ON A FUNDAMENTAL THEOREM OF THE CALCULUS OF VARIATIONS

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Consider the simplest kind of multiple integral variational problem, that of minimizing an integral $I[u]$ of the form

$$I[u] = \int_R F(x, y, u, p, q)\, dA, \tag{1}$$

where the admissible functions $u = u(x, y)$ are continuously differentiable in R and take on given continuous values on the boundary of R. Here R denotes an open bounded region of the plane and $dA = dx\, dy$.

It has been known for more than one hundred and fifty years that a *twice differentiable* minimizing function must satisfy the Euler-Lagrange equation

$$\frac{\partial}{\partial x}(F_p) + \frac{\partial}{\partial y}(F_q) = F_u. \tag{2}$$

A function $u(x, y)$ which is continuous in some region A and satisfies (2) at all interior points of A is called an *extremal*. Let us denote by $\mathcal{E}$ an extremal defined over the closure of R which continuously takes on the given boundary values. A major task of the calculus of variations is to give conditions under which $\mathcal{E}$ will minimize $I[u]$. This problem is not completely settled even today, in spite of the efforts of many investigators. We shall consider here one aspect of this problem, having its genesis in the classical field theory of Weierstrass and Hilbert. The latter, in his famous paper "Mathematische Probleme" and in a later paper "Zur Variationsrechnung", proved essentially the following theorem.

If the extremal $\mathcal{E}$ can be imbedded in a field $\mathfrak{F}$ with slope functions $\mathfrak{p}(x, y, z)$, $\mathfrak{q}(x, y, z)$, such that

$$E(x, y, z, \mathfrak{p}, \mathfrak{q}; P, Q) \geqslant 0$$

for each set of values (x, y, z) *in the field, then* $\mathcal{E}$ *minimizes the integral* $I[u]$ *relative to all admissible functions whose values lie in the field.*(1)

The kind of imbedding used in Hilbert's theorem requires the entire extremal surface corresponding to $\mathcal{E}$ to lie in the field $\mathfrak{V}$; as a consequence $\mathcal{E}$ must be continuously differentiable in the *closure* of R. Thus Hilbert's theorem, in spite of its elegance, is significantly restricted in its usefulness and range of application. Indeed, it is a commonplace today that the solution of a boundary value problem in partial differential equations need not be differentiable up to the boundary, even when the boundary values themselves are differentiable. On the other hand, if one considers extremals $\mathcal{E}$ which are not differentiable n the closure of R, then the conclusion of Hilbert's theorem comes in doubt, since $\mathcal{E}$ in this case need not give $I[u]$ a finite value.(2) There the matter rests at present, for other authors who have studied the problem from a more or less related point of view have always placed some kind of restriction on the behavior of $\mathcal{E}$ at the boundary (see, for example, the papers of Lichtenstein, Miranda, Karush, and Hestenes listed in the references).

The main purpose of this paper is to give a reformulation of Hilbert's theorem which avoids both the difficulties mentioned above. In the new formulation (Theorem 4) it is assumed that there is at least one admissible function which gives I a finite value, and the field is required to be an extremal field. On the other hand, by virtue of an extension of the notion of imbedding, it is not necessary to make any assumption concerning the behavior of $\mathcal{E}$ at the boundary, beyond simply requiring that it take on the given boundary values. The proof is a modification and extension of Hilbert's classical argument: at no stage is it necessary to resort to deep theorems of integration. Finally the proof applies

(1) The first complete statement of this theorem seems to be due to Bolza ([2], 683); see also [1], § 11. For completeness, we add the definition

$$E(x, y, u, p, q; P, Q) = F(P, Q) - F(p, q) - (P - p)F_p(p, q) - (Q - q)F_q(p, q),$$

in which the arguments (x, y, u) of F have been uniformly suppressed.

(2) This objection (usually attributed to Hadamard) was first pointed out in 1871 by F. Prym. Prym's example is so elegant that I cannot resist reproducting it here: Let R be the circle $|3z - 1| < 1$, and consider the function

$$u = \operatorname{Im}\{\sqrt{\log z}\}, \quad z = x + iy.$$

By direct calculation one finds that the Dirichlet integral of u over R is divergent, while, on the other hand, u is continuous in the closure of R. In other words, u is an extremal for a regular variational problem, and at the same time $I[u] = \infty$.

equally in n dimensions and to extremals satisfying only the Haar equations. Further results of the paper are discussed in sections 1 and 2.

The paper is divided into three parts. The first part contains preparatory material and a discussion of the special integrand $F(x, y, p, q)$, the second part treats the full integrand $F(x, y, u, p, q)$, and the final part is devoted to certain subsidiary matters.

PART I

1. Preliminaries: The purpose of this section is to define precisely the class of integrands $F(x, y, u, p, q)$ which we shall treat, and the class of functions which will be admitted to competition in minimizing the integral I.

We assume that F is defined and continuous for all values of u, p, q and for all $(x, y) \in R$. Furthermore, we suppose that the partial derivatives F_p, F_q, and F_u exist and are continuous. Finally, *in all of our results it is supposed that*

$$F(x, y, u, p, q) \geqslant 0, \tag{3}$$

and that the Weierstrass function is non-negative,

$$E(x, y, u, p, q; P, Q) \geqslant 0, \quad (p, q) \neq (P, Q). \tag{4}$$

It would be possible to lighten these assumptions somewhat in certain cases, but we leave such refinements to the reader. By a simple change of integrand, condition (3) can be attained for any integrand F which is bounded below by a fixed integrable function of x, y. Additional requirements will occasionally be placed on F, but these will be stated in the hypotheses of the individual theorems.

A real-valued function $v = v(x, y)$ defined on the closure of R will be said to be in the class $\mathfrak{A}$ of admissible functions if and only if it satisfies the following three conditions:

$\mathfrak{A}$ 1. v continuously takes on assigned (continuous) values on the boundary of R.

$\mathfrak{A}$ 2. v is continuously differentiable in R. The integral $I[v]$ is understood to be the improper Riemann integral obtained by exhaustion of the region R.

$\mathfrak{A}$ 3. $I[v]$ is finite.

Various generalizations of $\mathfrak{A}$ 2 can be treated, the simplest being piecewise continuously differentiable functions. The class of admissible functions secured from this particular extension of $\mathfrak{A}$ 2 is not especially interesting, and accordingly we shall set it aside without further discussion. More important is the class $\mathfrak{A}^*$ of admissible functions obtained by replacing $\mathfrak{A}$ 2 by the condition

$\mathfrak{A}$ 2*. v is continuous in R. The integral $I[v]$ is to be understood in the following generalized sense: Let $\{v_n\}$ be a sequence of continuously differentiable functions converging uniformly to v in any proper subregion of R. Then

$$I[v] = \text{g.l.b.} \liminf_{n\to\infty} I[v_n], \tag{5}$$

where the g.l.b. is taken over all sequences $\{v_n\}$ converging to v as above. It is tacitly assumed that each function v_n is defined in some subregion R_n of R, and $I[v_n] = I[v_n, R_n]$.

The generalized integral defined by (5) is somewhat analogous to the generalized area in Lebesgue's theory of surface area, and indeed can be shown equivalent to the latter when $F = \sqrt{1+p^2+q^2}$. In order for (5) to be meaningful and logically consistent, it must still be proved that if v is continuously differentiable in R, then $I[v]$, as defined by (5), has the natural value. This we shall do in the following paper in this journal. An important subclass $\mathfrak{L}$ of the admissible functions $\mathfrak{A}^*$ is obtained by restricting v to be Lipschitz continuous in any closed subregion of R.

With the preceding definitions in mind, we can now state the problem which will concern us throughout the paper.

Variational Problem I. Among all functions satisfying the boundary condition $\mathfrak{A}$ 1 and for which the integral I has a meaning, to determine a function u such that $I[u] = \text{Minimum}$.

Let $u = u(x, y)$ be a continuous function defined in some region A. We shall say that u is an *extremal* if and only if

1. u is continuously differentiable in the interior of A.
2. For every piecewise smooth function ζ which vanishes on and near the boundary of A, we have

$$\int_A (\zeta_x F_p + \zeta_y F_q + \zeta F_u)\, dA = 0. \tag{6}$$

(A function is said to be *piecewise smooth* in a region A if it is continuous in A, and has bounded continuous first derivatives except on a finite number of smooth arcs and at a finite number of isolated points.)

This definition of an extremal, which is equivalent to the Haar equations, will be used henceforth in the paper; it is obviously satisfied by any solution of the Euler-Lagrange equation (2). To simplify the wording of later theorems, we shall say that a function u is an *extremal for the variational problem I* if u is an extremal defined over the closure of R which continuously takes on the given boundary values.

Remark. The reader who prefers to deal with integrands and extremals which are of class C^2, and who is content to consider only the class $\mathfrak{A}$ of admissible functions, will find the proofs immediately applicable to his case; moreover, he may omit reading sections 4 and 5, the lemma in section 9, the last portion of section 10, and all of Part III.

2. The special integrand $F(x, y, p, q)$. In sections 2 through 5 we shall consider integrands which are independent of u. Recalling the definitions and terminology of the preceding sections, we can state our first result.

THEOREM 1. *Let F be independent of u, and suppose $u = u(x, y)$ is an extremal for the variational problem I. Let there exist at least one function v in the class $\mathfrak{A}$ (or $\mathfrak{A}^*$) of admissible functions. Then $I[u]$ exists and*

$$I[u] \leqslant I[v] \tag{7}$$

for each v in $\mathfrak{A}$ (or $\mathfrak{A}^*$).

If the equality sign in (4) *is excluded, then $I[u] < I[v]$ for each v different from u.*

This result is a corollary of Theorem 4 below. It has an independent proof, however, which is enough different to justify being given separately (section 3). This proof applies equally well in any number of dimensions.

Two special cases of Theorem 1 deserve notice. First, if $F_p(u)$ and $F_q(u)$ are bounded,(1) and if the boundary of R is rectifiable, then (7) follows in a well-known way simply by integration by parts. The merit of Theorem 1 lies precisely in eliminating these hypotheses.

Secondly, for the integrand $F = p^2 + q^2$ (the Dirichlet integral) a result equivalent to Theorem 1 was given by Lebesgue in 1913. It is interesting to examine the connection between this result and what is usually known as Dirichlet's principle. The latter may be interpreted as saying that, under certain circumstances, the Dirichlet integral may be minimized and the minimizing function solves the boundary value (Dirichlet) problem for the Laplace equation. Sufficient conditions for the existence of a minimum were first given by Hilbert, using direct methods of the calculus of variations, and the problem from this point of view has been extensively treated by Courant in a well-known monograph. On the other hand, if the result of Lebesgue is combined with the known fact that a minimizing function must necessarily be harmonic, then we obtain the following result: *a minimizing function exists if and only if* (*i*) *there exists at least one admissible function and* (*ii*) *the Dirichlet problem is solvable for the given boundary values.* This elegant formulation of the Dirichlet principle seems to be due originally to Kamke & Lorentz.

The preceding example has been discussed in some detail because of its close bearing on the problems considered here. Indeed, Theorem 1 shows that the "if" part of the italicized statement holds also for the variational problem I. In section 4 we shall prove the "only if"

(1) Boundedness is assured if u has bounded derivatives in R (this is the case usually found in the literature), or it may be an inherent property of the integrand, as in the case $F = \sqrt{1 + p^2 + q^2}$.

part for a large class of integrands. In combination, these results can appropriately be called a generalized Dirichlet principle.

The theorems of section 4 can also be interpreted as conditions for a minimizing function to be of class C^2. In particular, Theorems 3.1 and 3.2 are special cases of known results of Morrey. They are included here because they apply to a fairly wide range of integrands and can be proved in a matter of a few lines.

Before proceeding to the proof, I wish to thank Professor Johannes Nitsche for several stimulating and helpful conversations concerning various aspects of the paper.

3. Proof of Theorem 1. We recall that the classical "proof" by integration by parts fails because a certain boundary integral may not have meaning. To overcome this difficulty we propose to truncate a given admissible function v by means of surfaces $z = u(x, y) \pm \varepsilon$, and to show that this process actually reduces the integral $I[v]$. It will then follow that $I[u] \leqslant I[v]$ by letting $\varepsilon \to 0$. This process must be handled in a somewhat roundabout fashion so as to make it perfectly rigorous, which accounts for the method finally adopted. (A somewhat similar idea occurs in the note of Lebesgue cited above.)

Now let $v(x, y)$ be any function in $\mathfrak{A}$, or in $\mathfrak{A}^*$. Let S be a fixed closed subregion of R with smooth boundary Σ, and let ε be an arbitrary positive number. Since u and v take the same boundary values, we can find a closed region B such that $S \subset B \subset R$, and

$$|u - v| < \varepsilon/3 \quad \text{in } R - B.$$

Furthermore, by virtue of condition $\mathfrak{A}$ 2 or $\mathfrak{A}$ 2* it is clear that there exists a polynomial w such that

$$|v - w| < \varepsilon/3 \quad \text{in } B,$$

and

$$I[w, B] \leqslant I[v] + \varepsilon. \tag{8}$$

Finally, there exists a sequence of polynomials u_n with the property

$$|u - u_n|, \ |\nabla u - \nabla u_n| \to 0 \text{ uniformly in } B,$$

where ∇ denotes the vector gradient. Without loss of generality one can assume that $|u - u_n| < \varepsilon/3$ for all n.(1)

We now define the function

(1) By careful use of the theory of measure, one can omit this preliminary approximation. Our preference, however, is for the course of proof actually adopted.

$$\phi = \begin{cases} u_n + \varepsilon & \text{if } w > u_n + \varepsilon \\ u_n - \varepsilon & \text{if } w < u_n - \varepsilon \\ w & \text{otherwise.} \end{cases}$$

Obviously ϕ is piecewise differentiable in B, and can be substituted in the integral I. Then, in virtue of the definition of the Weierstrass E function and the fact that F is independent of u, we have

$$I[w, B] - I[\phi, B] = \int_B [\zeta_x F_p + \zeta_y F_q]_\phi \, dA + K, \tag{9}$$

where $$\zeta = w - \phi, \quad K = \int_B E(x, y, \phi_x, \phi_y; w_x, w_y)\, dA,$$

and the subscript ϕ denotes evaluation of F_p and F_q for the arguments (x, y, ϕ_x, ϕ_y). Now at all points of B at which $\zeta \neq 0$ we have $\phi = u_n \pm \varepsilon$, hence

$$\begin{aligned}\int_B [\zeta_x F_p + \zeta_y F_q]_\phi \, dA &= \int_B [\zeta_x F_p + \zeta_y F_q]_{u_n} \, dA \\ &= \int_B [\zeta_x F_p + \zeta_y F_q]_u \, dA + \varepsilon_1,\end{aligned}$$

where $\varepsilon_1 \to 0$ as $n \to \infty$. In virtue of the preceding construction, $\zeta = 0$ on and near the boundary of B, so that by hypothesis the last integral vanishes. Combining this result with (9) yields the formula

$$I[\phi, B] = I[w, B] - K - \varepsilon_1. \tag{10}$$

Turning next to an estimate for the integral $I[u, S]$, we have in the same way as (9),

$$I[\phi, S] - I[u, S] = \int_S [\eta_x F_p + \eta_y F_q]_u \, dA + K_1, \tag{11}$$

where $\eta = \phi - u$ and $K_1 \geqslant 0$.

Using condition (6), the integral appearing on the right in (11) is easily shown to equal

$$\oint_\Sigma \eta \,(F_p \, dy - F_q \, dx).$$

But $|\eta| \leqslant |\phi - u_n| + |u_n - u| \leqslant \frac{4}{3}\varepsilon$, whence from (11),

$$I[u, S] \leqslant I[\phi, S] + \frac{4}{3}\varepsilon \oint_\Sigma |F_p \, dy - F_q \, dx|.$$

Noting that $I[\phi, S] \leqslant I[\phi, B]$, and using (10) and (8) to estimate this last integral, we obtain

$$I[u, S] \leqslant I[v] - K - \varepsilon_1 + \text{Const.}\,\varepsilon.$$

In this inequality we may let $n \to \infty$; setting $\Lambda = \lim_{n\to\infty} K$ there results the main estimate

$$I[u, S] \leqslant I[v] - \Lambda + \text{Const.}\,\varepsilon. \tag{12}$$

Since Λ is certainly $\geqslant 0$, we conclude that $I[u, S] \leqslant I[v]$, and (7) follows at once.

In order to prove the remainder of Theorem 1 one can proceed as in sections 9 and 10 below. We shall omit the details, however, and merely accept the final statement of Theorem 1 as a corollary of Theorem 4.

Remark. The preceding theorem applies also to certain integrands in which the unknown function appears explicitly. Consider in particular the integrand

$$F(x, y, p, q) + G(x, y, u),$$

in which G and its first partial derivative G_u are continuous functions defined over the *closure* of R. *We suppose that G_u is a non-decreasing function of u.* Then by carrying out the steps in the preceding proof one finds in place of (10),

$$I[\phi, B] = I[w, B] - K' - \varepsilon_1 - \varepsilon_2, \tag{10$'$}$$

where ε_2 tends to zero with ε, and K' has a slightly different meaning than K but still remains $\geqslant 0$. The main estimate (12) continues to hold, except that the error term Const. ε must be replaced by a quantity which merely tends to zero as $\varepsilon \to 0$, $S \to R$.

Similar remarks hold for the integrand

$$F(x, y, p, q) + L(x, y, u)p + M(x, y, u)q + N(x, y, u),$$

in which L, M, N, and their partial derivatives L_x, M_y, and N_u are continuous functions defined in the closure of R. The condition on G_u in the preceding example is here replaced by the condition that $N_u - L_x - M_y$ should be a non-decreasing function of u. [The reason for this requirement may be seen from the identity

$$Lp + Mq = \left(\int L\,du\right)_x + \left(\int M\,du\right)_y - \int (L_x + M_y)\,du,$$

from which it follows that the function $N - \int (L_x + N_y)\,du$ plays the same role as G did before.] Assuming finally that the total integrand $F + Lp + Mq + N$ is bounded from below, one can again carry out the necessary steps in the proof of Theorem 1. In case the boundary of R is rectifiable we can dispense with the boundedness condition, for in this case the divergence terms above contribute only a fixed boundary integral.

4. Differentiability of minimizing functions. By using Theorem 1 one can obtain several simple conditions guaranteeing the differentiability of a minimizing function (even should the minimum be known only in the general class $\mathfrak{A}^*$). For simplicity we shall consider in this respect only integrands $F(x, y, p, q)$ which are of class C^2. Corresponding results for more general integrands can, however, easily be formulated.

Let $\mathfrak{K}$ denote a covering of R by open circular disks. We shall say that an integrand $F(x, y, p, q)$ is *tame* if it satisfies conditions (3) and (4), the latter with equality excluded, and if the Dirichlet problem for the Euler-Lagrange equation

$$F_{pp}r + 2F_{pq}s + F_{qq}t + F_{px} + F_{qy} = 0 \tag{13}$$

is solvable for all circles of some covering $\mathfrak{K}$.(1) The functions $p^2 + q^2$ and $\sqrt{1 + p^2 + q^2}$ are simple examples of tame integrands; in Theorems 3.1 and 3.2 we shall note some others.

THEOREM 2. *Let the integrand $F(x, y, p, q)$ be tame, and suppose $u = u(x, y)$ minimizes the integral I among all functions of class $\mathfrak{A}$, or $\mathfrak{L}$, or $\mathfrak{A}^*$. Then u is of class C^2 in R.*

Proof. Let P be any point of R, and let K be a circle about P in which the Dirichlet problem is solvable. Suppose U is the solution of (13) which agrees with u on the boundary of K. We assert that $U \equiv u$ in K. For if not, then we shall be able to define a new admissible function u^* such that $I[u^*] < I[u]$, contradicting the fact that u is a minimizing function. Granting this step for a moment, from the equality $U \equiv u$ it follows that u is of class C^2 in the neighborhood of P, and the theorem is proved.

It remains therefore to construct the function u^*. This will take a slightly different form in each of the three cases of the theorem. Suppose first that u minimizes in the class $\mathfrak{L}$. We define

$$u^* = \begin{cases} U + \delta & \text{if } u > U + \delta \\ U - \delta & \text{if } u < U - \delta \\ u & \text{otherwise.} \end{cases}$$

Then u^* is actually different from u, provided that δ is chosen small enough. Clearly $u^* \in \mathfrak{L}$; moreover, by Theorem 1

(1) There may be some ambiguity in this definition. To be precise, in saying that the Dirichlet problem is solvable for a region K we mean that, given any continuous data on the boundary of K, there exists a corresponding solution of (13), of class C^2 in the interior of K, and continuously taking on the given values on the boundary of K.

$$I[u] = I[u, R'] + I[u, R - R'] > I[U \pm \delta, R'] + I[u, R - R'] = I[u^*], \tag{14}$$

where R' denotes the (open) set where $u^* \neq u$. In case u minimizes in the class $\mathfrak{A}$, the function u^* defined above is only piecewise smooth, and therefore not admissible to competition. This difficulty is avoided by smoothing u^* at its ridges, taking care not to destroy the inequality (14).

Finally, if u minimizes in the class $\mathfrak{A}^*$, the preceding argument fails since the two equalities in (14) may no longer hold. In this case we set

$$u^* = \begin{cases} U & \text{in } K \\ u & \text{in } R - K. \end{cases}$$

Using a modified version of the proof of Theorem 1, in which the function u is truncated by means of the surfaces $z = U \pm \varepsilon$, it is easily shown that $u^* \in \mathfrak{A}^*$ and $I[u^*] < I[u]$. This completes the proof of Theorem 2.

While the above result is valid in n-dimensional space, the following is so far known to hold *only in the plane*. To extend it to higher dimensions will require a significant advance in the theory of partial differential equations (cf. [22], [5] and [19]).

THEOREM 3.1. *Let the integrand $F(x, y, p, q)$ satisfy the following conditions*

$$k(\xi^2 + \eta^2) \leqslant F_{pp}\xi^2 + 2F_{pq}\xi\eta + F_{qq}\eta^2 \leqslant K(\xi^2 + \eta^2)$$
$$F_{px}^2 + F_{qy}^2 \leqslant K(1 + p^2 + q^2),$$

where k and K are positive continuous functions of x, y. Suppose also that the five partial derivatives of F which appear in the above conditions are Hölder continuous. Let $u = u(x, y)$ minimize the integral I among all functions of class $\mathfrak{A}^*$. Then u is of class C^2 in R.

Proof. It is clear that F is bounded below by some fixed function of x, y, whence by a simple change of integrand we can suppose without loss of generality that $F \geqslant 0$. Also $F_{pp}F_{qq} - F_{pq}^2 > 0$, so that by Taylor's formula $E > 0$ for $(P, Q) \neq (p, q)$. Theorem 3.1 will therefore follow from Theorem 2 if the Dirichlet problem is solvable for equation (13). But the conditions of the theorem are sufficient for the solvability of the Dirichlet problem over circles in R (and in fact over much more general regions) for arbitrary continuous boundary data.(1) This completes the proof.

THEOREM 3.2. *Let the integrand $F = F(p, q)$ be of class C^3 and satisfy the conditions*

$$F \geqslant 0, \quad F_{pp}F_{qq} - F_{pq}^2 > 0.$$

(1) This has been proved by P. C. Rosenbloom and the author.

Let $u = u(x, y)$ minimize the integral I among all functions of class $\mathfrak{L}$. Then u is of class C^2 in R.

Proof. The conditions placed on F are not enough to conclude that F is tame according to the definition given at the opening of this section. On the other hand, from the proof of Theorem 2 it is apparent that we actually need to solve the Dirichlet problem only for Lipschitzian boundary values. Now the Euler-Lagrange equation takes the form

$$ar + 2bs + ct = 0, \tag{15}$$

where $ac - b^2 > 0$ and a, b, c are Hölder continuous functions of p and q. Under these conditions it can be shown(1) that (15) is solvable for convex regions with Lipschitzian boundary data, and the proof is complete.

Theorem 3.1 and 3.2 are contained in known results of Morrey, but the simplicity of their proofs (granted certain existence theorems for quasi-linear partial differential equations) justifies their inclusion here.

5. Example: the area integral. Let us set $W = \sqrt{1 + p^2 + q^2}$, and consider the (generalized) integral

$$A[u] = \int_R W\,dA.$$

It is clear that $A[u]$ is exactly the Lebesgue area of the surface $z = u(x, y)$ over R. Moreover, since $W \geqslant 0$, $W_{pp}W_{qq} - W_{pq}^2 > 0$, the conclusions of Theorems 1 and 2 apply unaltered in the present case.

Consider, then, the problem of spanning a curve with a surface of least area, area being understood in the sense of Lebesgue. From Theorem 1, *a minimal surface of the form $z = u(x, y)$ which spans a given curve Γ has smaller area than any other surface of the same form which spans Γ*. From Theorem 2, since W is tame (see [21], 101), *any surface of least area which has the form $z = u(x, y)$ is a minimal surface*, and, more generally, *any surface of least area which locally can be represented in the form $z = u(x, y)$, for some orientation of coordinates, is a minimal surface*. It happens that these results are not new, but have already been proved by E. J. McShane (except that in his work Γ was required to have a convex projection). On the other hand, it seems that the present proof is both simpler and more general in application than that of McShane.

Part II

6. Fields and imbedding. Before stating our main theorem, it is first necessary to define precisely the nature of the fields which we shall consider.

Consider a family of surfaces, each of which can be represented in the form $z = f(x, y)$.

A region $\mathcal{V}$ in (x, y, z) space is said to be *smoothly covered* by such a family if (i) through each point of $\mathcal{V}$ there passes exactly one surface of the family, (ii) each surface is continuously differentiable, and (iii) the slope functions $\mathfrak{p}\ (x, y, z)$ and $\mathfrak{q}\ (x, y, z)$ of the family are continuous in $\mathcal{V}$ and have continuous first partial derivatives $\mathfrak{p}_z$ and $\mathfrak{q}_z$. We shall say that $\mathcal{V}$ is z-simple if its intersection with any straight line $x = x_0$, $y = y_0$ is a connected set. The principal definitions of field and imbedding can now be stated.

Definition 1. A field of extremals is a z-simple region $\mathcal{V}$ of (x, y, z) space together with a family $\mathcal{S}$ of extremal surfaces which smoothly covers $\mathcal{V}$.

Definition 2. Suppose that $u = u(x, y)$ is an extremal, and let D denote the interior of its domain of definition. Also let $\mathcal{E}$ denote the extremal surface $z = u(x, y)$ for $(x, y) \in D$. We shall say that u is *weakly imbedded* in a field of extremals $(\mathcal{V}, \mathcal{S})$ if and only if

1. $\mathcal{E}$ is a member (or a portion of a member) of the family $\mathcal{S}$, and
2. If $\mathcal{E}_1$ is any other surface in $\mathcal{S}$, then the vertical distance from $\mathcal{E}$ to $\mathcal{E}_1$ is positive.

It is important to observe that the concept of weak imbedding requires no differentiability of u at the boundary of its domain of definition. This is in contrast with the usual situation (Bolza, Bliss) in which u must be continuously differentiable over its entire domain of definition.

7. Some properties of the Hilbert integral. Let S be a closed region in the plane with piecewise smooth boundary Σ. Let U be a piecewise smooth function in S, whose values lie in the region $\mathcal{V}$ of a field of extremals.(1) Then the Hilbert invariant integral $I^*[U, S]$ is defined by

$$I^*[U, S] = \int_S \{F + (P - \mathfrak{p}) F_p + (Q - \mathfrak{q}) F_q\}\, dA,$$

where the arguments of F, F_p, and F_q are $(x, y, U, \mathfrak{p}, \mathfrak{q})$ and

$$\mathfrak{p} = \mathfrak{p}(x, y, U), \quad \mathfrak{q} = \mathfrak{q}(x, y, U).$$

If V is another function satisfying the same conditions as U, then we have the familiar result

$$I^*[U, S] = I^*[V, S] \tag{16}$$

whenever $U = V$ on Σ. Formula (16) is not difficult to prove when the surfaces of the extremal family $\mathcal{S}$ and the integrand F are both of class C^2 (cf. [2] and [9]). The general case under consideration here requires a more sophisticated argument which we shall give in § 11. If the condition $U = V$ on Σ is not satisfied, then (16) must be replaced by

(1) By this we mean simply that the points (x, y, U) for $(x, y) \in S$ lie in $\mathcal{V}$.

$$I^*[U, S] = I^*[V, S] + \oint_{\Sigma} (U - V)(\tilde{F}_p\, dy - \tilde{F}_q\, dx), \tag{17}$$

where the tildes denote evaluation at an intermediate value between U and V. Formula (17) is proved in exactly the same way as (16), except that a certain boundary integral which vanishes in the derivation of the former result must now be carried along. Finally, let us note the obvious equality

$$I^*[u, S] = I[u, S] \tag{18}$$

when u is an extremal of the field $(\mathfrak{V}, S)$.

8. The main theorem. With the terminology and results of the preceding two sections understood, the main result of the paper may now be stated.

THEOREM 4. *Let $u = u(x, y)$ be an extremal for the variational problem I. Suppose that u can be weakly imbedded in an extremal field $(\mathfrak{V}, S)$, and that there exists at least one function $v \in \mathfrak{A}^*$ such that the values of v over R lie in $\mathfrak{V}$. Then $I[u]$ exists, and*

$$I[u] \leqslant I[v] \tag{19}$$

for each $v \in \mathfrak{A}^*$ whose values over R lie in $\mathfrak{V}$.

If the equality sign in (4) *is excluded, then $I[u] < I[v]$ for each v different from u.*

Proof. For simplicity we shall assume that the boundary of $\mathfrak{V}$ contains no points of the extremal surface

$$z = u(x, y), \quad (x, y) \in R.$$

It will be clear from the proof that this is no essential restriction.

Now let v be any function in $\mathfrak{A}^*$ whose values over R lie in $\mathfrak{V}$, let S be a fixed closed subregion of R with smooth boundary Σ, and let ε be an arbitrary positive number. In virtue of the assumption made in the preceding paragraph, we can find extremals u' and u'' in S such that

$$u - \varepsilon < u'' < u < u' < u + \varepsilon \quad \text{in } S.$$

Moreover, by condition 2 in the definition of weak imbedding there exists a number $\delta > 0$ such that

$$|u - u'|,\ |u - u''| > \delta.$$

Next, let B be a closed region such that $S \subset B \subset R$ and

$$|u - v| < \delta/3 \quad \text{in } R - B.$$

By hypothesis $\mathfrak{A}\ 2^*$ one can find a polynomial w with the properties

$$|v-w|<\delta/3 \quad \text{in } B$$

and
$$I[w, B] \leqslant I[v]+\varepsilon. \tag{20}$$

The following two paragraphs are devoted to proving the fundamental inequality (24). In order for the demonstration to be completely rigorous it is necessary to approximate both u' and u'' by polynomials. Unfortunately, the idea of the proof is somewhat obscured by this procedure, so we prefer instead to work formally, leaving the details of the approximation process to a later footnote.

We turn now to the computational aspect of the proof. Let the function ϕ be defined by

$$\phi=\begin{cases} u' & \text{if } w>u' \\ u'' & \text{if } w<u'' \\ w & \text{otherwise.} \end{cases}$$

Our first task is to find a formula for the quantity

$$\Lambda=I[w, B]-I[\phi, B].$$

To this end, let C' denote the set of points in B where $w>u'$ and C'' the set where $w<u''$. Setting $C=C'+C''$ we have

$$\begin{aligned} \Lambda &= I[w, C]-I[u', C']-I[u'', C''] \\ &= I[w, C]-I^*[u', C']-I^*[u'', C''] \\ &= I[w, C]-I^*[w, C]. \end{aligned}$$

In the last two equalities we have used properties (18) and (16) of the invariant integral, and the fact that $w=u'$ on the boundary of C', $w=u''$ on the boundary of C''. Using the Weierstrass function to represent the last expression yields finally

$$\Lambda=\int_C E(w)\, dA, \tag{21}$$

where $E(w) \equiv E(x, y, w, \mathfrak{p}, \mathfrak{q}; w_x, w_y)$.

We may now obtain the main estimate for the integral $I[u, S]$. By virtue of (18) and (17) we have

$$I[u, S]=I^*[u, S]=I^*[\phi, S]+\oint_\Sigma (u-\phi)(\tilde{F}_p\, dy-\tilde{F}_q\, dx). \tag{22}$$

Moreover, using respectively inequalities (4) and (3),

$$I^*[\phi, S] \leqslant I[\phi, S] \leqslant I[\phi, B]=I[w, B]-\Lambda \tag{23}$$

Thus, since $|u-\phi| \leqslant \varepsilon$ on Σ, it follows from (22), (23) and (20) that

$$I[u, S] \leqslant I[v] - \Lambda + \text{Const}\cdot\varepsilon, \qquad \Lambda = \int_C E(w)\,dA. \tag{24}$$

This result is fundamental for the further conclusions of the paper.[1]

To establish the first part of Theorem 4 we let $\varepsilon \to 0$ in (24). Using the fact that $\Lambda \geqslant 0$ there results simply

$$I[u, S] \leqslant I[v],$$

and (19) follows at once. The final statement of Theorem 4 requires a fairly elaborate treatment of its own, which will occupy us in the following two sections.

9. Some inequalities. We begin by deriving a more explicit form of the fundamental inequality (24). Let D denote an arbitrary closed subregion of R in which $v \neq u$. Choosing S and ε respectively so that $D \subset S$ and $|u-v| \geqslant 2\varepsilon$ in D, it is clear that the set C defined in the preceding section contains D. Using the fact that $E \geqslant 0$ then allows us to write in place of (24),

$$I[v] - I[u, S) \geqslant \int_D E(w)\,dA - \text{Const. } \varepsilon. \tag{25}$$

Consider now a function $v \in \mathfrak{A}$. The polynomial w can, under these circumstances, be chosen so that $|\nabla w - \nabla v| < \varepsilon$ in B. Then letting $\varepsilon \to 0$ in (25), and afterwards $S \to R$, yields

$$I[v] - I[u] \geqslant \int_D E(v)\,dA, \tag{26}$$

where D is an arbitrary subregion of R in which $v \neq u$.

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(1) To make the above argument rigorous, we approximate u' and u'' by sequences $\{u_n'\}$ and $\{u_n''\}$ of polynomials such that

$$|u' - u_v'|, \ |\nabla u' - \nabla u_n'| \to 0 \quad \text{uniformly in } B,$$

with similar relations for u_n''. It may be supposed that $|u' - u_n'|$ and $|u'' - u_n''|$ are less than $\delta/3$ for all n.

In the definition of ϕ one then replaces u' and u'' by their approximants; letting Λ_n stand for the resulting difference $I[w, B] - I[\phi, B]$, formula (21) becomes

$$\Lambda_n = \int_{C_n} E(w)\,dA + \varepsilon_1, \tag{21$'$}$$

in which $\varepsilon_1 \to 0$ as $n \to \infty$. Finally, in place of (24) we have

$$I[u, S] \leqslant I[v] - \Lambda_n + \text{Const. } \varepsilon. \tag{24$'$}$$

Letting $n \to \infty$ in this formula gives exactly the present estimate (24).

The following lemma, of a kind originally due to E. E. Levi, will be instrumental in the sequel.

Suppose $E > 0$ *for all* $(P, Q) \neq (p, q)$. *Let* $\mathfrak{D}$ *denote a bounded set in the five-dimensional space* (x, y, u, p, q). *Then there exists a convex function* $f(t)$ *such that* $f(0) = 0$, $f(t) > 0$ *if* $t > 0$, and

$$E(x, y, u, p, q; P, Q) \geqslant f(X), \quad X = \sqrt{(P-p)^2 + (Q-q)^2}, \tag{27}$$

for all $(x, y, u, p, q) \in \mathfrak{D}$.

Proof. For fixed (x, y, u) consider the surface $F = F(x, y, u, P, Q)$ over the P, Q plane. Let this surface (the figuratrix) be denoted by S, and let T denote the tangent plane to S at the fixed point p, q. Evidently the Weierstrass E-function expresses the vertical distance from T to S. Therefore, according to hypothesis, S lies everywhere above T, and since this holds for arbitrary (p, q), S is seen to be a *strictly convex surface.*

This being shown, it is easy to conclude the existence of the required function $f(t)$. For completeness we shall indicate an explicit construction. Set

$$g(t) = \underset{\mathfrak{D}}{\text{g.l.b.}} \ \underset{X=t}{\text{Min}} \ E(x, y, u, p, q; P, Q).$$

Obviously $g(0) = 0$ and $g(t) > 0$ for $t > 0$; also, because the surface S is convex one sees that

$$g(t) \geqslant g(1)t \quad \text{for } t \geqslant 1.$$

We may now choose for $f(t)$ the function defining the lower boundary of the convex hull of the graph of $g(t)$, $0 \leqslant t < \infty$. That $f(t)$ has the required properties is easily verified, and the lemma is proved.

10. Completion of the proof. It must still be shown that $I[u] < I[v]$, provided that v is different from u.

For simplicity, consider first the case of a function $v \in \mathfrak{A}$, $v \not\equiv u$. Supposing that $I[u] = I[v]$, then by virtue of (26) and the fact that $E \geqslant 0$ we have

$$E(v) \equiv 0 \ \text{ for } \ (x, y) \in D. \tag{28}$$

Since (4) is assumed to hold with the equality excluded, this implies

$$v_x = \mathfrak{p}(x, y, v), \quad v_y = \mathfrak{q}(x, y, v).$$

It follows that $z = v(x, y)$ is an integral surface of the slope functions $\mathfrak{p}$ and $\mathfrak{q}$. However, since $\mathfrak{p}_z$ and $\mathfrak{q}_z$ are continuous, there is but one solution surface through each point. Consequently, the surface $z = v(x, y)$ over D coincides with a member of the family $\mathcal{S}$. Since D

was an arbitrary set where $v \neq u$, it follows that v is an extremal of $\mathfrak{S}$ for all $(x, y) \in R$. Finally, by condition 2 in the definition of weak imbedding, the surfaces $z = v(x, y)$ and $z = u(x, y)$ must be a finite distance apart (recall that we have assumed $v \not\equiv u$). But this is impossible, for then v could not satisfy the given boundary condition. This contradiction proves that $I[u] < I[v]$.

Consider next the case when v is contained in $\mathfrak{A}^*$. It is no longer possible to establish (26), of course, because v need not be differentiable. However, if $I[u] = I[v]$ we do have

$$\lim_{\varepsilon \to 0} \int_D E(w)\, dA = 0, \tag{29}$$

as follows at once from (25). We shall suppose in the sequel that D is *convex*.

Now for $(x, y) \in D$ the arguments $(x, y, w, \mathfrak{p}, \mathfrak{q})$ in $E(w)$ are confined to some bounded set $\mathfrak{D}$. Therefore, using (29) and (27), together with Jensen's inequality,(1) we find

$$\lim \int_D X\, dA = 0; \tag{30}$$

here $X = \sqrt{(w_x - \mathfrak{p})^2 + (w_y - \mathfrak{q})^2}$, and the limit is evaluated for $\varepsilon \to 0$.

It is now an easy exercise to show that for any two points (x_1, y) and (x_0, y) in D, we have

$$v(x_1, y) - v(x_0, y) = \int_{(x_0, y)}^{(x_1, y)} \mathfrak{p}(x, y, v)\, dx. \tag{31}$$

Consequently, the derivative v_x exists and satisfies the first of the following two equations,

$$v_x = \mathfrak{p}(x, y, v), \quad v_y = \mathfrak{q}(x, y, v); \tag{32}$$

the second is derived by similar reasoning. Equations (32) being verified, the remainder of the proof is exactly the same as in the case $v \in \mathfrak{A}$. This completes the demonstration of Theorem 4.

It remains for us to show that Theorem 1 can be obtained as a corollary of Theorem 4. To do this, it is sufficient to show that the extremal $u(x, y)$ of Theorem 1 can be weakly imbedded in an extremal field for which $\mathfrak{V}$ is the entire cylinder $R \times \{z \mid -\infty < z < +\infty\}$. The required field can be obtaining by taking $\mathfrak{S}$ to be the family of vertical translates of the given surface $z = u(x, y)$. It is clear that $\mathfrak{S}$ smoothly covers the cylinder in question, and that the imbedding conditions are satisfied. Thus Theorem 1 is proved.

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(1) Cf. Polya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, p. 53. We require only the case $p = 1$.

PART III

11. A proof of Hilbert's formula. It has already been remarked that Hilbert's formula (16) can be proved without difficulty if suitable differentiability conditions are available. In the present circumstances these conditions do not hold, and an alternate proof must be found. We shall follow an argument of Bliss ([1] § 18), except for a few essential changes.

Let the surfaces of S be represented in the form

$$z = u(x, y, a), \tag{33}$$

where a is a real parameter (the surface corresponding to a given value of the parameter may consist of several disjoint parts, but this causes no difficulty). Because of the continuity of $\mathfrak{p}_z$ and $\mathfrak{q}_z$ it can be assumed that $u(x, y, a)$ is of class C^1 in all variables, with

$$\frac{\partial u}{\partial a} > 0. \tag{34}$$

An explicit construction of the family (33), so that (34) holds, can be obtained by choosing the parameter a to satisfy

$$u(x_0, y_0, a) = a \tag{35}$$

for some convenient constants x_0, y_0. Considering $u(x, y, a)$ to be the solution of the system

$$u_x = \mathfrak{p}(x, y, u), \quad u_y = \mathfrak{q}(x, y, u)$$

with (35) as initial condition, it follows from a well-known theorem of ordinary differential equations that u is of class C^1 in all variables. Moreover, $\partial u/\partial a > 0$. For otherwise, suppose $\partial u/\partial a = 0$ for some values x_1, y_1, a_1. Then we could reparametrize the surfaces so that $u(x_1, y_1, b) = b$, and it would follow that $\partial u/\partial b = \infty$ for the values $x_0, y_0, b(a_1)$. This is impossible. Because of (34), the transformation (33) has a *differentiable* inverse $a(x, y, z)$.

There is clearly no loss of generality in supposing that the surfaces $z = U$ and $z = V$ lie entirely interior to $\mathfrak{V}$. This being the case, there then exists a proper subregion $\mathfrak{V}'$ of $\mathfrak{V}$ which contains both surfaces, as well as the volume enclosed between them.

These preliminaries taken care of, let us define a function $\mathcal{A} = \mathcal{A}(x, y, a)$ by means of the integral average

$$\mathcal{A} = \frac{3}{4\pi\delta^3} \iiint F_p(\xi, \eta, u, u_x, u_y)\, d\xi\, d\eta\, d\alpha,$$

in which the arguments of u, u_x, u_y in the integrand are (ξ, η, α), and the domain of integration is a sphere of radius δ about the point (x, y, a). The constant δ is chosen so (small) that the integral is well-defined whenever (x, y, a) corresponds to a point in $\mathcal{V}'$. We define $\mathcal{B}$, $\mathcal{C}$ by similar formulae, but with F_q and F_u respectively replacing F_p. Since (6) holds identically in a, a well-known argument (cf. [3], or [1], § 15) shows that

$$\mathcal{A}_x + \mathcal{B}_y = \mathcal{C}. \tag{36}$$

This equation may be thought of as a kind of averaged version of the Euler-Lagrange equation.

For convenience in carrying out the remaining steps in the proof we define the two functions

$$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}(x, y, z) \equiv \mathcal{A}(x, y, a(x, y, z)), \quad \tilde{\mathcal{B}} = \tilde{\mathcal{B}}(x, y, z) \equiv \mathcal{B}(x, y, a(x, y, z)). \tag{37}$$

The functions $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are obviously differentiable, and indeed we have the useful relations

$$\tilde{\mathcal{A}}_x + \tilde{\mathcal{A}}_z\, \mathfrak{p} = \mathcal{A}_x, \qquad \tilde{\mathcal{B}}_x + \tilde{\mathcal{B}}_z\, \mathfrak{q} = \mathcal{B}_x, \tag{38}$$

obtained by differentiating (37) while keeping a fixed. Now let us put

$$\mathfrak{J}^*[U, S] = \int_S \{F + (P - \mathfrak{p})\tilde{\mathcal{A}} + (Q - \mathfrak{q})\tilde{\mathcal{B}}\}\, dA,$$

where the arguments of F, $\mathfrak{p}$, and $\mathfrak{q}$ are the same as in the integral $I^*[U, S]$ of section 7, while the arguments of $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are (x, y, U). Then by virtue of the limit properties of the integral average, it can be seen that $\mathfrak{J}^*[U, S]$ tends to $I^*[U, S]$ as δ tends to zero. Therefore, in order to prove (16) it is enough to show that

$$\mathcal{J} = \mathfrak{J}^*[V, S] - \mathfrak{J}^*[U, S] \to 0$$

as δ tends to zero. Let $W = U + t(V - U)$, $0 \leqslant t \leqslant 1$, and write $\mathfrak{J}^*(t)$ in place of $\mathfrak{J}^*[W, S]$. Then evidently

$$\mathcal{J} = \int_0^1 \frac{d\mathfrak{J}^*}{dt}\, dt.$$

The completion of the proof rests on a computation of $d\mathfrak{J}^*/dt$. For simplicity in making this computation we write the integrand of $\mathfrak{J}^*(t)$ in the form

$$W_x \tilde{\mathcal{A}}(x, y, W) + W_y \tilde{\mathcal{B}}(x, y, W) + \Phi(x, y, W),$$

where $\Phi = F - \mathfrak{p}\tilde{\mathcal{A}} - \mathfrak{q}\tilde{\mathcal{B}}$. Then, setting $\zeta = V - U$, we have

$$\frac{d}{dt}\mathcal{J}^*(t) = \int_S \{\zeta_x \tilde{\mathcal{A}} + \zeta_y \tilde{\mathcal{B}} + \zeta(\Phi_W + W_x \tilde{\mathcal{A}}_W + W_y \tilde{\mathcal{B}}_W)\} \, dA$$

$$= \int_S \{(\zeta \tilde{\mathcal{A}})_x + (\zeta \tilde{\mathcal{B}})_y + \zeta[\Phi_W - \tilde{\mathcal{A}}_x - \tilde{\mathcal{B}}_y]\} \, dA.$$

The divergence terms in this formula transform to a vanishing boundary integral. Also, by a straightforward calculation, making use of (36) and (38), the quantity in brackets turns out to be $O(\delta)$. Thus $\mathcal{J}$ itself is $O(\delta)$, and the proof is completed.

12. Concluding remarks. The main features of the paper, which I should like to mention in summary, are these. First, for an extremely wide class of integrands $F(x, y, p, q)$ an extremal can be shown to be a *minimizing* function, while for a somewhat smaller (but still significant) class, every minimizing function is an extremal. These results can appropriately be called a generalized Dirichlet principle. For integrands $F(x, y, u, p, q)$ the results are not so complete, yet in conjunction with the differentiability theorems of Morrey and others, they give a fairly satisfactory analogue of the earlier case. Recent work in variational problems and quasi-linear partial differential equations in more than two independent variables promises further applications.

Second, the class $\mathfrak{A}^*$ of admissible functions is of considerable interest in itself, both because it includes all the usual classes of admissible functions(1) and because of its connections and analogies to the theory of surface area. An interesting class of admissible functions, *not* included in $\mathfrak{A}^*$, is suggested by work of Cesari and Goffman on the theory of surface area. We shall say that a real-valued function $v = v(x, y)$ defined in the closure of R is in the class $\mathfrak{G}$ of admissible functions if and only if

$\mathfrak{G}$ 1. v continuously takes on assigned (continuous) values on the boundary of R.

$\mathfrak{G}$ 2. v is summable in R. The integral $I[v]$ is to be understood in the following generalized sense. Let $\{v_n\}$ be a sequence of continuously differentiable functions such that

$$\int_B |v - v_n| \, dA \to 0,$$

where B is an arbitrary closed subregion of R. Then

$$I[v] = \text{g.l.b.} \liminf I[v_n], \tag{40}$$

(1) The class of admissible functions treated by Hestenes is not included in $\mathfrak{A}^*$, nor does it include $\mathfrak{A}^*$. However, except in respect to boundary conditions, Hestenes' class *is* included in the class $\mathfrak{G}$ defined below.

the g.l.b. being taken over all sequences $\{v_n\}$ converging to v as above.

$\mathfrak{G}$ 3. $I[v]$ is finite.

For the integrand $\sqrt{1+p^2+q^2}$, definition (40) reduces to that of Cesari and Goffman for the area of the surface $z = v(x, y)$. In order for (40) to be logically consistent it must, of course, still be proved that if v is continuously differentiable in R, then $I[v]$, as defined by (40), has the classical value. This will be done in the following paper.

Theorems 1 and 4 remain true for the class $\mathfrak{G}$ of admissible functions, though the proofs need some modification. In the statement of these theorems, it is naturally understood that a function $v \in \mathfrak{G}$ is different from u if and only if $v \neq u$ at a set of positive measure. The validity of Theorem 2 is another question, and probably requires a stronger definition of *tameness*. We intend to treat these matters in more detail in a later paper, giving special attention to the area integrand.

References

[1]. G. A. Bliss, *The Calculus of Variations: Multiple Integrals*, Lecture Notes. University of Chicago, 1939.

[2]. O. Bolza, *Vorlesungen über Variationsrechnung*, B. G. Teubner, Leipzig, 1909.

[3]. M. Coral, On a necessary condition for the minimum of a double integral. *Duke Math. J.*, 3 (1937), 585–592.

[4]. R. Courant, *Dirichlet's Principle*. Interscience, New York, 1950.

[5]. E. De Giorgi, Sull'analiticità delle extremali degli integrali multipli. *Rend. Accad. Naz. Lincei*, 20 (1956), 438–441.

[6]. C. Goffman, Lower semi-continuity and area functionals. *Rend. Circ. Mat. Palermo* (2), 2 (1953), 203–235.

[7]. A. Haar, Zur Variationsrechnung. *Abh. Math. Sem. Univ. Hamburg*, 8 (1931), 1–27.

[8]. M. R. Hestenes, Sufficient conditions for multiple integral problems in the calculus of variations. *Amer. J. Math.*, 70 (1948), 239–276.

[9]. D. Hilbert, Mathematische Probleme. *Arch. Math. und Phys.* (3), 1 (1901), 213–237. Abhandlungen, vol. 3, 323–328.

[10]. ——, Zur Variationsrechnung. *Math. Ann.*, 62 (1906), 351–370. Abhandlungen, vol. 3, 38–55.

[11]. E. Kamke & G. G. Lorentz, Über das Dirichletsche Prinzip. *Math. Z.*, 51 (1947–1949), 217–232.

[12]. W. Karush, A semi-strong minimum for a multiple integral problem in the calculus of variations. *Trans. Amer. Math. Soc.*, 63 (1948), 439–451.

[13]. H. Lebesgue, Sur l'équivalence du problème du Dirichlet et du problème du calcul des variations considéré par Riemann. *C. R. Societé Math. de France* (1913), 48–50.

[14]. E. E. Levi, Sui criterii sufficienti per il massimo e per il minimo nel calcolo delle variazione. *Ann. Mat. Pura Appl.* (2), 21 (1913), 173–218.

[15]. L. Lichtenstein, Untersuchungen über zweidimensionale reguläre Variationsprobleme. *Monatsh. Math. und Phys.*, 28 (1917), 3–51.

[16]. E. J. McShane, On a certain inequality of Steiner. *Ann. of Math.*, 33 (1932), 125–138.

[17]. C. MIRANDA, Condizioni sufficienti per il minimo degli integrali doppi. *Memorie della reale accademia d'Italia*, 5 (1934), 159–172.

[18]. C. B. MORREY, JR., *Multiple Integral Problems in the Calculus of Variations and Related Topics*. University of California Publications in Mathematics, 1943. Especially Chap. VII.

[19]. J. NASH, Parabolic equations. *Proceedings Nat. Acad. Sci. U.S.A.*, 43 (1957), 754–758.

[20]. F. PRYM, Zur Integration der Differentialgleichung $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. *J. Reine Angew. Math.*, 73 (1871), 340–364 (in particular, 361–364).

[21]. G. RADÓ, *On the Problem of Plateau*. Springer, Berlin, 1932.

[22]. T. STAMPACCHIA, Sistimi di equazioni di tipo ellittico a derivate parziali del primo ordine e proprieta delle estremali degli integrali multipli. *Ricerche Mat.*, 1 (1951), 200–226.

SUPPLEMENTARY REFERENCES

(*Not quoted in the paper*)

G. A. BLISS, The calculus of variations for multiple integrals. *Amer. Math. Monthly*, 49 (1942), 77–89.

L. CESARI, *Surface area*. Ann. of Math. Studies No. 35. Princeton University Press, 1955. Especially Appendix B.

R. FINN, On equations of minimal surface type. *Ann. of Math.*, 60 (1952), 397–416.

E. J. MC SHANE, On the minimizing property of the harmonic function. *Bull. Amer. Math. Soc.*, 60 (1934), 593–598.

M. SHIFFMAN, Differentiability and analyticity of solutions of double integral variational problems. *Ann. of Math.*, 48 (1947), 274–284.

A. G. SIGALOV, Two dimensional problems of the calculus of variations. *Uspehi Matem. Nauk* (1951), 16–101.

——, Two dimensional problems of the calculus of variations in non-parametric form. *Trudy Moskov. Mat. Obšč.*, 2 (1953), 201–233.

——, Two dimensional problems of the calculus of variations in non-parametric form transformed into parametric form. *Mat. Sbornik*, 34 (1954), 385–406.

A. SILOVA, Existence of an absolute minimum of multiple integrals in the calculus of variations. *Doklady Akad. Nauk SSSR*, 102 (1955), 699–702.

L. TONELLI, *Fondamenti di calcolo delle variazione*. Bologna, 1922. Especially vol. I, 350–359.