the field. (We must of course arrange—what is easily done—that the field constructed over each component vanishes over all other.) The estimates on $|\boldsymbol{\psi}|$ and on its derivatives can be obtained directly from the method of construction.

2 a. Estimation of the Dirichlet Integral in a bounded region

We consider first the interior region bounded by a single closed surface Σ. We suppose that Σ is smooth, and that prescribed data $\mathbf{w}^*$ of class $C^{(3)}$ are given on Σ which satisfy condition (3). Applying Lemma 2.1, we obtain a vector field $\boldsymbol{\psi}(\mathbf{x})$ defined throughout space, such that curl $\boldsymbol{\psi} = \mathbf{w}^*$ on Σ. We may assume that $\boldsymbol{\psi}(\mathbf{x})$ and its derivatives up to third order are bounded, the bounds depending only on Σ and on $\mathbf{w}^*$.

Let 2δ be chosen smaller than any of the radii of curvature at points of Σ, and also so small that all points on a normal line of length 2δ originating from an arbitrary point P of Σ are closer to P than to any other boundary point. Then in the shell region $\mathcal{A}_\delta$ determined by the inequality $0 \leqslant s \leqslant \delta$, a non-singular coordinate system is defined by the normals to Σ and the surfaces Σ_s of constant distance s from Σ along the normals, with local surface coordinates on Σ_s obtained from those of Σ by constancy along the normals.

LEMMA 2.2 (Hopf [5]). *For any prescribed $\varepsilon > 0$, there is a real function $\lambda(\mathbf{x})$ with the following properties:*

a) *$\lambda(\mathbf{x})$ is defined in a neighborhood of Σ and has continuous derivatives up to the third order which are bounded, the bounds depending only on Σ, on $\boldsymbol{\psi}(\mathbf{x})$ and on ε.*
b) *$\lambda(\mathbf{x}) = 1$ on Σ, $\lambda(\mathbf{x}) = 0$ outside $\mathcal{A}_\delta$,*
c) *$\nabla \lambda(\mathbf{x}) = 0$ on Σ,*
d) *$|\text{curl } \lambda \boldsymbol{\psi}| < \varepsilon s^{-1}$ throughout $\mathcal{A}_\delta$.*

Such a $\lambda(\mathbf{x})$ can be obtained as a non-negative function of s alone. A possible construction is as follows:

Let $M = \max_{\mathcal{A}_\delta} |\boldsymbol{\psi}|$, $M_1 = \max_{\mathcal{A}_\delta} |\text{curl } \boldsymbol{\psi}|$. Choose $\delta_0 < \delta$ and sufficiently small that $2 M_1 \delta_0 < \varepsilon$, and define

$$\lambda(s) = \frac{\varepsilon}{2M} \int_s^{\delta_0} \frac{1}{\sigma} \left(1 - \frac{\sigma}{\delta_0}\right)^4 d\sigma$$

for all $s \geqslant s_0$. Here s_0 is the unique value of s determined by the conditions

$$\lambda(s) = 1 - \frac{s}{\delta_0}, \quad 0 < s < \delta_0.$$

It is then clear that for $s \leqslant s_0$, $\lambda(s)$ can be defined with continuous third derivatives in such a way that $\lambda(0) = 1$, $\lambda'(0) = 0$, $0 \leqslant |\lambda(s)| \leqslant 1$, and $|\lambda'(s)| < \frac{\varepsilon}{2 M s_0}$ in the interval $0 \leqslant s \leqslant s_0$. In particular, $|\lambda'(s)| < \frac{\varepsilon}{2 M s}$ in the entire range $0 \leqslant s \leqslant \delta_0$, and, using the identity $\text{curl } \lambda \boldsymbol{\psi} = \lambda \text{ curl } \boldsymbol{\psi} - \boldsymbol{\psi} \times \nabla \lambda$, we easily obtain the desired estimates.

Consider now a solution $\mathbf{w}(\mathbf{x})$ of (4) defined in the region G bounded by a smooth closed surface Σ, such that $\mathbf{w}(\mathbf{x}) = \mathbf{w}^*$ on Σ.

THEOREM 2.3 (Leray [8]). *Let $\mathbf{w}^*$ be of class $C^{(3)}$ on Σ, and let $\mathbf{w}(\mathbf{x})$ be a solution of (4) in G such that $w(\mathbf{x}) = \mathbf{w}^*$ on Σ. Then $D[\mathbf{w}] = \int_G |\nabla \mathbf{w}|^2 dV$ is bounded by a quantity which depends only on Σ and on the derivatives of $\mathbf{w}^*$ up to second order, and not on the particular solution considered.*

Proof. Corresponding to the data $\mathbf{w}^*$ we introduce a field $\mathbf{v}(\mathbf{x}) = \operatorname{curl} \lambda \boldsymbol{\psi}$ with the properties indicated in Lemmas 2.1 and 2.2. Let $\boldsymbol{\eta} = \mathbf{w} - \mathbf{v}$. We rewrite (4) in the form

$$\begin{aligned} \triangle \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} - \nabla p &= -\triangle \mathbf{v} + \boldsymbol{\eta} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \boldsymbol{\eta} + \mathbf{v} \cdot \nabla \mathbf{v}, \\ \nabla \cdot \boldsymbol{\eta} &= 0, \end{aligned} \tag{16}$$

multiply the first equation by $\boldsymbol{\eta}$, and integrate over G. We obtain, since $\boldsymbol{\eta} = 0$ on Σ and since $\mathbf{v} = 0$ outside A_δ,

$$\int_G |\nabla \boldsymbol{\eta}|^2 dV = -\int_{A_\delta} \nabla \boldsymbol{\eta} \cdot \nabla \mathbf{v}\, dV + \int_{A_\delta} \mathbf{v} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta}\, dV + \int_{A_\delta} \mathbf{v} \cdot \mathbf{v} \cdot \nabla \boldsymbol{\eta}\, dV. \tag{17}$$

By Lemma 2.2 and the Schwarz Inequality, we find

$$\int_G |\nabla \boldsymbol{\eta}|^2 dV \leqslant K \left(\int_G |\nabla \boldsymbol{\eta}|^2 dV \right)^{\frac{1}{2}} + \varepsilon \left(\int_{A_\delta} \frac{\boldsymbol{\eta}^2}{s^2} dV \right)^{\frac{1}{2}} \left(\int_G |\nabla \boldsymbol{\eta}|^2 dV \right)^{\frac{1}{2}}, \tag{18}$$

where K denotes a majorant for quantities which depend only on Σ, on $\mathbf{w}^*$, and on ε.

Integrating along a normal to Σ, we find

$$\begin{aligned} \int_0^\delta \frac{\boldsymbol{\eta}^2}{s^2} ds &= -\frac{\boldsymbol{\eta}^2}{s} \Big|_0^\delta + 2 \int_0^\delta \frac{\boldsymbol{\eta} \cdot \boldsymbol{\eta}'}{s} ds \\ &\leqslant 2 \left(\int_0^\delta \frac{\boldsymbol{\eta}^2}{s^2} ds \right)^{\frac{1}{2}} \left(\int_0^\delta \boldsymbol{\eta}'^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

hence

$$\int_0^\delta \frac{\boldsymbol{\eta}^2}{s^2} ds \leqslant 4 \int_0^\delta |\nabla \boldsymbol{\eta}|^2 ds,$$

from which we conclude immediately

$$\int_{A_\delta} \frac{\boldsymbol{\eta}^2}{s^2} dV \leqslant K_1 \int_{A_\delta} |\nabla \boldsymbol{\eta}|^2 dV, \tag{19}$$

K_1 depending only on the maximum curvature of Σ (and not on $\mathbf{w}^*$). Inserting this result in (18) and choosing $\varepsilon = 1/2 K_1^{\frac{1}{2}}$, we obtain

$$\int_G |\nabla \boldsymbol{\eta}|^2 dV \leqslant 4 K^2.$$

But from $\mathbf{w} = \boldsymbol{\eta} + \mathbf{v}$ we find

$$\int_G |\nabla \mathbf{w}|^2 dV \leqslant 2 \int_G |\nabla \boldsymbol{\eta}|^2 dV + 2 \int_G |\nabla \mathbf{v}|^2 dV$$

from which Theorem 2.3 follows immediately.

2b. Estimation of the Dirichlet Integral in an exterior region; case of zero outflux

It cannot be expected that an estimate on Dirichlet Integral in an exterior region depending only on prescribed data can be obtained, even for solutions which tend to a limit at infinity and for which this integral is finite, cf. the example in § 6. Under suitable assumptions, however, such an estimate does exist. We consider here two such cases. The region of definition for the solution is assumed to be the exterior $\mathcal{E}$ of a smooth closed surface Σ, Σ to consist of a finite number of connected components carrying prescribed data $\mathbf{w}^*$ of class $C^{(3)}$. Denote by $\mathcal{E}_R$ the region bounded by Σ and by the surface of a sphere Σ_R centered at the origin and with radius R sufficiently large that Σ lies interior to Σ_R. Let $\mathbf{w}_0$ be a prescribed (constant) vector.

Theorem 2.4 (Leray [8]). *Let $\mathbf{w}(\mathbf{x})$ be a solution of (4) in $\mathcal{E}_R$ such that $\mathbf{w}(\mathbf{x}) = \mathbf{w}^*$ on Σ and $\mathbf{w}(\mathbf{x}) = \mathbf{w}_0$ on Σ_R. Then $D[\mathbf{w}] = \int_{\mathcal{E}_R} |\nabla \mathbf{w}|^2 dV$ is bounded by a quantity which depends only on $\mathbf{w}^*$, on Σ, and on $\mathbf{w}_0$ (and not on R).*

Theorem 2.4 is true in any number of dimensions. To prove it, we introduce a new field $\mathbf{u}(\mathbf{x}) = \mathbf{w}(\mathbf{x}) - \mathbf{w}_0$. In terms of $\mathbf{u}(\mathbf{x})$, equations (4) become

$$\triangle \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{w}_0 \cdot \nabla \mathbf{u} - \nabla p = 0 \qquad (20)$$
$$\nabla \cdot \mathbf{u} = 0$$

and the boundary conditions become

$$\mathbf{u} = \mathbf{u}^* = \mathbf{w}^* - \mathbf{w}_0 \text{ on } \Sigma$$
$$\mathbf{u} = 0 \text{ on } \Sigma_R.$$

The remainder of the proof follows very closely the proof of Theorem 2.3. We choose $\mathbf{v}(\mathbf{x}) = \operatorname{curl} \lambda \boldsymbol{\psi}$ so that $\mathbf{v}(\mathbf{x}) = \mathbf{u}^*$ on Σ, $\mathbf{v}(\mathbf{x}) = 0$ outside $\mathcal{A}_\delta$, and set $\boldsymbol{\eta} = \mathbf{u} - \mathbf{v}$, The relations (16) become

$$\triangle \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} - \mathbf{w}_0 \cdot \nabla \boldsymbol{\eta} - \nabla p = -\triangle \mathbf{v} + \boldsymbol{\eta} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \boldsymbol{\eta} + \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{w}_0 \cdot \nabla \mathbf{v},$$
$$\nabla \cdot \boldsymbol{\eta} = 0,$$

and (17) becomes, since $\boldsymbol{\eta} = 0$ on Σ and on Σ_R,

$$\int_{\mathcal{E}_R} |\nabla \boldsymbol{\eta}|^2 dV = -\int_{A_\delta} \nabla \boldsymbol{\eta} \cdot \nabla \mathbf{v}\, dV + \int_{A_\delta} \mathbf{v} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta}\, dV + \int_{A_\delta} \mathbf{v} \cdot \mathbf{v} \cdot \nabla \boldsymbol{\eta}\, dV + \int_{A_\delta} \mathbf{v} \cdot \mathbf{w}_0 \nabla \boldsymbol{\eta}\, dV.$$

We obtain (18) and (19) as before. Combining these inequalities and choosing again $\varepsilon = 1/2K_1^{\frac{1}{2}}$, the required estimate follows.(1)

The sense in which Theorem 2.4 applies to solutions defined throughout the exterior of Σ is illustrated by the following corollary, which we shall apply in § 5.

COROLLARY. *Let $\{R_j\}$ be a sequence of values tending to infinity and let $\mathbf{w}_{R_j}$ denote a sequence of solutions of (4) in $\mathcal{E}_{R_j}$ such that $\mathbf{w}_{R_j} = \mathbf{w}^*$ on Σ and $\mathbf{w}_{R_j} = \mathbf{w}_0$ on Σ_{R_j}. We suppose further that the sequence $\{\mathbf{w}_{R_j}\}$, together with all partial derivatives up to first order, converges at all points in the exterior $\mathcal{E}$ of Σ, uniformly in any compact subregion of $\mathcal{E}$, to a vector field $\mathbf{w}(\mathbf{x})$. Then $D[\mathbf{w}] = \int_{\mathcal{E}} |\nabla \mathbf{w}|^2 dV \leqslant M < \infty$, where M depends only on $\mathbf{w}^*$, on Σ, and on $\mathbf{w}_0$.*

Let R_0 be arbitrary but fixed. By Theorem 2.4, we have

$$\int_{\mathcal{E}_{R_0}} |\nabla \mathbf{w}_{R_j}|^2 dV \leqslant \int_{\mathcal{E}_{R_j}} |\nabla \mathbf{w}_{R_j}|^2 dV < M < \infty \quad \text{for all } R_j > R_0.$$

By uniform convergence in $\mathcal{E}_{R_0}$, there follows $\int_{\mathcal{E}_{R_0}} |\nabla \mathbf{w}|^2 dV < M$. Since R_0 is arbitrary, we must have $\int_{\mathcal{E}} |\nabla \mathbf{w}|^2 dV \leqslant M$, q.e.d.

(1) *Note added in proof:* In order to obtain an indication of the suitability of this method for finding energy integral estimates in a practical case, Mr. Paul L. Patterson has used the method to estimate the Dirichlet Integral for the explicitly known solution of (6) which vanishes on a sphere of radius a and which tends to a limit $\mathbf{w}_0$ at ∞. From the choice,

$$\boldsymbol{\psi} = (0, -\mathbf{w}_0 x_3, 0),$$
$$\delta_0 = 4a,$$
$$\lambda = \delta_0^{-4} (\delta_0 - s)^3 (3s + \delta_0),$$

he has obtained the estimate, $\int_{\mathcal{E}} |\nabla \mathbf{w}|^2 dV < 31\pi a \mathbf{w}_0^2$. The actual value, computed from the known solution, is $6\pi a \mathbf{w}_0^2$. We remark however, that in the non-linear case, the estimate tends rapidly to infinity as the Reynolds' number increases. We do not know in what sense this situation reflects reality.

The second case we consider is that in which the solution is defined throughout the exterior $\mathcal{E}$ of Σ and tends to a limit at a suitable rate.

THEOREM 2.5. *Let* $\mathbf{w}(\mathbf{x})$ *be a solution of* (4) *in* $\mathcal{E}$, *and let* $\mathbf{w}(\mathbf{x}) = \mathbf{w}^*$ *on* Σ. *Suppose there exists an* $\varepsilon > 0$ *such that as* $r \to \infty$, $|\mathbf{w}(\mathbf{x}) - \mathbf{w}_0| < C r^{-\frac{1}{2}-\varepsilon}$ *for some constant* C *and constant vector* $\mathbf{w}_0 \neq 0$. *Then* $D[\mathbf{w}] = \int_{\mathcal{E}} |\nabla \mathbf{w}|^2 dV < M < \infty$, *where* M *depends only on* Σ, *on* $\mathbf{w}^*$, *and on* $\mathbf{w}_0$.

The main burden for the proof of Theorem 2.5 rests on estimates given in [2] for asymptotic behavior of solutions of (4) in $\mathcal{E}$. The demonstrations are too complicated to reproduce here. It is shown in [2] that the hypotheses of Theorem 2.5 imply, in particular, the estimates

$$\oint_{\Sigma_R} |\mathbf{u}|^2 dS \to 0, \quad \oint_{\Sigma_R} |\nabla \mathbf{u}|^2 dS \to 0, \quad \oint_{\Sigma_R} p^2 dS \to 0$$

for $\mathbf{u}(\mathbf{x}) = \mathbf{w}(\mathbf{x}) - \mathbf{w}_0$ and $p(\mathbf{x})$ as $R \to \infty$, provided $p(\mathbf{x})$ is modified by a suitable additive constant. Using these estimates, we introduce a sphere Σ_R and apply the reasoning in the proof of Theorem 2.4 to $\mathbf{w}(\mathbf{x})$ in the region $\mathcal{E}_R$. The only change that occurs is in the surface integral over Σ_R, which no longer vanishes but assumes the value

$$\oint_{\Sigma_R} \left[\mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial n} - |\mathbf{u}|^2 (\mathbf{u} \cdot \mathbf{n}) - |\mathbf{u}|^2 (\mathbf{w}_0 \cdot \mathbf{n}) - p(\mathbf{u} \cdot \mathbf{n}) \right] dS.$$

The above estimates show that this term vanishes in the limit as $R \to \infty$. This fact established, the proof of Theorem 2.5 then coincides with that of Theorem 2.4.

2 c. Estimation of the Dirichlet Integral in an exterior region; general case

We show now that in the case of an exterior region in three or more dimensions, the conditions (3) $\oint_{\Sigma} (\mathbf{w}^* \cdot \mathbf{n})\, dS = 0$ which we have imposed on the boundary data can be relinquished, and a bound on Dirichlet Integral obtained nevertheless for any possible solution of a suitable class, provided that the net outflux through Σ is sufficiently small. For simplicity, we restrict the discussion to the three dimensional case in which Σ consists of a single connected component. The method fails in two dimensions, but in every other respect these restrictions are unnecessary. Let Q be the net flux across Σ,

$$Q = \oint_{\Sigma} (\mathbf{w}^* \cdot \mathbf{n})\, dS. \tag{21}$$

Then since $\nabla \cdot \mathbf{w} = 0$ in $\mathcal{E}$, Q is necessarily the net flux across any surface Σ_R enclosing Σ, and hence if $Q \neq 0$ the boundary condition $\mathbf{w} = \mathbf{w}_0$ on Σ_R cannot be fulfilled by any solution. The simplest choice available to us is obtained by adjoining to $\mathbf{w}_0$ the velocity field of a potential source flow whose singularity lies interior to Σ.

THEOREM 2.6. *Let* $\mathbf{w}(\mathbf{x})$ *be a solution of* (4) *in* $\mathcal{E}_R$ *such that* $\mathbf{w}(\mathbf{x}) = \mathbf{w}^*$ *on* Σ *and* $\mathbf{w}(\mathbf{x}) = \mathbf{w}_0 - \frac{Q}{4\pi} \nabla \left(\frac{1}{r}\right)$ *on* Σ_R. *Then if Q is sufficiently small (depending only on* Σ*), the Dirichlet Integral* $D[\mathbf{w}] = \int_{\mathcal{E}_R} |\nabla \mathbf{w}|^2 dV$ *is bounded by a quantity which depends only on* $\mathbf{w}^*$, *on* Σ, *on* $\mathbf{w}_0$ *and on* Q *(and not on* R*).*

We choose the origin of coordinates interior to Σ, and let $\gamma(\mathbf{x}) = -\frac{Q}{4\pi} \nabla \left(\frac{1}{r}\right)$. Then on Σ, the data $\mathbf{u}^* - \boldsymbol{\gamma}$ satisfy condition (3), hence for any $\varepsilon > 0$ there exists a field $\mathbf{h}(\mathbf{x}) = \operatorname{curl} \lambda \boldsymbol{\psi}$ of the type described in Lemmas 2.1 and 2.2, such that $\mathbf{h}(\mathbf{x}) = \mathbf{u}^* - \boldsymbol{\gamma}$ on Σ. Let $\mathbf{v}(\mathbf{x}) = \boldsymbol{\gamma} + \mathbf{h}$ in $\mathcal{E}_R$. Then $\mathbf{v}(\mathbf{x}) = \mathbf{u}^*$ on Σ, and setting $\boldsymbol{\eta} = \mathbf{u} - \mathbf{v}$, we have $\boldsymbol{\eta} = 0$ on Σ and on Σ_R. From (20) we find

$$\triangle \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} - \mathbf{w}_0 \cdot \nabla \boldsymbol{\eta} - \nabla p = -\triangle \mathbf{v} + \boldsymbol{\eta} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \boldsymbol{\eta} + \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{w}_0 \cdot \nabla \mathbf{v},$$
$$\nabla \cdot \boldsymbol{\eta} = 0.$$

Multiplying by $\boldsymbol{\eta}$ and integrating over $\mathcal{E}_R$ yields, since $\triangle \boldsymbol{\gamma} = 0$,

$$D[\boldsymbol{\eta}] = \int_{A_\delta} \nabla \boldsymbol{\eta} \cdot \nabla \mathbf{h} \, dV + \int_{\mathcal{E}_R} \mathbf{v} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} \, dV$$
$$- \int_{\mathcal{E}_R} \mathbf{v} \cdot \mathbf{v} \cdot \nabla \boldsymbol{\eta} \, dV - \int_{\mathcal{E}_R} \mathbf{v} \cdot \mathbf{w}_0 \cdot \nabla \boldsymbol{\eta} \, dV.$$

We study the right hand side term by term:

$$\left| \int_{A_\delta} \nabla \boldsymbol{\eta} \cdot \nabla \mathbf{h} \, dV \right| \leqslant \left(\int_{A_\delta} |\nabla \mathbf{h}|^2 dV \right)^{\frac{1}{2}} \left(\int_{\mathcal{E}_R} |\nabla \boldsymbol{\eta}|^2 dV \right)^{\frac{1}{2}}$$

$$\int_{\mathcal{E}_R} \mathbf{v} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} \, dV = \int_{A_\delta} \mathbf{h} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} \, dV + \int_{\mathcal{E}_R} \boldsymbol{\gamma} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} \, dV,$$

$$\left| \int_{A_\delta} \mathbf{h} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} \, dV \right| \leqslant \varepsilon K_1 \int_{A_\delta} |\nabla \boldsymbol{\eta}|^2 dV$$

(cf. the proof of Theorem 2.3), and an integration by parts, using the relations $\nabla \cdot \boldsymbol{\eta} = 0$, $\boldsymbol{\gamma} = -\frac{Q}{4\varphi} \nabla \left(\frac{1}{r}\right)$, yields

$$\int_{\mathcal{E}_R} \boldsymbol{\gamma} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} \, dV = \frac{Q}{4\pi} \int_{\mathcal{E}_R} \frac{1}{r} (\triangle \boldsymbol{\eta} \cdot \nabla^* \boldsymbol{\eta}) \, dV,$$

where $\nabla^* \boldsymbol{\eta}$ denotes the transpose of the matrix $\nabla \boldsymbol{\eta}$. Also,

$$\left|\int_{\mathcal{E}_R} \mathbf{v}\cdot\mathbf{v}\cdot\nabla\boldsymbol{\eta}\,dV\right| \leqslant 2\int_{\mathcal{E}_R} \boldsymbol{\gamma}^2|\nabla\boldsymbol{\eta}|\,dV + 2\int_{A_\delta} \mathbf{h}^2|\nabla\boldsymbol{\eta}|\,dV$$
$$\leqslant K\left(\int_{\mathcal{E}_R}|\nabla\boldsymbol{\eta}|^2\,dV\right)^{\frac{1}{2}},$$

K depending only on prescribed data, since $\boldsymbol{\gamma}^4$ is integrable over the exterior of Σ. Similar reasoning shows that

$$\left|\int_{\mathcal{E}_R} \mathbf{v}\cdot\mathbf{w}_0\cdot\nabla\boldsymbol{\eta}\,dV\right| \leqslant K\left(\int_{\mathcal{E}_R}|\nabla\boldsymbol{\eta}|^2\,dV\right)^{\frac{1}{2}}.$$

Collecting these estimates we obtain

$$D[\boldsymbol{\eta}] \leqslant K(D[\boldsymbol{\eta}])^{\frac{1}{2}} + \varepsilon K_1 D[\boldsymbol{\eta}] + \frac{Q}{4\pi r_0} D[\boldsymbol{\eta}], \tag{22}$$

where r_0 is the shortest distance from the origin to Σ. From (22) we see that an estimate for $D[\eta]$, independent of R and the particular solution considered, can be obtained whenever $Q < 4\pi r_0$. Using this estimate, we can find a bound for $D[\mathbf{u}]$ from the inequality

$$D[\mathbf{u}] \leqslant 2D[\boldsymbol{\eta}] + 2D[\mathbf{v}] \leqslant 2D[\boldsymbol{\eta}] + 4D[\mathbf{h}] + 4D[\boldsymbol{\gamma}]$$

and the fact that $\boldsymbol{\gamma}$ has finite Dirichlet Integral over the region exterior to Σ.

We remark finally that *the Corollary to Theorem 2.4 and the conclusions of Theorem 2.5 are valid also in this case.*

3. A priori estimation of the solution

The estimates on Dirichlet Integral obtained in the preceeding section are applied here to find pointwise estimates for any possible solution, depending only on prescribed data. We base these estimates on general properties of the Green's Tensor associated with the linearized equations

$$\triangle\mathbf{w} - \nabla p = 0 \tag{6}$$
$$\triangle\cdot\mathbf{w} = 0$$

in a bounded region. We consider in detail only the three dimensional case. It seems certain that similar estimates hold for two dimensional solutions. but the necessary estimates on the Green's Tensor have not yet been formally verified in this case. In higher than three dimensions, there appears to be an intrinsic difficulty in the method.

THEOREM 3.1 (Odqvist [11, p. 365]). *For any interior region $\mathcal{G}$ bounded by a sufficiently smooth closed surface Σ, there is a unique tensor $\mathbf{G}(\mathbf{x}, \mathbf{y}) = (G_{ij})$, such that each column vector, considered as function of* $\mathbf{x}$, *is a solution of* (6) *for all* $\mathbf{x}$ *in* $\mathcal{G}$ *with* $\mathbf{x} \neq \mathbf{y}$, *such that* $\mathbf{G}(\mathbf{x}, \mathbf{y}) = 0$ *for* $\mathbf{x}$ *on* Σ *and* $\mathbf{y}$ *in* $\mathcal{G}$, *and such that at* $\mathbf{x} = \mathbf{y}$, $\mathbf{G}(\mathbf{x}, \mathbf{y})$ *has the singularity of the fundamental solution tensor* $\boldsymbol{\chi}(\mathbf{x}, \mathbf{y})$ *defined in* (7). *Throughout* $\mathcal{G} + \Sigma$, *the tensor* $\mathbf{G}(\mathbf{x}, \mathbf{y})$ *and the (suitably normalized) associated pressure vector* $\mathbf{P}(\mathbf{x}, \mathbf{y})$ *admit, as function of* $\mathbf{x}$, *uniformly in* $\mathbf{y}$, *the estimates*

$$|\mathbf{G}(\mathbf{x}, \mathbf{y})| < \frac{C}{r_{\mathbf{x}\mathbf{y}}} \qquad |\mathbf{G}(\mathbf{x}, \mathbf{y}) - \mathbf{G}(\mathbf{x}', \mathbf{y})| < C \frac{r_{\mathbf{x}\mathbf{x}'}^{1-\varepsilon}}{r_0^2} \tag{23}$$

$$|\nabla \mathbf{G}(\mathbf{x}, \mathbf{y})| < \frac{C}{r_{\mathbf{x}\mathbf{y}}^2}, \qquad |\mathbf{P}(\mathbf{x}, \mathbf{y})| < \frac{C}{r_{\mathbf{x}\mathbf{y}}^2}$$

$$|\nabla (\mathbf{G}(\mathbf{x}, \mathbf{y}) - \mathbf{G}(\mathbf{x}', \mathbf{y}))| < C \frac{r_{\mathbf{x}\mathbf{x}'}^{1-\varepsilon}}{r_0^3} \qquad |\mathbf{P}(\mathbf{x}, \mathbf{y}) - \mathbf{P}(\mathbf{x}', \mathbf{y})| < C \frac{r_{\mathbf{x}\mathbf{x}'}^{1-\varepsilon}}{r_0^3}$$

for any $\varepsilon > 0$, *provided* $\mathbf{x}$ *and* $\mathbf{x}'$ *lie exterior to a sphere of radius* r_0 *centered at* $\mathbf{y}$. $\mathbf{G}(\mathbf{x}, \mathbf{y})$ *has the symmetry property*, $G_{ij}(\mathbf{x}, \mathbf{y}) = G_{ji}(\mathbf{y}, \mathbf{x})$.

We shall assume Theorem 3.1 and use it in what follows. Using the Green's Tensor, any solution of (6) in $\mathcal{G}$ which equals $\mathbf{w}^*$ on Σ admits the representation

$$\mathbf{w}(\mathbf{x}) = \oint_\Sigma \mathbf{w}^* \cdot T\,\mathbf{G}\,dS \tag{24}$$

$$p(\mathbf{x}) = \oint_\Sigma \mathbf{w}^* \cdot T\,\mathbf{P}\,dS,$$

where the "pressure" used in forming the expression $T\mathbf{P}$ is the identically vanishing function. Conversely, the representation (24) leads to the construction of a solution in $\mathcal{G}$ which assumes boundary data $\mathbf{w}^*$ on Σ. We have, in fact:

THEOREM 3.2 (Odqvist [11]). *Let* $\mathbf{w}^*(x)$ *be boundary data of class* $C^{(3)}$ *and satisfying condition* (3) *on the smooth closed surface* Σ. *Then there is a unique solution* $\mathbf{w}(\mathbf{x})$ *of the linearized equations* (6) *in the region* $\mathcal{G}$ *bounded by* Σ, *such that* $\mathbf{w}(\mathbf{x}) = \mathbf{w}^*$ *on* Σ. *Throughout* $\mathcal{G} + \Sigma$, $\mathbf{w}(x)$ *and the suitably normalized associated pressure* $p(\mathbf{x})$ *satisfy, for any* $\varepsilon > 0$, *the inequalities*

$$|\mathbf{w}(\mathbf{x})| < C, \quad |\nabla \mathbf{w}(\mathbf{x})| < C, \quad |p(\mathbf{x})| < C, \tag{25}$$

$$|\nabla \mathbf{w}(\mathbf{x}) - \nabla \mathbf{w}(\mathbf{y})| < C r_{\mathbf{x}\mathbf{y}}^{1-\varepsilon}, \qquad |p(\mathbf{x}) - p(\mathbf{y})| < C r_{\mathbf{x}\mathbf{y}}^{1-\varepsilon},$$

where C depends only on $\mathbf{w}^*$, *on* Σ, *and on* ε. *At any interior point of* $\mathcal{G}$, *the derivatives of* $\mathbf{w}(x)$ *of all orders are bounded, depending only on* Σ, *on* $\mathbf{w}^*$, *and on distance from the point to* Σ.

Proof. The function $\mathbf{w}(\mathbf{x})$ defined by (24) is clearly a solution of (6) in $\mathcal{G}$. Applying Lemma 2.1, we may extend $\mathbf{w}^*$ into the interior $\mathcal{G}$ as a divergence-free vector field $\mathbf{v}(\mathbf{x})$ with bounded second derivatives, depending only on the boundary data. Let $\boldsymbol{\eta} = \mathbf{w} - \mathbf{v}$. Transforming (6) by the identities (12) yields, for $\mathbf{x}$ in $\mathcal{G}$,

$$\boldsymbol{\eta}(\mathbf{x}) = \int \mathbf{G} \cdot \triangle \mathbf{v}\, dV$$

$$p(\mathbf{x}) = \int \mathbf{P} \cdot \triangle \mathbf{v}\, dV.$$

The assumption of boundary data by $\mathbf{w}(\mathbf{x})$ and the estimates (25) then follow from the estimates (23) of Odqvist by standard potential-theoretic arguments. To obtain the interior estimates, we represent $\mathbf{w}(\mathbf{x})$ by means of the fundamental solution tensor (7), which is explicitly known,

$$\mathbf{w}(\mathbf{x}) = \oint_{\Sigma} (\mathbf{w}^* \cdot T\boldsymbol{\chi} - \boldsymbol{\chi} \cdot T\mathbf{w}^*). \tag{26}$$

Interior bounds then follow directly by differentiation of (26) under the integral sign and use of the estimates (15).

We shall need also the following lemma, due to Payne and Weinberger [13]:

LEMMA 3.3. *Let* $\boldsymbol{\gamma}(\mathbf{x})$ *be a vector valued function defined and piecewise continuously differentiable* (1) *in a neighborhood* N *of infinity. Then there exists a constant vector* γ_0 *such that for any sphere* Σ_R *which lies, together with its exterior* E_R, *in* N,

$$\frac{1}{R} \oint_{\Sigma_R} |\boldsymbol{\gamma} - \boldsymbol{\gamma}_0|^2\, dS \leq \int_{E_R} |\nabla \boldsymbol{\gamma}|^2\, dV. \tag{27}$$

From Lemma 3.3 we can obtain the following result:

LEMMA 3.4. *Let* $\boldsymbol{\gamma}(\mathbf{x})$ *be a vector valued function defined and piecewise continuously differentiable throughout space. Then there exists a constant vector* $\boldsymbol{\gamma}_0$ *such that, for any choice of the origin of coordinates,*

(1) That is, continuous in N and continuously differentiable except on a finite number of smooth surfaces.

$$\int_E \frac{|\gamma-\gamma_0|^2}{r^2}\,dV \leqslant 4\int_E |\nabla\gamma|^2\,dV, \tag{28}$$

the integration being taken over the entire space E.

Proof. We may assume that $\int_E |\nabla\gamma|^2\,dV < \infty$, for otherwise Lemma 3.4 is trivially correct. Let γ_0 be the vector whose existence is asserted in Lemma 3.3. Integration by parts in a sphere V_R of radius R and boundary Σ_R yields

$$\int_{V_R} \frac{|\gamma-\gamma_0|^2}{r^2}\,dV = -\int_{V_R} \frac{1}{r}\nabla r\cdot\nabla(\gamma-\gamma_0)^2\,dV + \frac{1}{R}\oint_{\Sigma_R}|\gamma-\gamma_0|^2\,dS. \tag{29}$$

Using Schwarz' Inequality and Lemma 3.3 yields

$$\int_{V_R} \frac{|\gamma-\gamma_0|^2}{r^2}\,dV \leqslant 2\left(\int_{V_R}\frac{|\gamma-\gamma_0|^2}{r^2}\,dV\right)^{\frac{1}{2}}\left(\int_{V_R}|\nabla\gamma|^2\,dV\right)^{\frac{1}{2}} + \varepsilon(R),$$

where $\varepsilon(R)\to 0$ as $R\to\infty$. Lemma 3.4 then follows by a passage to the limit.

Remark 1. In the special case that the function $\gamma(\mathbf{x})$ vanishes outside a compact set, the surface integral in (29) vanishes for sufficiently large R, and it is unnecessary to apply Lemma 3.3 in the proof.

Remark 2. In case it is known that γ tends to a limit at infinity, this limit necessarily coincides with γ_0. For otherwise we would have $R^{-1}\oint_{\Sigma_R}|\gamma-\gamma_0|^2\,dS\to\infty$ as $R\to\infty$, a contradiction.

3 a. Estimation of the solution in a bounded region

We assume as given a solution $\mathbf{w}(\mathbf{x})$ in a finite region $\mathcal{G}$, which takes on values $\mathbf{w}^*$ on the boundary Σ of $\mathcal{G}$. It is then possible to extend $\mathbf{w}^*$ to the exterior of $\mathcal{G}$ in such a way that the extension vanishes outside a compact set and has finite Dirichlet Integral, depending only on $\mathbf{w}^*$. By Theorem 2.3, $\mathbf{w}(\mathbf{x})$ has interior to $\mathcal{G}$ a finite Dirichlet Integral, depending only on $\mathbf{w}^*$. Denote the sum of these two integrals by D. Using the identities (13, 14), we find for $\mathbf{w}(\mathbf{x})$, $p(\mathbf{x})$ the representation,

$$\mathbf{w}(\mathbf{x}) = \oint_\Sigma \mathbf{w}^*\cdot T\mathbf{G}\,dS + \int_{\mathcal{G}} \mathbf{G}\cdot\mathbf{w}\cdot\nabla\mathbf{w}\,dV, \tag{30}$$

$$p(\mathbf{x}) = \oint_\Sigma \mathbf{w}^*\cdot T\mathbf{P}\,dS + \int_{\mathcal{G}} \mathbf{P}\cdot\mathbf{w}\cdot\nabla\mathbf{w}\,dV.$$

The first term on the right is a solution of the linearized system (6) which assumes the boundary data w^*. This solution, which we denote by $w_1(x)$, satisfies the inequalities (25) of Theorem 3.2. Using Theorem 3.1 and Lemma 3.4, we find (cf. the remarks at the end of § 1),

$$|w(x)| \leqslant |w_1(x)| + C\left(\int_G \frac{w^2}{r^2}\,dV\right)^{\frac{1}{2}}\left(\int_G |\nabla w|^2\,dV\right)^{\frac{1}{2}} \leqslant C + D \leqslant C,$$

constants depending only on prescribed data. Further,

$$|\nabla w(x)| \leqslant |\nabla w_1(x)| + \int_G \frac{|w(y)|\,|\nabla w(y)|}{r_{xy}^2}\,dV_y \leqslant C + C\int_G \frac{|\nabla w(y)|}{r_{xy}^2}\,dV_y. \tag{31}$$

Multiplying by r_{xz}^{-2} and integrating with respect to x, we obtain

$$\int_G \frac{|\nabla w(x)|}{r_{zx}^2}\,dV_x \leqslant C + C\int_G \frac{|\nabla w(y)|}{r_{zy}}\,dV_y \leqslant C + C\left(\int_G |\nabla w|^2\right)^{\frac{1}{2}} dV \leqslant C$$

by Schwarz' Inequality. Inserting this result in (31), we obtain

$$|\nabla w(x)| \leqslant C.$$

Placing these results in the relation (30) for the pressure and using Theorems 3.1 and 3.2, we obtain immediately

$$|p| \leqslant C,$$

and the estimates (23) for the Green's Tensor imply, by standard potential theoretic arguments,

$$|\nabla w(x) - \nabla w(y)| < C\,r_{xy}^{1-\varepsilon} \qquad |p(x) - p(y)| < C\,r_{xy}^{1-\varepsilon}.$$

Collecting these results, we obtain:

THEOREM 3.5 (Leray [8]). *For any bounded region $\mathcal{G}$, the estimates* (25) *of Theorem* 3.2 *are valid also for any solution of the Navier-Stokes equations* (4), *with constants independent of the particular solution considered.*

3 b. Estimation of the solution in an exterior region

We base the estimates for an exterior region on the following general property of solutions of the Navier-Stokes equations:

THEOREM 3.6. *Let* $\mathbf{w}(\mathbf{x})$ *be a solution of* (4) *in a region* $\mathcal{G}$ (*finite or infinite*) *for which the Dirichlet Integral is finite,* $\int_{\mathcal{G}} |\nabla \mathbf{w}|^2 dV < \infty$. *Suppose it is possible to extend* $\mathbf{w}(\mathbf{x})$ *to a piecewise continuously differentiable vector field defined in the entire space, possessing finite Dirichlet Integral and tending*(1) *to a limit* $\mathbf{w}_0$ *as* $\mathbf{x} \to \infty$. *Then in every compact subregion of* $\mathcal{G}$, $\mathbf{w}(\mathbf{x})$ *and the associated pressure* $p(\mathbf{x})$ *satisfy the inequalities* (25) *of Theorem* (3.2), *with constants depending only on the Dirichlet Integral of* $\mathbf{w}(\mathbf{x})$ *and of its extension, on distance to the boundary* Σ *of* $\mathcal{G}$, *and on* $\mathbf{w}_0$. *The estimate for* $p(\mathbf{x})$ *depends also on the choice of an additive constant, and on the particular subregion considered.*

Proof. Let $\mathbf{x}$ be a point of $\mathcal{G}$, let $4d$ be the distance from $\mathbf{x}$ to the boundary. Describe spheres S_1 and S_2 of radii d and $2d$, respectively, with $\mathbf{x}$ as center. We then have the representation

$$\mathbf{w}(\mathbf{x}) = \oint_{S_2} \mathbf{w} \cdot T\mathbf{G}\, dS + \int_{V_2} \mathbf{G} \cdot \mathbf{w} \cdot \nabla \mathbf{w}\, dV, \tag{32}$$

where V_2 is the interior of S_2 and $\mathbf{G}(\mathbf{x}, \mathbf{y})$ is the Green's Tensor associated with the system (6) in V_2. We rewrite (32) in the form

$$\mathbf{u}(\mathbf{x}) = \oint_{S_2} \mathbf{u} \cdot T\mathbf{G}\, dS + \int_{V_2} \mathbf{G} \cdot \mathbf{u} \cdot \nabla \mathbf{u}\, dV + \int_{V_2} \mathbf{G} \cdot \mathbf{w}_0 \cdot \nabla \mathbf{u}\, dV,$$

where we have set $\mathbf{u}(\mathbf{x}) = \mathbf{w}(\mathbf{x}) - \mathbf{w}_0$.

For $\mathbf{x}$ interior to S_1, $|T\mathbf{G}(\mathbf{x}, \mathbf{y})| < Cd^{-2}$ on S_2. Thus,

$$\left(\oint_{S_2} \mathbf{u} \cdot T\mathbf{G}\, dS\right)^2 \leqslant \oint_{S_2} \mathbf{u}^2\, dS \oint_{S_2} (T\mathbf{G})^2\, dS \leqslant Cd^{-2} \oint_{S_2} (\mathbf{w} - \mathbf{w}_0)^2\, dS.$$

By Lemma 3.3, $\oint_{S_2} (\mathbf{w} - \mathbf{w}_0)^2\, dS < 2dD$, where D is the sum of Dirichlet Integrals of $\mathbf{w}(\mathbf{x})$ and its extension. Also, by Theorem 3.1,

$$\left(\int_{V_2} \mathbf{G} \cdot \mathbf{u} \cdot \nabla \mathbf{u}\, dV\right)^2 \leqslant C \int_{V_2} \frac{u^2}{r^2}\, dV \int_{V_2} |\nabla \mathbf{w}|^2\, dV \leqslant C$$

by Lemma 3.4. Finally,

$$\left(\int_{V_2} \mathbf{G} \cdot \mathbf{w}_0 \cdot \nabla \mathbf{u}\, dV\right)^2 \leqslant C |\mathbf{w}_0| \int_{V_2} |\nabla \mathbf{w}|^2\, dV \leqslant C.$$

Collecting these inequalities, we obtain an estimate of the form

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(1) It is sufficient that $\mathbf{w} \to \mathbf{w}_0$ in the sense of Lemma 3.3.

$$|\mathbf{w}(\mathbf{x})| \leqslant C\left(D + \sqrt{\frac{D}{d}} + \sqrt{(D|\mathbf{w}_0|)} + |\mathbf{w}_0|\right)$$

which establishes the interior bound on $|\mathbf{w}(\mathbf{x})|$.

Now we permit $\mathbf{x}$ to vary within S_1. We observe first that $\oint_{S_2} \mathbf{u}\cdot T\,\mathfrak{G}\,dS$ can be differentiated under the sign for $\mathbf{x}$ in V_1, resulting in a uniformly bounded function, depending only on the bound for $|\mathbf{u}|$ on S_2. We then obtain from (32),

$$|\nabla \mathbf{w}(\mathbf{x})| \leqslant C + C\int_{V_2} \frac{|\nabla \mathbf{w}|}{r^2}\,dV \tag{33}$$

for all $\mathbf{x}$ in V_1. Multiplication by $r_{\mathbf{xz}}^{-2}$ and integration over V_1 yields

$$\int_{V_1} \frac{|\nabla \mathbf{w}|}{r^2}\,dV \leqslant C + C\int_{V_2} |\nabla \mathbf{w}|^2\,dV \leqslant C.$$

Repeating this reasoning with V_1 replaced by V_2 and V_2 by V_3 shows that

$$\int_{V_2} \frac{|\nabla \mathbf{w}|}{r^2}\,dV \leqslant C.$$

Insertion in (33) yields the bound on $|\nabla \mathbf{w}|$. The estimates (23), together with (32) and the corresponding relation for $p(\mathbf{x})$, imply by the usual methods of potential theory the remaining estimates (25). The function $p(\mathbf{x})$ is, however, determined in a given sphere only up to an additive constant. If determinations in overlapping spheres are to coincide, one of them must be adjusted by a suitable constant. Thus (for example), along any path of length L covered by spheres in $\mathcal{G}$ of radius bounded from zero, the determination of $p(\mathbf{x})$ may conceivably change by an amount CL. In particular, Theorem 3.6 does not provide a uniform bound for $p(\mathbf{x})$ in an infinite region.

We study next a particular case. The region $\mathcal{G}$ has as boundary component a smooth connected closed surface Σ. It is assumed that on Σ, $\mathbf{w}$ takes on data $\mathbf{w}^*$ of class $C^{(3)}$. The same assumptions on Dirichlet Integral are made as in Theorem 3.6. Let $\mathcal{A}_0$ be a neighborhood of Σ in $\mathcal{G}$ which contains no boundary points and which is bounded by Σ and by a smooth closed surface Σ_0 in $\mathcal{G}$. Let $\mathcal{A}_1$ be another such neighborhood, such that Σ_1 lies in $\mathcal{A}_0$. On Σ_0, estimates on $|\mathbf{w}|$ are available from Theorem 3.6. Let $\mathbf{x}$ lie in $\mathcal{A}_1$. We may write

$$\mathbf{w}(\mathbf{x}) = \oint_{\Sigma+\Sigma_0} \mathbf{w}\cdot T\,\mathfrak{G}\,dS + \int_{\mathcal{A}_0} \mathfrak{G}\cdot\mathbf{w}\cdot\nabla\mathbf{w}\,dV,$$

where $\mathfrak{G}(\mathbf{x},\mathbf{y})$ is the Green's Tensor for the system (6) relative to $\mathcal{A}_0$.

For $\mathbf{x}$ in $\mathcal{A}_1$, the integral over Σ_0 can be differentiated under the sign and has bounded derivatives up to second order. The integral over Σ may be studied by the method of proof of Theorem 3.2, and leads to estimates of the form (25), valid up to Σ. Finally, we apply to the volume integral the analysis in the proof of Theorem 3.5. We obtain:

THEOREM 3.7. *Under the hypotheses of Theorem* 3.6, *suppose* $\mathcal{G}$ *has as boundary component a smooth closed surface* Σ *carrying data* $\mathbf{w}^*$ *of class* $C^{(3)}$. *Then the estimates* (25) *are valid for* $\mathbf{w}(\mathbf{x})$ *in a neighborhood of* Σ *and up to* Σ *itself.*

LEMMA 3.8. *Let* Σ *consist of a finite number of smooth closed components carrying data* $\mathbf{w}^*$ *of class* $C^{(3)}$. *Let* $\mathcal{E}_R$ *be the region bounded by* Σ *and by a sphere* Σ_R *about the origin of large radius R. Let* $\mathbf{w}_0$ *be a prescribed constant vector, and let* $\{\mathbf{w}_R(\mathbf{x})\}$ *denote a family of solutions, depending on R, such that,*

i) $\mathbf{w}_R(\mathbf{x}) = \mathbf{w}^*$ *on* Σ, $\mathbf{w}_R(\mathbf{x}) = \mathbf{w}_0 - \frac{Q}{4\pi}\nabla\left(\frac{1}{r}\right)$ *on* Σ_R ($Q =$ const.),

ii) $\{\mathbf{w}_R(\mathbf{x})\}$ *have Dirichlet Integrals uniformly bounded in R. Then there is a subsequence of the* $\{\mathbf{w}_R(\mathbf{x})\}$ *which converges uniformly in the closure of any fixed region* $\mathcal{E}_{R_0}$, *together with its derivatives up to first order, to a solution of* (4) *in the exterior* $\mathcal{E}$ *of* Σ. *The Dirichlet Integral of the limit solution* $\mathbf{w}(\mathbf{x})]$ *has the same bound, and* $[\mathbf{w}(\mathbf{x}), p(\mathbf{x})$ *satisfy the estimates* (25) *uniformly throughout* $\mathcal{E}$.

To prove Lemma 3.8, we note that the $\{\mathbf{w}_R(\mathbf{x})\}$ can obviously be extended to all space with Dirichlet Integrals bounded independent of R. Theorems 3.6 and 3.7 then show that the $\{\mathbf{w}_R(\mathbf{x})\}$ are equicontinuous and have equicontinuous derivatives of first order. Hence there is a subsequence which converges, uniformly together with its derivatives of first order, in any $\mathcal{E}_{R_0}$. But for any fixed sphere V with surface S in $\mathcal{E}_{R_0}$, we have

$$\mathbf{w}_R(\mathbf{x}) = \oint_S \mathbf{w}_R \cdot T\,\mathfrak{G}\,dS + \int_V \mathfrak{G}\cdot\mathbf{w}_R\cdot\nabla\,\mathbf{w}_R\,dV$$

and the uniform convergence then shows that this relation holds also for the limit field $\mathbf{w}(\mathbf{x})$. A formal calculation then shows that $\mathbf{w}(\mathbf{x})$ satisfies (4) The bounds on the solution follow from Theorems 3.6 and 3.7 except for the bound on $p(\mathbf{x})$, which must be replaced by $|p(\mathbf{x})| < C|\mathbf{x}|$. The estimate $|p(\mathbf{x}) < C$ is, however, correct. We shall prove it in § 4.

Remark: If Q satisfies the hypotheses of Theorem 2.6 *it is unnecessary to assume that the Dirichlet Integrals are bounded.*

4. Behavior at infinity; the representation formula

Let Σ consist of a finite number of smooth closed surfaces carrying data $\mathbf{w}^*$ of class $C^{(3)}$. Let $\mathbf{w}(\mathbf{x})$ be a solution of (4) in the exterior $\mathcal{E}$ of Σ, such that $D[\mathbf{w}] = \int_{\mathcal{E}} |\nabla \mathbf{w}|^2 dV < \infty$.

THEOREM 4.1 (*cf* [1]). *There exists a constant vector* $\mathbf{w}_0$ *and a scalar* p_0 *such that* $\mathbf{w}(\mathbf{x}) \to \mathbf{w}_0$, $p(\mathbf{x}) \to p_0$ *as* $\mathbf{x} \to \infty$ *in any way.*

To prove this result, we first apply Lemma 3.3 to obtain the existence of a vector $\mathbf{w}_0$ with the properties described in that lemma. Next, we rewrite (4) as a system for $\mathbf{u}(\mathbf{x}) = \mathbf{w}(\mathbf{x}) - \mathbf{w}_0$,

$$\begin{aligned} \triangle \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{w}_0 \cdot \nabla \mathbf{u} - \nabla p &= 0 \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \tag{34}$$

and introduce a fundamental solution tensor $\boldsymbol{\chi}(\mathbf{x}, \mathbf{y})$ associated with the linearized system

$$\begin{aligned} \triangle \mathbf{u} - \mathbf{w}_0 \cdot \nabla \mathbf{u} - \nabla p &= 0 \\ \nabla \cdot \mathbf{u} &= 0. \end{aligned} \tag{5}$$

The tensor $\boldsymbol{\chi}(\mathbf{x}, \mathbf{y})$ has the properties described in § 1. In terms of it, we obtain the representation

$$\mathbf{u}(\mathbf{x}) = \oint_{\Sigma + \Sigma_R} [\mathbf{u} \cdot T \boldsymbol{\chi} - \boldsymbol{\chi} \cdot T \mathbf{u} + (\boldsymbol{\chi} \cdot \mathbf{u})(\mathbf{w}_0 \cdot \mathbf{n})] \, dS + \int_{\mathcal{E}_R} \boldsymbol{\chi} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, dV \tag{13}$$

for a sphere Σ_R enclosing Σ and corresponding annular region $\mathcal{E}_R$. (By Theorem 3.7, $T\mathbf{u}$ is continuous up to Σ, hence the integral over Σ has a meaning.) Choose $\varepsilon > 0$, choose R_0 sufficiently large that $\int_{E_{R_0}} |\nabla \mathbf{w}|^2 dV < \varepsilon$, where E_{R_0} is the exterior of Σ_{R_0}. Next, choose $|\mathbf{x}|$ sufficiently large that at all points of $\mathcal{E}_{R_0}$, $|\boldsymbol{\chi} \cdot \mathbf{u} \cdot \nabla \mathbf{u}| < \frac{3\varepsilon}{4\pi R_0^3}$. This can be done because of the property (15i) of $\boldsymbol{\chi}(\mathbf{x}, \mathbf{y})$. Then for $R > |\mathbf{x}|$, we have

$$\begin{aligned} \left| \int_{\mathcal{E}_R} \boldsymbol{\chi} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, dV \right| &= \left| \int_{R_0} \boldsymbol{\chi} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, dV \right| + \left| \int_{\mathcal{E}_R - \mathcal{E}_{R_0}} \boldsymbol{\chi} \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, dV \right| \\ &\leqslant \varepsilon + \left(\int_{\mathcal{E}'} \boldsymbol{\chi}^2 \mathbf{u}^2 \, dV \right)^{\frac{1}{2}} \left(\int_{E_{R_0}} |\nabla \mathbf{w}|^2 \, dV \right)^{\frac{1}{2}}. \end{aligned}$$

By (15i) we have, in particular, $|\boldsymbol{\chi}|^2 < C/r^2$. Hence, using Lemma 3.4, we obtain

$$\left|\int_{\varepsilon_R} \boldsymbol{\chi}\cdot\mathbf{u}\cdot\nabla\,\mathbf{u}\,d\,V\right| \leqslant C\varepsilon$$

for all sufficiently large $|\mathbf{x}|$, uniformly in R for $R > |\mathbf{x}|$. Further, we note that for any fixed $\mathbf{x}$, $\left|\int_{E_R} \boldsymbol{\chi}\cdot\mathbf{u}\cdot\nabla\,\mathbf{u}\,d\,V\right| \leqslant C\int_{E_R} |\nabla\,\mathbf{w}|^2\,d\,V$, which $\to 0$ as $R\to\infty$. Hence, for fixed $\mathbf{x}$, $\int_{\varepsilon_R} \boldsymbol{\chi}\cdot\mathbf{u}\cdot\nabla\,\mathbf{u}\,d\,V$ tends to a finite limit as $R\to\infty$.

Using (15) again, we see that the integral over Σ tends to zero as $\mathbf{x}\to\infty$. Thus, from (13),

$$\mathbf{u}(\mathbf{x}) = \eta_1(\mathbf{x}) + \eta_2(x;\,R) + \oint_{\Sigma_R} [\quad]\,d\,S.$$

Here $\eta_1(\mathbf{x})\to 0$ as $\mathbf{x}\to\infty$, $\eta_2(\mathbf{x};\,R)\to\bar{\eta}_2(\mathbf{x})$ as $R\to\infty$ for fixed $\mathbf{x}$, and $\bar{\eta}_2(\mathbf{x})\to 0$ as $\mathbf{x}\to\infty$.

Fixing $\mathbf{x}$ and letting $R\to\infty$, we see that the integral over Σ_R must tend to a limit. Let

$$\mathbf{F}(\mathbf{x}) = \lim_{R\to\infty}\oint_{\Sigma_R} [\mathbf{u}\cdot T\,\boldsymbol{\chi} - \boldsymbol{\chi}\cdot T\,\mathbf{u} + (\boldsymbol{\chi}\cdot\mathbf{u})\,(\mathbf{w}_0\cdot n)]\,d\,S. \tag{35}$$

Then
$$\mathbf{u}(\mathbf{x}) = \eta_1(\mathbf{x}) + \bar{\eta}_2(\mathbf{x}) + \mathbf{F}(\mathbf{x})$$

and we see that it will be sufficient to prove that $\mathbf{F}(\mathbf{x})\to 0$ as $\mathbf{x}\to\infty$. This may be verified directly, using the estimates (15) and Lemma 3.3, for all terms of the integrand except $\boldsymbol{\chi}.\ T\,\mathbf{u}$, for which these estimates do not suffice. We can overcome this difficulty by noting that by differentiating $\boldsymbol{\chi}(\mathbf{x},\mathbf{y})$ a sufficient number of times, in arbitrary directions, the resulting derivatives can be made to decay in magnitude to zero faster than any preassigned negative power of $|\mathbf{x}-\mathbf{y}|$. Thus, since $|T\,\mathbf{u}| < C\,R$ on Σ_R by Theorem 3.8, we should expect that a derivative of $\mathbf{F}(\mathbf{x})$ to a sufficiently high order would tend to zero as a limit. We cannot differentiate (35) under the sign, but it is legitimate to form successive difference quotients of $\mathbf{F}(\mathbf{x})$ as the limit of the integrals involving the corresponding difference quotients of $\boldsymbol{\chi}$. A simple application of the mean value theorem shows that for fixed differences, these quotients have the same order of decay for $|\mathbf{x}-\mathbf{y}|\to\infty$ as the analogous derivatives. Let N be the smallest integer such that if $\boldsymbol{\chi}$ in (35) is replaced by the result $\delta\boldsymbol{\chi}^{(N)}$ of taking N (fixed) differences in arbitrary directions, the integral over Σ_R tends to zero. Then

$$\delta\,\mathbf{F}^{(N)}(\mathbf{x}) = \lim_{R\to\infty}\oint_{\Sigma_R} [\mathbf{u}\cdot T\,\delta\boldsymbol{\chi}^{(N)} - \delta\boldsymbol{\chi}^{(N)}\cdot T\,\mathbf{u} + (\delta\boldsymbol{\chi}^{(N)}\cdot\mathbf{u})\,(\mathbf{w}_0\cdot n)]\,d\,S$$

vanishes identically in x. But the differences are arbitrary, and we conclude that $\mathbf{F}(\mathbf{x})$ is necessarily a polynomial in x of degree at most $N-1$, $\mathbf{F}(\mathbf{x}) \equiv \mathbf{P}_{N-1}(\mathbf{x})$ (see, for example, Lemma 3 in [1]). Therefore,

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\eta}_1(\mathbf{x}) + \bar{\boldsymbol{\eta}}_2(\mathbf{x}) + \mathbf{P}_{N-1}(\mathbf{x})$$

with $\boldsymbol{\eta}_1(\mathbf{x})$, $\bar{\boldsymbol{\eta}}_2(\mathbf{x})$ tending to zero at infinity. On the other hand,

$$\lim_{R\to\infty} \frac{1}{R} \oint_{\Sigma_R} \mathbf{u}^2(\mathbf{x})\, dS = 0$$

by Lemma 3.3, from which we see that $\mathbf{P}_{N-1}(\mathbf{x})$ must vanish identically. This completes the proof that $\mathbf{w}(\mathbf{x}) \to \mathbf{w}_0$. The corresponding result for $p(\mathbf{x})$ is obtained by an analogous, although more technical, discussion, based on (14) (cf [1]). We omit details. We have incidentally proved:

THEOREM 4.2: *Let* $\mathbf{w}(\mathbf{x})$ *satisfy the conditions of Theorem* 4.1. *Then* $\mathbf{w}(\mathbf{x})$, $p(\mathbf{x})$, *admit the representations*

$$\mathbf{w}(\mathbf{x}) = \mathbf{w}_0 + \oint_\Sigma [\mathbf{u}\cdot T\boldsymbol{\chi} - \boldsymbol{\chi}\cdot T\mathbf{u} + (\boldsymbol{\chi}\cdot\mathbf{u})(\mathbf{w}_0\cdot\mathbf{n})]\, dS + \int_{\mathcal{E}} \boldsymbol{\chi}\cdot\mathbf{u}\cdot\nabla\mathbf{u}\, dV$$

$$p(\mathbf{x}) = p_0 + \oint_\Sigma [\mathbf{u}\cdot T\boldsymbol{\psi} - \boldsymbol{\psi}\cdot T\mathbf{u} + (\boldsymbol{\psi}\cdot\mathbf{u})\,\mathbf{w}_0\cdot\mathbf{n})]\, dS + \int_\Sigma \boldsymbol{\psi}\cdot\mathbf{u}\cdot\nabla\mathbf{u}\, dV,$$

where $\mathbf{u}(\mathbf{x}) = \mathbf{w}(\mathbf{x}) - \mathbf{w}_0$.

Remark. The outflux condition (3) is not assumed in Theorems 4.1 and 4.2.

THEOREM 4.3. *Let* $\mathbf{w}(\mathbf{x})$ *be the limit, uniform up to* Σ *and in every compact subregion of* $\mathcal{E}$, *of solutions of a sequence of interior problems:* $\mathbf{w}^{(j)}(\mathbf{x}) = \mathbf{w}^*$ *on* Σ,

$$\mathbf{w}^{(j)}(\mathbf{x}) = \mathbf{w}_0 - \frac{Q}{4\pi}\nabla\left(\frac{1}{r}\right) \quad on\ R_j,\ R_j \to \infty,$$

$\mathbf{w}^{(j)}(\mathbf{x})$ *satisfies* (4) *in* $\mathcal{E}_{R_j}$, $\int_{\mathcal{E}_{R_j}} |\nabla \mathbf{w}^{(j)}|^2\, dV < M < \infty$, *M independent of j. Then* $\mathbf{w}(\mathbf{x}) \to \mathbf{w}_0$ *as* $\mathbf{x} \to \infty$.

Proof. It suffices to show that the vector $\mathbf{w}_0$ determined in Theorem 4.1 coincides with the data imposed on $\mathcal{E}_{R_j}$. But for a fixed region $\mathcal{E}_{R_0}$, $\mathbf{w}^{(j)} + Q\,\nabla\,(r^{-1}/4\pi$ has bounded Dirichlet Integral in $\mathcal{E}_{R_0}$, hence $\int_{\mathcal{E}_{R_0}} |\mathbf{w}^{(j)} - \mathbf{w}_0|^2 r^{-2}\, dV < C$ by Lemma 3.4, hence by uniform convergence $\int_{\mathcal{E}_{R_0}} |\mathbf{w} - \mathbf{w}_0|^2 r^{-2}\, dV < C$. Since R_0 is arbitrary,

$$\int_{\mathcal{E}} |\mathbf{w}-\mathbf{w}_0|^2 r^{-2} < C.$$

This inequality cannot hold for two distinct values of $\mathbf{w}_0$, and the result follows.

5. Existence theorems

5 a. Existence of a solution in a finite region

We apply now the fixed point theorem (1) of Leray and Schauder [9] to obtain the existence of at least one solution of (4) corresponding to prescribed data $\mathbf{w}^*$ of class $C^{(3)}$ on a smooth closed surface Σ. It is assumed that $\mathbf{w}^*$ satisfies the outflux condition, $\oint_\Sigma (\mathbf{w}^* \cdot \mathbf{n})\, dS = 0$. If such a solution exists, it admits the representation

$$\mathbf{w}(\mathbf{x}) = \oint_\Sigma \mathbf{w}^* \cdot T\, \mathbf{G}\, dS + \int_{\mathcal{G}} \mathbf{G} \cdot \mathbf{w} \cdot \nabla\, \mathbf{w}\, dV,$$

where $\mathbf{G}(\mathbf{x}, \mathbf{y})$ is the Green's Tensor for the linearized system (6) in the region $\mathcal{G}$ bounded by Σ. This representation can be considered as an integral equation for the unknown $\mathbf{w}(\mathbf{x})$.

Consider now the functional equation

$$\mathbf{w}(\mathbf{x}) - \lambda \int_{\mathcal{G}} \mathbf{G} \cdot \mathbf{w} \cdot \nabla\, \mathbf{w}\, dV + \oint_\Sigma \mathbf{w}^* \cdot T\, \mathbf{G}\, dS = \mathbf{w} - \mathfrak{J}(\mathbf{w}; \lambda) = 0 \tag{36}$$

for a "point" $\mathbf{w}$ of a suitable Banach space $\mathcal{B}$; here λ is a real parameter such that $0 \leqslant \lambda \leqslant 1$. We seek to demonstrate the existence of a solution of (36) for the parameter value $\lambda = 1$. To do so, we show that the space $\mathcal{B}$ can be so chosen that the hypotheses of Leray and Schauder [9] are satisfied.

Choose $\mathcal{B}$ to be the linear manifold of all vectors $\mathbf{w}(\mathbf{x})$ defined and continuously differentiable in $\mathcal{G} + \Sigma$, such that

i) $|\mathbf{w}(\mathbf{x})| < C$

ii) $|\nabla\, \mathbf{w}(\mathbf{x}) < C$

iii) $|\nabla\, \mathbf{w}(\mathbf{x}) - \nabla\, \mathbf{w}(\mathbf{y})| < C |\mathbf{x}-\mathbf{y}|^{\frac{1}{2}}$.

As norm of $\mathbf{w}(\mathbf{x})$, we set

$$\|\mathbf{w}\| = \text{glb}\, \{C\} \tag{37}$$

(1) Actually, only a relatively simple form of this theorem is used here, see Schaefer, H., "Über die Methode der a-priori Schranken". *Math. Ann.* 129 (1955), 415–416.

for all constants C such that i), ii) and iii) are satisfied. Under this definition, $\mathcal{B}$ becomes a normed, linear, complete Banach space of vector functions.

1. $\mathfrak{J}(\mathbf{w}; \lambda)$ is completely continuous for each λ on the range $0 \leqslant \lambda \leqslant 1$. In fact, setting $\overline{\mathbf{w}} = \mathfrak{J}(\mathbf{w}; \lambda)$, and the estimates (23) on $\mathfrak{G}(\mathbf{x}, \mathbf{y})$ in Theorem 3.1 imply that $\overline{\mathbf{w}}$ satisfies i), ii), iii) whenever $\mathbf{w}$ satisfies i), ii). It follows that the image of a bounded set is bounded. But in the norm (37), every bounded set has equicontinuous first derivatives, hence contains a convergent subset.

2. If $\lambda = 0$, $\mathfrak{J}(\mathbf{w}; 0) = \oint_{\Sigma} \mathbf{w}^* \cdot T\, \mathfrak{G}\, dS$, where $\mathbf{w}^*$ is prescribed. That is, $\mathfrak{J}(\mathbf{w}; 0)$ maps all of $\mathcal{B}$ into a single point. Thus, the transformation $\Phi(\mathbf{w}) = \mathbf{w} - \mathfrak{J}(\mathbf{w}; 0)$ is a uniform translation of $\mathcal{B}$, and it follows that the index of the (unique) solution of $\Phi(\mathbf{w}) = 0$ is one.

3. For $0 \leqslant \lambda \leqslant 1$, all solutions of (36) are bounded in $\mathcal{B}$. This result is contained in Theorem 3.5 for the case $\lambda = 1$. But if $0 \leqslant \lambda \leqslant 1$, all estimates leading to the proof of Theorem 3.5 remain in force, hence the theorem remains correct, uniformly in λ.

These properties of $\mathfrak{J}(\mathbf{w}; \lambda)$ imply, by the theorem of Leray and Schauder, the existence of a continuum of solutions of (36) corresponding to the segment $0 \leqslant \lambda \leqslant 1$. But formal calculation verifies that any solution of (36) for the parameter value $\lambda = 1$ is necessarily a solution of (4) which attaches continuously to $\mathbf{w}^*$ on Σ. Hence:

THEOREM 5.1 (Leray [8]). *Let* $\mathbf{w}^*$ *be of class* $C^{(3)}$ *on a smooth closed surface* Σ *which bounds a region* $\mathcal{G}$. *Then there is at least one solution of* (4) *in* $\mathcal{G}$ *such that* $\mathbf{w} = \mathbf{w}^*$ *on* Σ.

The question of uniqueness of the solution is discussed in § 8.

5 b. Existence of a solution in an exterior region

Let Σ consist of a finite number of connected closed components, and let Q be so small that the hypotheses of Theorem 2.6 are satisfied. Let Σ_R be a sphere about the origin of radius R so large that Σ lies in its interior. By Theorem 5.1, there is at least one solution of the problem: $\mathbf{w}(\mathbf{x}) = \mathbf{w}^*$ on Σ, $\mathbf{w}(\mathbf{x}) = \mathbf{w}_0 - Q \nabla (r^{-1})/4\pi$ on Σ_R, $\mathbf{w}(\mathbf{x})$ a solution of (4) $\mathcal{E}_R$. By theorem 2.6, the Dirichlet Integrals of all such solutions are uniformly bounded, independent of R. By Lemma 3.8, the solutions form a bounded set in $\mathcal{B}$, and for any sequence $R_j \to \infty$, there is a subsequence of solutions $\mathbf{w}^{(j)}$ which converges uniformly together with all first order derivatives in any compact subregion to a solution of (4). By Theorem 4.3, $\mathbf{w}(\mathbf{x}) \to \mathbf{w}_0$ as $\mathbf{x} \to \infty$. Hence:

THEOREM 5.2 (cf. Leray [8]; Finn [1]).

Let $\mathbf{w}^*$ *be prescribed data of class* $C^{(3)}$ *on a smooth closed surface* Σ, *and suppose the net outflux* Q *is sufficiently small (depending only on* Σ). *Then there is at least one solution* $\mathbf{w}(\mathbf{x})$ *of* (4) *defined in the exterior* $\mathcal{E}$ *of* Σ, *such that* $\mathbf{w}(\mathbf{x}) = \mathbf{w}^*$ *on* Σ *and such that* $\mathbf{w}(\mathbf{x})$ *tends to a prescribed vector* $\mathbf{w}_0$ *at infinity.*

6. Remarks on the preceding sections; an example

In connection with the solution in an exterior region $\mathcal{E}$, there remain several questions which must be answered before the theory can be considered in any reasonable sense as complete. We mention some of these.

1) Does the solution whose existence we have demonstrated necessarily admit an asymptotic development at infinity in inverse powers of r (or suitable functions of r) analogous to the classical expansion for harmonic functions, or more generally, the known developments at infinity for solutions of the equations of potential compressible flow (cf. [3])? The answer to this question is of importance in the determination of the forces and moments exerted on Σ by any possible solution in $\mathcal{E}$. Various recent investigations have been devoted to the determination of such developments under the assumption that they exist and can be obtained by iterative procedures whose convergence is not easily demonstrated, hence it seems worth-while to point out that, at least in two dimensions, not every solution which is regular in a neighborhood of infinity and tends to a limit at infinity can be represented asymptotically by an expansion in reasonable functions of r with coefficients independent of r. An example is provided by the family of vector fields $\mathbf{w} = (u, v) = \mathbf{w}(x, y)$ defined by the following relations:

$$u = (1-\alpha)\, r^{-\alpha} \frac{y}{r} - (1+\alpha) \frac{x}{r^2}$$

$$v = -(1-\alpha)\, r^{-\alpha} \frac{x}{r} - (1+\alpha) \frac{y}{r^2}.$$

For any real α, $\mathbf{w}(x, y)$ is a solution of the two-dimensional system (4) at all points except at the origin, and if $\alpha > 0$, $\mathbf{w} \to 0$ as $r \to \infty$. In the range $0 < \alpha < 1$, $r^{\alpha} |\mathbf{w}| \to 1 - \alpha$ as $r \to \infty$, that is, $|\mathbf{w}|$ behaves asymptotically as $r^{-\alpha}$. Since α is arbitrary in this range, any possible expansion would be in terms of functions which vanish more slowly than any negative power of r.

Letting $\alpha \to 0$, we obtain a family of solutions, each member of which has limiting value zero at infinity. These solutions converge uniformly together with their derivatives of all orders in any compact subregion excluding the origin, to the solution $\bar{\mathbf{w}}$ defined by

$$\bar{u} = \frac{y}{r} - \frac{x}{r^2} \qquad \bar{v} = -\frac{x}{r} - \frac{y}{r^2}.$$

This solution is discontinuous at infinity. Thus, there exist solutions which are bounded in the exterior $\mathcal{E}$ of a circle but discontinuous at infinity, and such solutions can even be obtained as the limit, uniform in compact subregions of $\mathcal{E}$, of solutions having limiting behavior at infinity.

The behavior exhibited in this example is in marked contrast to the known properties of velocity fields arising from equations of potential fluid flow (cf. [3]).

It is not known whether pathological behavior is possible in three dimensions. We can assert, however, that no example of the type described above can exist in this case. In fact, *let* $\mathbf{w}(\mathbf{x})$ *be any vector valued function having finite Dirichlet Integral* D *in a three-dimensional neighborhood* $\mathcal{E}$ *of infinity, and such that* $\mathbf{w}(\mathbf{x}) \to \mathbf{w}_0$ *as* $\mathbf{x} \to \infty$. *Then given any* $\varepsilon > 0$, *there is a constant* $C(\varepsilon)$ *and a set* E_ε *of measure less then* ε *on a unit sphere* S_0, *such that all points of* S_0 *not in* E_ε *are intersection points of lines extending to infinity from the center of* S_0, *along which* $|\mathbf{w}(\mathbf{x}) - \mathbf{w}_0| < C r^{-\frac{1}{2}}$.

Proof. Let S by any sphere which lies, with its exterior, in $\mathcal{E}$, and let its radius be ϱ. Clearly

$$\int_\Omega \int_\varrho^\infty \mathbf{w}_r^2 r^2 \, dr \, d\Omega < D$$

where Ω denotes the surface of a concentric unit sphere S_0. Hence for any $\varepsilon > 0$,

$$\int_\varrho^\infty \mathbf{w}_r^2 r^2 \, dr < \frac{1}{\varepsilon} D$$

along rays through the center of S, except perhaps for a set of measure less than ε on S_0. Now along such a ray,

$$|\mathbf{w}(R) - \mathbf{w}_0|^2 = \left(\int_R^\infty \mathbf{w}_r \, dr \right)^2 \leqslant \int_R^\infty \mathbf{w}_r^2 r^2 \, dr \int_R^\infty \frac{1}{r^2} \, dr \leqslant \frac{1}{R} \frac{1}{\varepsilon} D$$

from which the assertion follows.

We remark that we have proved in [1] that if a three dimensional solution $\mathbf{w}(\mathbf{x})$ tends to a non-zero limit $\mathbf{w}_0$, and if for some $\varepsilon > 0$, $|\mathbf{w}(\mathbf{x}) - \mathbf{w}_0| < C r^{-\frac{1}{2}-\varepsilon}$, then $\mathbf{w}(\mathbf{x})$ necessarily has at infinity, up to higher order terms, the asymptotic structure of the fundamental solution tensor $\boldsymbol{\chi}(\mathbf{x}, \mathbf{y})$ corresponding to the linearized system (5). This result, together with the above property of general vector fields, suggests that solutions with finite Dirichlet Integral necessarily exhibit at infinity the behavior of $\chi(x, y)$. We have, however, been unable to prove this. It seems not inconceivable that solutions exist which exhibit singular behavior in one critical direction, say the direction of the vector w_0.

2. Our second question concerns the method by which the solution is constructed. It seems to me inelegant and unsatisfactory to obtain the solution of an exterior problem as the limit of a sequence of solutions of interior problems. It would be preferable to find the solution directly in a suitable class of functions defined throughout $\mathcal{E}$. I have been unable to determine such a class. To begin with, it seems doubtful that an a-priori estimate on Dirichlet Integral depending only on boundary data, can be found on the single assumption that this integral is finite, even though this assumption implies that the solution is continuous at infinity. For example, we may consider the family of solutions discussed under 1). As $\alpha \to 0$, all derivatives of $w(x)$ remain smaller than fixed bounds on the unit circumference, but the Dirichlet Integrals tend to infinity. (1) It would seem natural to seek the solution in a class of vector functions satisfying an inequality of the form $|w(x) - w_0| < Cr^{-1}$ as $r \to \infty$, since this is the expected behavior of the solution and implies the desired a-priori bound on Dirichlet Integral. It is shown in [2] that an integral operator equivalent to the one defined by (13) transforms such a class into itself. For an existence theorem it would, however, be necessary to obtain an a-priori estimate on the constant C. Such an estimate is not yet available. Except for the potential flows, which from the point of view of this paper are the trivial solutions, it is not known whether there exists a single solution of an exterior boundary value problem which decays to its limit at the (expected) rate $|w - w_0| < Cr^{-1}$.

3) Although the necessary estimates on the Green's Tensor analogous to (23) have never been formally demonstrated, there seems little doubt that also in two dimensions, the procedure of § 5 will lead to the construction of a solution defined in the exterior of Σ which assumes the given boundary data and has finite Dirichlet Integral. Whether every such solution necessarily assumes the prescribed data at infinity is uncertain. That the answer to this question is not obvious is already indicated by the example discussed under 1). The problem seems not accessible to methods which are presently available.

(1) We point out, however, that in this example there is a net outflux across the circumference, and in this case we are unable to find any construction which yields an a-priori bound on Dirichlet integral in two dimensions. On the other hand, the outflux does not seem to be the essential source of singular behavior, since throughout the range of α considered, it remains between fixed positive bounds.

4) Finally, we may ask under what circumstances the solution is unique. It is known (see, e.g. [16]) that a sufficiently small solution in a bounded region is unique among all competing solutions with the same boundary data. (We give an improved version of this theorem in § 8.) It seems likely that solutions which are large in magnitude are not unique, but no examples are known. In the case of an exterior region $\mathcal{E}$, experimental evidence indicates that the solution is again unique if $|\mathbf{w}|$ is everywhere sufficiently small. A strict mathematical proof of this is yet to be given. We do prove in § 7, however, that solutions corresponding to sufficiently small prescribed data differ by arbitrarily small amounts, depending only on the given data.

7. Transition to zero Reynolds' Number

We study in this section a family of solutions of (4) defined in a fixed region (exterior or interior), and corresponding to boundary data which transform to zero in a prescribed manner. We show that these solutions necessarily tend uniformly, in the appropriate sense, to the solution of the corresponding problem for the equations (6), which are the equations obtained from (4) by linearizing about the solution $\mathbf{w}(\mathbf{x}) \equiv 0$.

7 a. Transition to zero Reynolds' Number; case of a bounded region

Let $\mathbf{w}^*$ be a prescribed vector function of class $C^{(3)}$ on a smooth closed surface Σ, and satisfying the outflux condition (3). Let λ be a parameter, $0 \leqslant \lambda \leqslant 1$. Let $\mathbf{w}(\mathbf{x}; \lambda)$ be a solution of (4) in the region $\mathcal{G}$ bounded by Σ, such that $\mathbf{w}(\mathbf{x}; \lambda) = \lambda \mathbf{w}^*$ on Σ. Such a solution exists for each λ, as was shown in § 5, and for sufficiently small λ the solutions are unique, as will be shown in the next section. (1) Let $\mathbf{w}_0(\mathbf{x})$ be the (unique) solution of the linear system (6) in $\mathcal{G}$ which assumes the data $\mathbf{w}^*(\mathbf{x})$ on Σ, let $\mathbf{W}(\mathbf{x}; \lambda) = \lambda^{-1} \mathbf{w}(\mathbf{x}; \lambda)$, and let $P(\mathbf{x}; \lambda) = \lambda^{-1} p(\mathbf{x}; \lambda)$.

THEOREM 7.1. *There exists a constant C, depending only on Σ and on $\mathbf{w}^*$, such that* $|\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{W}_0(\mathbf{x})| < C\lambda$ *throughout* $\mathcal{G} + \Sigma$.

To prove this result, it will be necessary to obtain an estimate on Dirichlet Integral as function of λ. To do so, we return to the considerations of § 2. We can introduce, by Lemma 2.1, a field $\mathbf{v}(\mathbf{x}) =$ curl $\boldsymbol{\psi}$ in $\mathcal{G} + \Sigma$ such that $\mathbf{v}(\mathbf{x}) = \mathbf{w}^*$ on Σ. Let $\boldsymbol{\eta}(\mathbf{x}; \lambda) = \mathbf{W}(\mathbf{x}; \lambda) - \mathbf{v}(\mathbf{x})$. Then $\boldsymbol{\eta}(\mathbf{x}; \lambda)$ is divergence-free, and $\boldsymbol{\eta}(\mathbf{x}; \lambda) = 0$ on Σ. We have from (4),

(1) In the proof that follows, we make no use of this fact.

$$\triangle \boldsymbol{\eta} - \lambda \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} - \frac{1}{\lambda} \nabla p = -\triangle \mathbf{v} + \lambda \boldsymbol{\eta} \cdot \nabla \mathbf{v} + \lambda \mathbf{v} \cdot \nabla \boldsymbol{\eta} + \lambda \mathbf{v} \cdot \nabla \mathbf{v},$$

$$\nabla \cdot \boldsymbol{\eta} = 0.$$

We multiply the first equation by $\boldsymbol{\eta}$ and integrate over $\mathcal{G}$, obtaining

$$\int_{\mathcal{G}} |\nabla \boldsymbol{\eta}|^2 \, dV = -\int_{\mathcal{G}} \nabla \boldsymbol{\eta} \cdot \nabla \mathbf{v} \, dV + \lambda \int_{\mathcal{G}} \mathbf{v} \cdot \boldsymbol{\eta} \cdot \nabla \boldsymbol{\eta} \, dV + \lambda \int_{\mathcal{G}} \mathbf{v} \cdot \mathbf{v} \cdot \nabla \boldsymbol{\eta} \, dV$$
$$\leqslant C \left(\int_{\mathcal{G}} |\nabla \boldsymbol{\eta}|^2 \, dV \right)^{\frac{1}{2}} + C\lambda \left(\int_{\mathcal{G}} \boldsymbol{\eta}^2 \, dV \right)^{\frac{1}{2}} \left(\int_{\mathcal{G}} |\nabla \boldsymbol{\eta}|^2 \, dV \right)^{\frac{1}{2}}$$

by Schwarz' Inequality, where C defends only on $\mathbf{w}^*$ and on Σ. Since $\boldsymbol{\eta} = 0$ on Σ,

$$\int_{\mathcal{G}} \boldsymbol{\eta}^2 \, dV \leqslant C \int_{\mathcal{G}} \frac{\eta^2}{r^2} \, dV = -C \int_{\mathcal{G}} \frac{1}{r} \nabla r \cdot \nabla \boldsymbol{\eta}^2 \, dV$$
$$\leqslant 2C \left(\int_{\mathcal{G}} \frac{\eta^2}{r^2} \, dV \right)^{\frac{1}{2}} \left(\int_{\mathcal{G}} |\nabla \boldsymbol{\eta}|^2 \, dV \right)^{\frac{1}{2}}$$

(cf. the proof of Lemma 3.4), hence

$$\left(\int_{\mathcal{G}} \boldsymbol{\eta}^2 \, dV \right)^{\frac{1}{2}} \leqslant C \left(\int_{\mathcal{G}} |\nabla \boldsymbol{\eta}|^2 \, dV \right)^{\frac{1}{2}},$$

C depending only on Σ, and we find

$$\int_{\mathcal{G}} |\nabla \boldsymbol{\eta}|^2 \, dV \leqslant C \left(\int_{\mathcal{G}} |\nabla \boldsymbol{\eta}|^2 \, dV \right)^{\frac{1}{2}} + \lambda C \int_{\mathcal{G}} |\nabla \boldsymbol{\eta}|^2 \, dV,$$

for a constant C depending only on $\mathbf{w}^*$ and on Σ. Thus, for all $\lambda < 1/2C$ we obtain the estimate

$$\int_{\mathcal{G}} |\nabla \boldsymbol{\eta}|^2 \, dV \leqslant C.$$

But
$$\int_{\mathcal{G}} |\nabla \mathbf{W}|^2 \, dV \leqslant 2 \int_{\mathcal{G}} |\nabla \boldsymbol{\eta}|^2 \, dV + 2 \int_{\mathcal{G}} |\nabla \mathbf{v}|^2 \, dV$$

and hence
$$\int_{\mathcal{G}} |\nabla \mathbf{W}(\mathbf{x}; \lambda)|^2 \, dV \leqslant C \tag{38}$$

uniformly for all sufficiently small λ, depending only on $\mathbf{w}^*$ and on Σ.

Let $\mathfrak{G}(\mathbf{x}, \mathbf{y})$ be the Green's Tensor for the linear system (6) in the region $\mathfrak{G}$. Then

$$\mathbf{W}(\mathbf{x}; y) = \oint_{\Sigma} \mathbf{w}^* \cdot T \, \mathfrak{G} \, dS - \lambda \int_{\mathcal{G}} \mathfrak{G} \cdot \mathbf{W} \cdot \nabla \mathbf{W} \, dV, \tag{39}$$

hence by the definition of $\mathbf{W}_0(\mathbf{x})$,

$$|\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{W}_0(\mathbf{x})|^2 \leqslant C^2 \lambda^2 \int \frac{W^2}{r^2} dV \int |\nabla \mathbf{W}|^2 dV,$$

where we have used the estimates (23) of Odqvist for $|\mathfrak{G}(\mathbf{x}, \mathbf{y})|$. Applying (38) and Lemma 3.4, we obtain the desired result,

$$|\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{W}_0(\mathbf{x})| \leqslant C\lambda \tag{40}$$

provided only that λ is sufficiently small, depending on Σ and on $\mathbf{w}^*$. For the remaining values of λ, the result follows from Theorem 3.5.

THEOREM 7.2. *Under the assumptions of Theorem* 7.1,

$$|\nabla \mathbf{W}(\mathbf{x}; \lambda) - \nabla \mathbf{W}_0(\mathbf{x})| < C\lambda, \qquad |P(\mathbf{x}; \lambda) - P_0(\mathbf{x})| < C\lambda,$$

and for any $\varepsilon > 0$,

$$|[\nabla \mathbf{W}(\mathbf{x}; \lambda) - \nabla \mathbf{W}_0(\mathbf{x})] - [\nabla \mathbf{W}(\mathbf{y}; \lambda) - \nabla \mathbf{W}_0(\mathbf{y})]| < C_1 \lambda r_{\mathbf{xy}}^{1-\varepsilon},$$

$$|[P(\mathbf{x}, \lambda) - P_0(\mathbf{x})] - [P(\mathbf{y}, \lambda) - P_0(\mathbf{y})]| < C_1 \lambda r_{\mathbf{xy}}^{1-\varepsilon}$$

uniformly in $\mathcal{G} + \Sigma$, *where* C *depends only on* Σ *and* $\mathbf{w}^*$, *and* C_1 *depends on* Σ, *on* $\mathbf{w}^*$, *and on* ε.

Proof. From (40) and Theorem 3.2 we obtain

$$\mathbf{W}(\mathbf{x}; \lambda) < C$$

for all sufficiently small λ. Hence from (39), using the estimates (23) on $\mathfrak{G}(\mathbf{x}, \mathbf{y})$,

$$|\nabla \mathbf{W}(\mathbf{x}; \lambda) - \nabla \mathbf{W}_0(\mathbf{x})| \leqslant C\lambda \int_{\mathcal{G}} \frac{\nabla \mathbf{W}}{r^2} dV.$$

In particular, by Theorem 3.2,

$$|\nabla \mathbf{W}(\mathbf{x}; \lambda)| \leqslant C + C\lambda \int_{\mathcal{G}} \frac{\nabla \mathbf{W}}{r^2} dV.$$

Multiplying by r^{-2} and integrating over $\mathcal{G}$ yields

$$\int_{\mathcal{G}} \frac{|\nabla \mathbf{W}|}{r^2} dV \leqslant C + C\lambda \int_{\mathcal{G}} \frac{|\nabla \mathbf{W}|}{r} dV \leqslant C + C\lambda \int_{\mathcal{G}} \frac{|\nabla \mathbf{W}|}{r^2} dV$$

from which, for small λ,

$$\int_{\mathcal{G}} \frac{|\nabla \mathbf{W}|}{r^2} dV < C,$$

hence $$|\nabla \mathbf{W}(\mathbf{x};\lambda)-\nabla \mathbf{W}_0(\mathbf{x})|<C\lambda.$$

The pressure term is estimated similarly, and the remainder of the proof is obtained by use of standard methods of potential theory, starting with the estimates (23) of Theorem 3.1.

7 b. *Transition to zero Reynolds' number; exterior region*

The method of § 7 a cannot be applied to an exterior region without modification, since the components of the Green's Tensor for the system (6) in an exterior region are in general not square integrable. In order to obtain estimates for the deviation of the given solution from the solution of the linearized equations, we consider first a finite region of special form.

LEMMA 7.3. *Let Σ consist of a finite number of smooth closed surfaces, and let $\mathcal{E}_R$ be the annular region bounded by Σ and by a sphere Σ_R about the origen of (large) radius R. Let $\mathbf{x}_0$ be a point on a fixed concentric sphere Σ_0 which contains Σ in its interior, and let $\mathfrak{G}_R(\mathbf{x},\mathbf{y})$ be the Green's Tensor for* (6) *in $\mathcal{E}_R$. Then on the surface Σ_R, $|T\,\mathfrak{G}_R(\mathbf{x}_0,\mathbf{y})|<C R^{-2}$, and uniformly for all $\mathbf{y}$ in $\mathcal{E}_R$, $|\mathfrak{G}_R(\mathbf{x}_0,\mathbf{y})|<C\,r_{\mathbf{x}_0\mathbf{y}}^{-1}$, where C depends only on Σ and on Σ_0 (and not on R).*

Proof. For the singular part $\boldsymbol{\chi}(\mathbf{x},\mathbf{y})$ of $\mathfrak{G}_R(\mathbf{x},\mathbf{y})$, these estimates are easy consequences of the defining relations (10), hence we need only prove them for the regular part $\boldsymbol{\gamma}_R(\mathbf{x},\mathbf{y})$. We obtain first a bound for the Dirichlet Integral of $\boldsymbol{\gamma}_R$. To do this, we introduce a comparison field $\mathbf{v}(\mathbf{x},\mathbf{y})$ which is divergence-free, equal to $\boldsymbol{\gamma}_R$ on Σ and on Σ_R, and vanishes outside a neighborhood of the boundary (of Lemma 2.1). It is clear that such a field can be constructed near Σ to have uniformly bounded Dirichlet Integral for all $\mathbf{x}$ on Σ_0. To construct the field near Σ_R, we exploit the homogeneity of the system (6). Thus, the values of $\boldsymbol{\chi}(\mathbf{x}_0,\mathbf{y})$ for $\mathbf{y}$ on Σ_R are exactly $R^{-1}\boldsymbol{\chi}^*(\boldsymbol{\zeta}_0,\boldsymbol{\eta})$, where $\boldsymbol{\chi}^*(\boldsymbol{\zeta}_0,\boldsymbol{\eta})$ are the values on the surface of a unit sphere Σ_1 of the fundamental solution tensor $\boldsymbol{\chi}(\boldsymbol{\zeta}_0,\boldsymbol{\eta})$ with singularity at $\boldsymbol{\zeta}=R^{-1}\mathbf{x}_0$. The values $\boldsymbol{\chi}^*$ can be extended to the interior of Σ_1 so that the extension vanishes outside a neighborhood $\mathcal{A}_1$ of Σ_1 and has Dirichlet Integral uniformly bounded, independent of R. Let $\mathbf{v}_1(\boldsymbol{\zeta},\boldsymbol{\eta})$ denote this extension. Then $\mathbf{v}_R(\mathbf{x}_0,\mathbf{y})=R^{-1}\mathbf{v}_1(R^{-1}\mathbf{x}_0,R^{-1}\mathbf{y})$ yields an extension of the given data on Σ_R to a neighborhood $\mathcal{A}_R$ of Σ_R. We have

$$\int_{\mathcal{A}_R}|\nabla \mathbf{v}_R|^2\,dV=\frac{1}{R}\int_{\mathcal{A}_1}|\nabla \mathbf{v}_1|^2\,dV<C R^{-1}.$$

We may suppose that the neighborhoods of Σ and of Σ_R do not intersect. Let $\mathbf{v}(\mathbf{x}_0, \mathbf{y})$ be the sum of the two comparison fields. Then $\mathbf{v}(\mathbf{x}_0, \mathbf{y})$ has bounded Dirichlet Integral, uniformly in R and in $\mathbf{x}_0$, for $\mathbf{x}_0$ on Σ_0. Let $\boldsymbol{\eta} = \boldsymbol{\gamma}_R - \mathbf{v}$. Then $\boldsymbol{\eta} = 0$ on Σ and on Σ_R, and, from (6),

$$\int_{\mathcal{E}_R} |\nabla \boldsymbol{\eta}|^2 \, dV = \int_{\mathcal{E}_R} |\nabla \boldsymbol{\eta} \cdot \nabla \mathbf{v}) \, dV \leqslant \left(\int_{\mathcal{E}_R} |\nabla \boldsymbol{\eta}|^2 \, dV\right)^{\frac{1}{2}} \left(\int_{\mathcal{E}_R} |\nabla \mathbf{v}|^2 \, dV\right)^{\frac{1}{2}}$$

from which
$$\int_{\mathcal{E}_R} |\nabla \boldsymbol{\eta}|^2 \, dV < C.$$

But
$$\int_{\mathcal{E}_R} |\nabla \boldsymbol{\gamma}_R|^2 \, dV \leqslant 2 \int_{\mathcal{E}_R} |\nabla \boldsymbol{\eta}|^2 \, dV + 2 \int_{\mathcal{E}_R} |\nabla \mathbf{v}|^2 \, dV,$$

hence
$$\int_{\mathcal{E}_R} |\nabla \boldsymbol{\gamma}_R|^2 \, dV < C,$$

which was to be proved.

We observe next that $\boldsymbol{\gamma}_R$ can be extended to the whole space as a piecewise continuously differentiable field which tends to zero at infinity and has bounded Dirichlet Integral, independent of R. In fact, a particular extension is provided by the singular part $\boldsymbol{\chi}(\mathbf{x}_0, y)$ of $\mathfrak{G}_R$. Hence by Lemma 3.3,

$$\frac{1}{R'} \oint_{\Sigma'} \boldsymbol{\gamma}_R^2 \, dS < C$$

for an arbitrary sphere Σ' of radius R'.

We may now use these estimates to find a bound for $\boldsymbol{\gamma}_R$ on Σ_0. Let $\mathbf{x}_0$ be a point on Σ_0, let V be a sphere of radius r_0 and surface S about $\mathbf{x}_0$. Let $\mathfrak{G}_V$ be the Green's Tensor for this sphere. Then

$$|\boldsymbol{\gamma}_R|^2 = \left| \oint_S \boldsymbol{\gamma}_R \cdot T \, \mathfrak{G}_V \, dS \right|^2 \leqslant \oint_S \boldsymbol{\gamma}_R^2 \, dS \oint (T \, \mathfrak{G}_V)^2 \, dS,$$

hence on Σ_0,
$$|\boldsymbol{\gamma}_R| < C.$$

Next, we estimate $T \boldsymbol{\gamma}_R$ on Σ. Let $\mathfrak{G}_0$ by the Green's Tensor for the region $\mathcal{E}_0$ bounded by Σ and by Σ_0. Since this region is fixed, we have the estimates (23) of Odqvist. In $\mathcal{E}_0$,

$$\boldsymbol{\gamma}_R = \oint_{\Sigma_0} \boldsymbol{\gamma}_R \cdot T \, \mathfrak{G} \, dS - \oint_{\Sigma} \boldsymbol{\chi}^* \cdot T \, \mathfrak{G} \, dS,$$

where χ^* denotes the boundary data arrising from the singularity at x_0, which is used to define γ_R. Since these data have bounded third derivatives on Σ, they can be extended by Lemma 2.1 to the interior as a divergence free vector field $v(x)$ with bounded second derivatives. Hence,

$$\gamma_R(x_0, x) = \oint_{\Sigma_0} \gamma_R \cdot T\,G\,dS + \int_{\mathcal{E}_0} G \cdot \triangle\, v\, dV.$$

The first term on the right has bounded derivatives up to Σ because of the above bound for $|\gamma_R|$ on Σ_0. The second term can be estimated by formal application of (23). A similar discussion establishes a bound for the pressure up to Σ_0.

Consider now the field γ_R in the entire region $\mathcal{E}_R$. Let $G(x, y)$ be the Green's Tensor for the sphere bounded by Σ_R. We have

$$\gamma_R(x_0, x) = \oint_{\Sigma} (\chi^* \cdot T\,G - G \cdot T\,\gamma_R)\,dS + \oint_{\Sigma_R} (\chi^* \cdot T\,G)\,dS$$

where χ^* denotes boundary data due to the singularity at x_0. The second term on the right is the regular part $\gamma(x_0, x)$ of the Green's Tensor for the entire sphere, with singularity at x_0. Homogeneity considerations, starting from the Green's Tensor for the unit sphere, show that $|\gamma(x_0, x)| < C R^{-1}$ and $|T\gamma| < C R^{-2}$ for x on Σ_R. In the first term on the right, which we denote by γ_Σ, χ^* and $T\gamma_R$ are known to be bounded. Again using the homogeneity of (6) to estimate G and $T\,G$, we find $|\gamma_\Sigma| < C|x|^{-1}$ in $\mathcal{E}_R$, and $|T\gamma_\Sigma| < C R^{-2}$ on Σ_R. Thus $\gamma_R = \gamma + \gamma_\Sigma$ has again these properties. This completes the proof of Lemma 7.3.

We are now prepared to estimate the deviation of the solutions of (4) from those of (6) in an exterior region. Again we consider a boundary Σ consisting of a finite number of smooth closed surfaces. Let w^* be prescribed data on Σ, and let w_0 be a prescribed constant vector. Let λ be a (small) positive parameter, and let $w(x; \lambda)$ be a solution of (4), such that $w = \lambda w^*$ on Σ, $w \to \lambda w_0$ at infinity. The existence of such a solution (1) is proved in § 5 and we assume that $w(x; \lambda)$ can be constructed by the method of that section. That is, $w(x; \lambda)$ is the limit, uniform in compact subregions, of solutions of the interior problem: $w = \lambda w^*$ on Σ, $w = \lambda w_0$ on Σ_R, where Σ_R is a sphere of large radius R. Let $W(x; \lambda) = \lambda^{-1} w(x; \lambda)$, $P(x; \lambda) = \lambda^{-1} p(x; \lambda)$, and

(1) It is unnecessary to assume the outflux condition (3) for w^* since if λ is sufficiently small, the conditions of Theorem (2.6) will automatically be satisfied.

let $\mathbf{W}_0(\mathbf{x})$, $P_0(\mathbf{x})$ be the solution [1] of (6) and corresponding pressure such that $\mathbf{W}_0(\mathbf{x}) = \mathbf{w}^*$ on Σ and $\mathbf{W}_0(\mathbf{x}) \to \mathbf{w}_0$ at infinity.

THEOREM 7.4. *There exist constants C and C_1, depending only on Σ, on $\mathbf{w}^*$ and on $\mathbf{w}_0$, such that* [2]

$$|\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{W}_0(\mathbf{x})| < C\lambda + C_1 r^{-1}\sqrt{\lambda}$$

uniformly in the closed exterior of Σ.

Proof. We start by obtaining an estimate for the Dirichlet Integral of $\mathbf{W}(\mathbf{x}; \lambda)$. Because of the assumed method of construction of $\mathbf{w}(\mathbf{x}; \lambda)$, this can be achieved by formal modification of the reasoning in the proof of Theorem 7.1, analogous to the change required from § 2 a to § 2 b, and we omit details. We find

$$\int_{\mathcal{E}} |\nabla \mathbf{W}|^2 dV < C$$

for all sufficiently small λ.

Consider a fixed sphere Σ_0 surrounding Σ and let Σ_R be a sphere of radius R surrounding Σ_0. Let $\mathcal{E}_R$ be the region bounded by Σ and by Σ_R, and $\mathfrak{G}_R$ be the Green's Tensor for (6) in $\mathcal{E}_R$. For any point $\mathbf{x}$ of Σ_0 we have

$$\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{w}_0 = \oint_{\Sigma} \mathbf{w}^* \cdot T\,\mathfrak{G}_R\, dS + \oint_{\Sigma_R} (\mathbf{W} - \mathbf{w}_0) \cdot T\,\mathfrak{G}_R\, dS$$
$$+ \lambda \int_{\mathcal{E}_R} \mathfrak{G}_R \cdot (\mathbf{W} - \mathbf{w}_0) \cdot \nabla \mathbf{W}\, dV + \lambda \int_{\mathcal{E}_R} \mathfrak{G}_R \cdot \mathbf{w}_0 \cdot \nabla \mathbf{W}\, dV. \tag{41}$$

We may rewrite (41) in the form

$$\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{W}_0(\mathbf{x}) = \oint_{\Sigma_R} (\mathbf{W}(\mathbf{x}) - \mathbf{w}_0) \cdot T\,\mathfrak{G}_R\, dS - \oint_{\Sigma_R} (\mathbf{W}_0 - \mathbf{w}_0) \cdot T\,\mathfrak{G}_R\, dS +$$
$$+ \lambda \int_{\mathcal{E}_R} \mathfrak{G}_R \cdot (\mathbf{W} - \mathbf{w}_0) \cdot \nabla \mathbf{W}\, dV + \lambda \int_{\mathcal{E}_R} \mathfrak{G}_R \cdot \mathbf{w}_0 \cdot \nabla \mathbf{W}\, dV.$$

Consider first the surface integrals over Σ_R. We have

$$\left[\oint_{\Sigma_R} (\mathbf{W} - \mathbf{w}_0) \cdot T\,\mathfrak{G}_R\, dS\right]^2 \leqslant \oint_{\Sigma_R} (\mathbf{W} - \mathbf{w}_0)^2 dS \oint_{\Sigma_R} (T\,\mathfrak{G}_R)^2 dS$$
$$\leqslant C R \cdot R^{-2} = C R^{-1}$$

---

(1) The existence of a solution $\mathbf{W}_0(\mathbf{x})$ and its uniqueness in a class of solutions which differ from $\mathbf{w}_0$ by $O(r^{-1})$ is proved in [11]. The uniqueness in the most general class of solutions which are continuous at infinity is proved in [4]. A still more general uniqueness theorem will appear in a forthcoming work of I. D. Chang and the author.

(2) The origin of coordinates is assumed interior to Σ. The result implies, in particular, the uniform inequality $|\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{W}_0(\mathbf{x})| < C\sqrt{\lambda}$ in $\mathcal{E} + \Sigma$.

by Lemma 3.3 and Lemma 7.3. Also, on Σ_R, $|\mathbf{W}_0 - \mathbf{w}_0| < C R^{-1}$ (see footnote (1), p. 237), hence by Lemma 7.3,

$$\left| \oint_{\Sigma_R} (\mathbf{W}_0 - \mathbf{w}_0) \cdot T\, \mathfrak{G}_R \, dS \right| < C R^{-1}.$$

We study the volume integrals, using Lemmas 3.4 and 7.3:

$$\left| \lambda \int_{\mathcal{E}_R} \mathfrak{G}_R \cdot (\mathbf{W} - \mathbf{w}_0) \cdot \nabla \mathbf{W} \, dV \right|^2 \leqslant C \lambda^2 \int_{\mathcal{E}_R} \frac{(\mathbf{W} - \mathbf{w}_0)^2}{r^2} dV \int_{\mathcal{E}_R} |\nabla \mathbf{W}|^2 dV \leqslant C \lambda^2$$

$$\left| \lambda \int_{\mathcal{E}_R} \mathfrak{G}_R \cdot \mathbf{w}_0 \cdot \nabla \mathbf{W} \, dV \right|^2 \leqslant C \lambda^2 \int_{\mathcal{E}_R} \frac{1}{r^2} dV \int_{\mathcal{E}_R} |\nabla \mathbf{W}|^2 dV \leqslant C \lambda^2 R.$$

Now choose $R = \lambda^{-1}$. The above estimates yield, for all $\mathbf{x}$ on Σ_0,

$$|\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{W}_0(\mathbf{x})| < C \sqrt{\lambda}.$$

This result established, we consider the fixed region $\mathcal{E}_0$ bounded by Σ and by Σ_0. The associated Green's Tensor $\mathfrak{G}_0$ satisfies the estimates (23) of Odqvist. For $\mathbf{x}$ in $\mathcal{E}_0$, we have

$$\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{W}_0(\mathbf{x}) = \oint_{\Sigma_0} (\mathbf{W} - \mathbf{W}_0) \cdot T\, \mathfrak{G}_0 \, dS$$
$$+ \lambda \int_{\mathcal{E}_0} \mathfrak{G}_0 \cdot (\mathbf{W} - \mathbf{w}_0) \cdot \nabla \mathbf{W} \, dV + \lambda \int_{\mathcal{E}_0} \mathfrak{G}_0 \cdot \mathbf{w}_0 \cdot \nabla \mathbf{W} \, dV.$$

Let us consider values of $\mathbf{x}$ in a neighborhood of Σ (bounded away from Σ_0). We see immediately that the two volume integrals admit uniform bounds, hence by the above estimates, we find that in this fixed neighborhood of Σ, $|\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{W}_0(\mathbf{x})| < C\sqrt{\lambda}$. A repetition of the reasoning which led to Theorem 7.2 then shows that throughout this neighborhood, $|\nabla \mathbf{W} - \nabla \mathbf{W}_0| < C \sqrt{\lambda}$. Similarly, we show that if the pressures p and P_0 are suitably normalized, then $|P(\mathbf{x}; \lambda) - P_0(\mathbf{x})| < C \sqrt{\lambda}$ up to Σ. Hence $|T\mathbf{W} - T\mathbf{W}_0| < C \sqrt{\lambda}$ up to Σ.

Now let $\boldsymbol{\chi}(\mathbf{x}, y; \sigma)$ be the fundamental solution tensor associated with the system (5). For any point $\mathbf{x}$ in the exterior $\mathcal{E}$ of Σ, we have by Theorem 4.2 the representation

$$\mathbf{W}(\mathbf{x}; \lambda) - \mathbf{w}_0 = \oint_{\Sigma} [\mathbf{w}^* \cdot T \boldsymbol{\chi} - \boldsymbol{\chi} \cdot T\mathbf{W} + \lambda \boldsymbol{\chi} \cdot (\mathbf{w}^* - \mathbf{w}_0)(\mathbf{w}_0 \cdot \mathbf{n})] \, dS$$
$$+ \lambda \int_{\mathcal{E}} \boldsymbol{\chi} \cdot (\mathbf{W} - \mathbf{w}_0) \cdot \nabla \mathbf{W} \, dV.$$

Letting $\boldsymbol{\chi}_0(\mathbf{x}, \mathbf{y})$ denote the fundamental solution tensor for the system (6), we have

$$\mathbf{W}_0(\mathbf{x}) - \mathbf{w}_0 = \oint_\Sigma [\mathbf{w}^* \cdot T\chi_0 - \chi_0 \cdot T\,\mathbf{W}_0]\, dS.$$

Thus, for all points of $\mathcal{E}$,

$$\mathbf{W}(\mathbf{x};\lambda) - \mathbf{W}_0(\mathbf{x}) = \oint_\Sigma [\mathbf{w}^* \cdot (T\chi - T\chi_0) - (\chi - \chi_0)\cdot T\,\mathbf{W}_0 - \chi\cdot(T\,\mathbf{W} - T\,\mathbf{W}_0)]\, dS$$
$$+ \lambda \oint_\Sigma \chi \cdot (\mathbf{w}^* - \mathbf{w}_0)(\mathbf{w}_0 \cdot \mathbf{n})\, dS + \lambda \int_{\mathcal{E}} \chi \cdot (\mathbf{W} - \mathbf{w}_0) \cdot \nabla \mathbf{W}\, dV. \qquad (42)$$

Because of what we have already shown, we need only study points $\mathbf{x}$ at large distance from Σ. Formal (although tedious) calculation, starting from the definition (7) for $\chi(\mathbf{x}, \mathbf{y}; \sigma)$, shows that for $r_{\mathbf{xy}}$ bounded from zero, $|\chi(\mathbf{x}, \mathbf{y}; \sigma) - \chi_0(\mathbf{x}, \mathbf{y}) < C\sigma$ $|T\chi(\mathbf{x}, \mathbf{y}; \sigma) - T\chi_0(\mathbf{x}, \mathbf{y})| < C\sqrt{\sigma}\, r_{\mathbf{xy}}^{-1}$. Also, for all $r_{\mathbf{xy}}$, $|\chi(\mathbf{x}, \mathbf{y}; \sigma)| < C\, r_{\mathbf{xy}}^{-1}$ uniformly in σ as $\sigma \to 0$. The desired result, $|\mathbf{W}(\mathbf{x};\lambda) - \mathbf{W}_0(\mathbf{x})| < C\lambda + C_1\sqrt{\lambda}\, r^{-1}$ uniformly in $\mathcal{E}$, then follows immediately from (42), from the above estimates, from Lemma 3.4, and from the definition of σ.

We can extend this result to obtain corresponding estimates on the derivatives of the solution-

THEOREM 7.5. *Under the assumptions of Theorem* 7.4,

$$|\nabla \mathbf{W}(\mathbf{x};\lambda) - \nabla \mathbf{W}_0(\mathbf{x})| < C\lambda + C_1\sqrt{\lambda}\, r^{-1},$$
$$|P(\mathbf{x};\lambda) - P_0(\mathbf{x})| < C\lambda + C_1\sqrt{\lambda}\, r^{-1},$$

and for any $\varepsilon > 0$,

$$|[\nabla \mathbf{W}(\mathbf{x};\lambda) - \nabla \mathbf{W}_0(\mathbf{x})] - [\nabla W(\mathbf{y};\lambda) - \nabla \mathbf{W}_0(\mathbf{y})]| < [\bar{C}\lambda + \bar{C}_1\sqrt{\lambda}\, r^{-1}]\, r_{\mathbf{xy}}^{1-\varepsilon}$$
$$|[P(\mathbf{x};\lambda) - P_0(\mathbf{x})] - [P(\mathbf{y};\lambda) - P_0(\mathbf{y})]| < [\bar{C}\lambda + \bar{C}_1\sqrt{\lambda}\, r^{-1}]\, r_{\mathbf{xy}}^{1-\varepsilon}$$

uniformly in $\mathcal{E} + \Sigma$, *where* C, C_1 *depend only on* Σ, *on* $\mathbf{w}^*$, *and on* $\mathbf{w}_0$, *and* $\bar{C}$, $\bar{C}_1$ *depend on* Σ, *on* $\mathbf{w}^*$, *on* $\mathbf{w}_0$, *and on* ε.

For any compact region containing Σ, the proof of this result is essentially contained in the proof of Theorem 7.2. For a point $\mathbf{x}$ at large distance r from Σ, we may enclose $\mathbf{x}$ in a unit sphere V of surface S, and write

$$\mathbf{W}(\mathbf{x};\lambda) = \oint_S \mathbf{W} \cdot T\,\mathbf{G}\, dS + \lambda \int_V \mathbf{G} \cdot \mathbf{W} \cdot \nabla \mathbf{W}\, dV,$$

where $\mathbf{G}(\mathbf{x}, \mathbf{y})$ is the Green's Tensor for (6) in V. Hence

$$\mathbf{W}(\mathbf{x};\lambda) - \mathbf{W}_0(\mathbf{x}) = \oint_S (\mathbf{W} - \mathbf{W}_0) \cdot T\,\mathfrak{G}\,dS + \lambda \int_V \mathfrak{G} \cdot \mathbf{W} \cdot \nabla\, \mathbf{W}\, dV,$$

and by Theorem 7.4,

$$|\nabla\, \mathbf{W}(\mathbf{x};\lambda) - \nabla\, \mathbf{W}_0(\mathbf{x})| \leqslant C\lambda + C_1 \sqrt{\lambda}\, r^{-1} + C_2 \lambda \int_V \frac{|\nabla W|}{r^2_{\mathbf{x}\mathbf{y}}}\, dV.$$

Since $|\nabla\, \mathbf{W}_0(\mathbf{x})| < C r^{-2}$, the remainder of the proof can be obtained by a repetition of the reasoning in the demonstration of Theorem 7.2.

7 c. Transition of the force exerted on a fluid interface

We consider a solution of (4) defined in a region $\mathcal{G}$ (or $\mathcal{E}$) bounded by Σ. Denote by Σ' an arbitrary smooth closed contour which lies in the flow region. The force exerted across the interface Σ' on the particles in the region $\mathcal{G}'$ interior to Σ is defined as the integral of the stress tensor over Σ',

$$\mathbf{F}' = -\oint_{\Sigma'} T\,\mathbf{w}\, dS.$$

That is,

$$F_i' = -\oint_{\Sigma'} \left[-p\,\delta_{ij} + \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) \right] n_j\, dS.$$

Consider again a family of solutions $\mathbf{w}(\mathbf{x};\lambda)$ with boundary data $\lambda\mathbf{w}^*$, $\lambda\mathbf{w}_0$, $0 < \lambda \leqslant 1$, and let $\mathbf{W}(\mathbf{x};\lambda) = \lambda^{-1}\mathbf{w}(\mathbf{x};\lambda)$, $P(\mathbf{x};\lambda) = \lambda^{-1} p(\mathbf{x};\lambda)$. Again let $\mathbf{W}_0(\mathbf{x})$, $P_0(\mathbf{x})$ denote the solution and corresponding pressure of the solution of (6) with boundary data $\mathbf{w}^*$, $\mathbf{w}_0$. For this solution, the force on Σ' is given by

$$\mathbf{F}_0' = -\oint_{\Sigma'} T\,\mathbf{W}_0\, dS.$$

Applying Theorems 7.2 and 7.5, we conclude:

THEOREM 7.6. *There exist constants C and λ_0, depending only on Σ, on $\mathbf{w}^*$, and (for an exterior region) on $\mathbf{w}_0$, such that whenever $\lambda < \lambda_0$,*

$$\left| \frac{1}{\lambda}\mathbf{F}' - \mathbf{F}_0' \right| < C\lambda$$

in an interior region, and

$$\left| \frac{1}{\lambda}\mathbf{F}' - \mathbf{F}_0' \right| < C\sqrt{\lambda}$$

for an exterior region $\mathcal{E}$.

A case of particular interest is that of a flow of a viscous fluid past an obstacle, i.e. of a solution $\mathbf{w}(\mathbf{x})$ of (1) exterior to Σ, such that $\mathbf{w}=0$ on Σ and $\mathbf{w}\to\mathbf{w}_0$ at infinity. Theorem 7.6 shows that for large values of the viscosity μ, the ratio of the force on Σ to μ can be approximated by the corresponding quantity for the solution of the linearized equations, with an error not larger than $C/\sqrt{\lambda}$, It is in principle possible to obtain an explicit value for C in particular cases, e.g., the flow past a sphere, and we plan to carry out these calculations in the near future.

8. Uniqueness and continuous dependence

Uniqueness theorems for time independent motions of a viscous fluid at small Reynolds number can be traced back to Osbourne Reynolds [16]. Except for improvements in detail and in exposition the available knowledge has remained unchanged since that time. Essentially, the result states that *if* $\mathbf{w}_1(\mathbf{x})$ *is a solution of* (4) *in a finite region* $\mathcal{G}$ *bounded by a smooth closed surface* Σ, *and if the maximum of* $|\mathbf{w}_1(\mathbf{x})|$ *in* $\mathcal{G}$ *is sufficiently small, then there is no other solution of* (4) *in* $\mathcal{G}$ *which assumes the same boundary data.* The general line of proof is as follows: Let $\mathbf{w}_2(\mathbf{x})$ be another solution, and set $\mathbf{W}(\mathbf{x})=\mathbf{w}_1-\mathbf{w}_2$, $P(\mathbf{x})=p_1-p_2$. Then $\mathbf{W}(\mathbf{x})=0$ on Σ, and

$$\triangle\,\mathbf{W}-\mathbf{W}\cdot\nabla\,\mathbf{W}-\nabla\,P=\mathbf{W}\cdot\nabla\,\mathbf{w}_1-\mathbf{w}_1\cdot\nabla\,\mathbf{W} \tag{43}$$
$$\nabla\cdot\mathbf{W}=0.$$

Scalar multiplication of (43) by $\mathbf{W}(\mathbf{x})$ and integration over $\mathcal{G}$ leads to

$$\int_{\mathcal{G}}|\nabla\,\mathbf{W}|^2\,d\,V=-\int_{\mathcal{G}}\mathbf{W}\cdot\mathbf{W}\cdot\nabla\,\mathbf{w}_1\,d\,V=\int_{\mathcal{G}}\mathbf{w}_1\cdot\mathbf{W}\cdot\nabla\,\mathbf{W}\,d\,V,$$

hence
$$\int_{\mathcal{G}}|\nabla\,\mathbf{W}|^2\,d\,V\leqslant\gamma\left(\int_{\mathcal{G}}\mathbf{W}^2\,d\,V\right)^{\frac{1}{2}}\left(\int_{\mathcal{G}}|\nabla\,\mathbf{W}|^2\,dV\right)^{\frac{1}{2}},$$

where $\gamma=\max_{\mathcal{G}}|\mathbf{w}_1(\mathbf{x})|$. Since $\mathbf{W}(\mathbf{x})=0$ on Σ, there is a constant C, depending only on Σ, such that $\int_{\mathcal{G}}\mathbf{W}^2\,d\,V\leqslant C^2\int_{\mathcal{G}}|\nabla\,\mathbf{W}|^2\,d\,V$. Hence

$$\int_{\mathcal{G}}|\nabla\,\mathbf{W}|^2\,d\,V\leqslant\gamma\,C\int_{\mathcal{G}}|\nabla\,\mathbf{W}|^2\,dV$$

and if γ is smaller than C^{-1} we conclude $|\nabla\,\mathbf{W}|\equiv 0$, hence $\mathbf{W}\equiv 0$ in $\mathcal{G}$, q.e.d.

We present in this section an improvement of this theorem, in the sense that the only knowledge assumed is of an a-priori character, on the boundary values $\mathbf{w}^*$ of the solution. Nothing is assumed about the behavior of the solution interior to $\mathcal{G}$.

THEOREM 8.1. *Let* $\mathbf{w}^*$ *be prescribed data of class* $C^{(3)}$ *on a smooth closed surface* Σ, *and let* μ *denote a bound for the magnitudes of* $\mathbf{w}^*$ *and of its second order derivatives on* Σ. *Then if* μ *is sufficiently small, depending only on* Σ, *there is at most one solution of* (4) *in* $\mathcal{G}$ *which is equal to* $\mathbf{w}^*$ *on* Σ.

We remark that if $\mathbf{w}^*$ satisfies condition (3) there is at least one solution (cf. § 5), hence in this case there is exactly one solution.

Theorem 8.1 appears as a special case of a more general result, which shows the continuous dependence of the solution on its boundary values.

THEOREM 8.2. *Let* $\mathbf{w}_1(\mathbf{x})$, $\mathbf{w}_2(\mathbf{x})$ *be two solutions of* (4) *in* $\mathcal{G}$ *which assume boundary data* $\mathbf{w}_1^*$, $\mathbf{w}_2^*$ *of class* $C^{(3)}$ *on* Σ. *Let* μ *be a common majorant for* $\mathbf{w}_1^*$, $\mathbf{w}_2^*$, *in the sense of Theorem* 8.1. *Let* $\mathbf{W}^* = \mathbf{w}_1^* - \mathbf{w}_2^*$ *and let* $\mathbf{W}_0(\mathbf{x})$ *be the (unique) solution of the linear system* (6) *such that* $\mathbf{W}_0(\mathbf{x}) = \mathbf{W}^*$ *on* Σ. *Then if* μ *is sufficiently small, depending only on* Σ, *we have*

$$|\mathbf{w}_1(\mathbf{x}) - \mathbf{w}_2(\mathbf{x})| < 2 \max_{\mathcal{G}+\Sigma} |\mathbf{W}_0(\mathbf{x})|$$

uniformly for all $\mathbf{x}$ *in* $\mathcal{G}+\Sigma$.

Proof: The difference $\mathbf{W}(\mathbf{x}) = \mathbf{w}_1(\mathbf{x}) - \mathbf{w}_2(\mathbf{x})$ satisfies

$$\triangle \mathbf{W} - \nabla P = \mathbf{W} \cdot \nabla \mathbf{w}_1 + \mathbf{w}_2 \cdot \nabla \mathbf{W}$$
$$\nabla \cdot \mathbf{W} = 0,$$

where $P(\mathbf{x}) = p_1 - p_2$. Let $\mathfrak{G}(\mathbf{x}, \mathbf{y})$ be the Green's Tensor for the system (6) in $\mathcal{G}$. Then

$$\mathbf{W}(\mathbf{x}) = \oint_\Sigma \mathbf{W}^* \cdot T\, \mathfrak{G}\, dS + \int_{\mathcal{G}} \mathfrak{G} \cdot \mathbf{W} \cdot \nabla \mathbf{w}_1\, dV + \int_{\mathcal{G}} \mathfrak{G} \cdot \mathbf{w}_2 \cdot \nabla \mathbf{W}\, dV$$

from which $$\mathbf{W}(\mathbf{x}) = \mathbf{W}_0(\mathbf{x}) + \int_{\mathcal{G}} \mathfrak{G} \cdot \mathbf{W} \cdot \nabla \mathbf{w}_1\, dV - \int_{\mathcal{G}} \mathbf{W} \cdot \mathbf{w}_2 \cdot \nabla \mathfrak{G}\, dV.$$

Let $M = \max_{\mathcal{G}+\Sigma} |\mathbf{W}(\mathbf{x})|$, $M_0 = \max_{\mathcal{G}+\Sigma} |\mathbf{W}_0(\mathbf{x})|$. We may assume that the value M is achieved at a point not on Σ, since otherwise the theorem is trivially correct. We find, using the estimates (23) of Odqvist,

$$M \leqslant M_0 + M \int_{\mathcal{G}} \frac{|\nabla \mathbf{w}_1|}{r}\, dV + M \int_{\mathcal{G}} \frac{|\mathbf{w}_2|}{r^2}\, dV. \tag{44}$$

But by Theorem 2.3,

$$\int_{\mathcal{G}} \frac{|\nabla \mathbf{w}_1|}{r} d V \leqslant \left(\int_{\mathcal{G}} \frac{1}{r^2} d V\right)^{\frac{1}{2}} \left(\int_{\mathcal{G}} |\nabla \mathbf{w}_1|^2 d V\right)^{\frac{1}{2}} \leqslant \varepsilon_1(\mu),$$

where $\varepsilon_1 \to 0$ as $\mu \to 0$. Also, by Lemma 3.4,

$$\int_{\mathcal{G}} \frac{|\mathbf{w}_2|}{r^2} d V \leqslant \left(\int_{\mathcal{G}} \frac{1}{r^2} d V\right)^{\frac{1}{2}} \left(\int_{\mathcal{G}} \frac{|\mathbf{w}_2|^2}{r^2} d V\right)^{\frac{1}{2}} \leqslant C \left(\int_{\mathcal{E}} |\nabla \overline{\mathbf{w}}_2|^2\right)^2,$$

where $\overline{\mathbf{w}}_2$ denotes an extension of $\mathbf{w}_2$ to the exterior of Σ, which vanishes outside a compact subregion. Applying Lemma 3.4 and Theorem 2.3, we find

$$\int_{\mathcal{G}} \frac{|\mathbf{w}_2|}{r^2} d V \leqslant \varepsilon_2(\mu),$$

where $\varepsilon_2(\mu) \to 0$ as $\mu \to 0$. Thus, if μ is chosen so small that $\varepsilon_1(\mu) + \varepsilon_2(\mu) < \frac{1}{2}$, we obtain from (44)

$$M \leqslant 2 M_0,$$

which was to be proved.

Finally, we state a result which does not differ essentially from one that we have used in § 5, but which seems worthwile to formulate explicitly.

THEOREM 8.3. *Let* $\{\mathbf{w}_n(\mathbf{x})\}$ *be a sequence of solutions of* (4) *defined in a finite region* $\mathcal{G}$ *with smooth boundary* Σ, *and suppose that the boundary values* $\{\mathbf{w}_n^*\}$ *are uniformly bounded and have third derivatives smaller in magnitude than a fixed bound. Then there is a subsequence of the* $\{\mathbf{w}_n\}$ *which converges uniformly in* $\mathcal{G} + \Sigma$, *together with all derivatives of first order, to a solution of* (4) *in*) $\mathcal{G}$.

Theorem 8.3 follows immediately from Theorem 3.5, from classical theorems on equicontinuous families, and from the representation formula (30).

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Received July 27, 1960