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Wiener's tauberian theorem for spherical functions on the automorphism group of the unit disk

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Abstract. Our main result gives necessary and sufficient conditions, in terms of Fourier transforms, for an ideal in the convolution algebra of spherical integrable functions on the (conformal) automorphism group of the unit disk to be dense, or to have as closure the closed ideal of functions with integral zero. This is then used to prove a generalization of Furstenberg's theorem, which characterizes harmonic functions on the unit disk by a mean value property, and a "two circles" Morera type theorem (earlier announced by Agranovskiĭ).

Introduction

If G is a locally compact abelian group, Wiener's tauberian theorem asserts that if the Fourier transforms of the elements of a closed ideal I of the convolution algebra $L^1(G)$ have no common zero, then $I=L^1(G)$.

In the non-abelian case, the analog of Wiener's theorem for two-sided ideals holds for all connected nilpotent Lie groups, and all semi-direct products of abelian groups [27]. However, Wiener's theorem does not hold for any non-compact connected semisimple Lie group [15], [27].

In their seminal series of papers on harmonic analysis on the Lie group $SU(1,1)$, Ehrenpreis and Mautner use the ideal structure on the disk algebra $A(\mathbf{D})$ to show that the analog of Wiener's theorem fails even for the commutative subalgebra $L^1(G//K)$ of spherical functions ([15], see also [5]). They realized that in addition to the non-vanishing of the Fourier transforms, a condition on the rate of decay of the Fourier transforms at infinity is needed as well. For technical reasons, it

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was necessary for them to impose various smoothness conditions on the Fourier transforms, in addition to the natural conditions of non-vanishing of the Fourier transforms and the "correct" rate of decay, in their analog of Wiener's theorem ([14], see also [5]).

It is known that smoothness conditions make Wiener's theorem much easier. See, *exempli gratia*, [23], for a trivial proof that if $f \in L^1(\mathbf{R})$, and its Fourier transform $\hat{f}$ is slightly more regular than a general function in the Fourier image of $L^1(\mathbf{R})$ (its first and second derivatives should also belong to the Fourier image of $L^1(\mathbf{R})$) and never vanishes, then the closure of the convolution ideal generated by f is all of $L^1(\mathbf{R})$. The main result of the present paper is *a genuine analog of Wiener's theorem without any superfluous smoothness condition.* We use the method of the resolvent transform, as developed by Gelfand, Beurling, and Carleman [11]. Gelfand's approach was later rediscovered by Domar [13], and applied and extended by Hedenmalm and Borichev in the study of harmonic analysis on the real line, the half-line, and the first quadrant in the plane [20], [21], [9], [10].

As applications of the "correct" version of Wiener's theorem, we follow the ideas of [5], and give a generalization of a theorem of Furstenberg [16], [17] characterizing bounded harmonic functions in the unit disk as the bounded solutions of certain convolution equations (in other words, μ-harmonic functions), and a "two circles" Morera type theorem characterizing holomorphic functions in the unit disk.

The results of this article were announced in [4].

1. Preliminaries

The basic references for this section are [22], [25], [30], [5], [31], [14], [3].

Let $G=SL(2,\mathbf{R})$, where $SL(2,\mathbf{R})$ is the multiplicative group of all 2×2 real matrices with determinant 1. We identify G with $SU(1,1)$, the group of all complex matrices

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \text{with determinant } 1,$$

so that $\widetilde{G}=G/\{\pm 1\}$ (the $\{\pm 1\}$ indicates that we mod out with respect to the equivalence relation $A\sim -A$) coincides with the group of all conformal automorphisms

$$g(z)=\frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}, \quad z\in\mathbf{D},\ |\alpha|^2-|\beta|^2=1,$$

of the unit disk $\mathbf{D}$. The polar decomposition of G is $G=KA^+K$, where K is the subgroup of all "rotations" in $G=SU(1,1)$, with typical element

$$k=\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta\in\mathbf{R},$$

and A^+ is the set of matrices

$$a_\zeta = \begin{pmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{pmatrix}, \quad \zeta\in\mathbf{R}_+,$$

which we identify with the half-line $\mathbf{R}_+=[0,+\infty[$. The associated $\widetilde{K}=K/\{\pm 1\}$ in $\widetilde{G}$ is then the subgroup of all rotations of $\mathbf{D}$.

The left and right invariant Haar measure on G is normalized so that $dg=\sinh 2\zeta\, d\zeta\, d\varphi\, d\theta$, where $d\zeta$ is the Lebesgue measure on the positive real axis $\mathbf{R}_+$, and $d\varphi$ and $d\theta$ both equal the Haar measure on the rotation subgroup K, which we identify with the unit circle.

The symmetric space G/K (which may be identified with $\widetilde{G}/\widetilde{K}$) is identified with *the Poincaré model of the hyperbolic plane* $\mathbf{H}^2$, that is, with the unit disk. It carries the quotient measure $\sinh 2\zeta\, d\zeta\, d\varphi$ on G/K and the Riemannian structure

$$\langle u,v\rangle_z = \frac{(u,v)}{(1-|z|^2)^2},$$

where u and v are any tangent vectors at $z\in\mathbf{D}$. This structure enables us to define the Riemannian metric and measure on G/K, and the Laplace–Beltrami operator

$$\Delta = (1-x^2-y^2)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

The space $K\backslash(G/K)$ is denoted by $G//K$. It is canonically identified with A^+, with metric

$$d(a_0,a_\zeta)=d(0,\tanh\zeta)=\zeta, \quad \zeta\in\mathbf{R}_+;$$

the second $d(\cdot,\cdot)$ above refers to the hyperbolic distance in the unit disk $\mathbf{D}$. We may also think of it as the points on the radius $[0,1[$. The Haar measure on G induces the measure $\sinh 2\zeta\, d\zeta$ on $G//K$. If we parametrize the space $G//K$ with the variable $\tau\in[1,+\infty[$ instead, where $\tau=\cosh 2\zeta$, the induced "Haar" measure on $G//K$ is simply $\frac{1}{2}d\tau$.

In this paper we study $L^1(G//K)$—the convolution algebra of all integrable functions on G which are invariant under rotations from left and right. This is a commutative Banach algebra, and our main goal is Theorem 1.3, which gives a version of Wiener's tauberian theorem for this algebra.

To give a precise formulation of our result we first need some notations: Let Σ be the strip

$$\Sigma=\{s\in\mathbf{C}:0\le \operatorname{Re} s\le 1\}.$$

The space of all maximal (modular) ideals of $L^1(G//K)$ is identified with the quotient of Σ modulo the equivalence relation $s\sim 1-s$. With this identification, the Gelfand transform $\hat{f}$ of $f\in L^1(G//K)$ is given by

$$\hat{f}(s)=\int_G f(g)\varphi(g,s)\,dg=\int_0^{+\infty} f(\zeta)\varphi(\zeta,s)\sinh 2\zeta\,d\zeta, \tag{1-1}$$

where $s\in\Sigma$, and

$$\varphi(\zeta,s)=\frac{1}{2\pi}\int_0^{2\pi}(\cosh 2\zeta+\sinh 2\zeta\cos\theta)^{-s}\,d\theta,\quad s\in\mathbf{C}.$$

The terms Fourier transform and spherical transform are also used for this Gelfand transform. It naturally extends to the finite Borel measures on $G//K$ (which we call the spherical measures on G),

$$\hat{\mu}(s)=\int_G \varphi(g,s)\,d\mu(g).$$

The functions $\varphi(\,\cdot\,,s)$ are defined for all $s\in\mathbf{C}$ and are called the zonal spherical functions. They may be thought of as the normalized rotation invariant (that is, left K-invariant) eigenfunctions of the Laplace–Beltrami operator on $G/K\simeq\mathbf{D}$.

The following two lemmas summarize the basic properties of the zonal spherical functions and of the Fourier transform, and we present them without proofs:

Lemma 1.1. *Let $\varphi(\,\cdot\,,s)$ be a zonal spherical function. Then*:

(1) *$\varphi(\zeta,s)=P_{s-1}(\cosh 2\zeta)$, where P_{s-1} is the Legendre function of the first kind.*

(2) *The relations $\varphi(\zeta,s)=\varphi(\zeta,1-s)=\bar{\varphi}(\zeta,\bar{s})$ hold for all $\zeta\in\mathbf{R}^+$, $s\in\mathbf{C}$.*

(3) *For each fixed $\zeta\in\mathbf{R}^+$, $\varphi(\zeta,s)$ is an entire function of the complex variable s, and it is of exponential type 2ζ.*

(4) *If $s\in\Sigma$, then for all $\zeta\in\mathbf{R}^+$,*

$$|\varphi(\zeta,s)|\le 1.$$

(5) *For each fixed ζ and every strip $X=\{s:a\le \operatorname{Re} s\le b\}$, $\varphi(\zeta,s)\to 0$ uniformly as $|s|\to\infty$ in X.*

For an $f\in L^1(G//K)$, let us write $\tilde{f}(\tau)=f(\zeta)$, where $\tau=\cosh 2\zeta$, and $\zeta\in\mathbf{R}_+$ is the standard parameter for $G//K$. Then

$$\hat{f}(s)=\int_1^{+\infty} P_{s-1}(\tau)\,\tilde{f}(\tau)\,\tfrac{1}{2}d\tau,\quad s\in\Sigma, \tag{1-1$'$}$$

where P_{s-1} is the Legendre function of the first kind (see Section 2), because $P_{s-1}(\tau)=\varphi(\zeta,s)$, and because the measure $\sin 2\zeta\,d\zeta$ transforms to $\frac{1}{2}d\tau$.

Lemma 1.2. *The following assertions are valid:*

(1) *For each* $f \in L^1(G//K)$, *its Fourier transform* $\hat{f}$ *is continuous in the strip* Σ *and is analytic in the interior of* Σ. *If* f *has a compact support, then* $\hat{f}$ *is an entire function of finite exponential type.*

(2) $|\hat{f}(s)| \le \|f\|_{L^1}$ *for all* $s \in \Sigma$ *and* $f \in L^1(G//K)$.

(3) $\hat{f}(s) = \hat{f}(1-s)$ *for all* $s \in \Sigma$.

(4) $\hat{f}(s) \to 0$ *when* $|s| \to \infty$, $s \in \Sigma$. *In fact,* $\hat{\mu}(s) \to \mu(\{e\})$ *for any spherical measure* μ *on* G.

The above-mentioned holomorphic behavior of the Fourier transforms of functions in $L^1(G//K)$ is in sharp contrast with the L^2 situation: the Fourier image of $L^2(G//K)$ can be identified with a weighted L^2 space on the middle line $\frac{1}{2} + i\mathbf{R}$ (subject to the symmetry condition $f(1-s) = f(s)$).

For functions $f \in L^1(G//K)$, we need quantitative measures of decay of the Fourier transforms at ∞ and at 0 for $f \in L^1(G//K)$:

$$\delta_\infty(f) = -\limsup_{t \to +\infty} e^{-\pi t} \log \left| \hat{f}\left(\tfrac{1}{2} + it\right) \right|, \tag{1-2}$$

$$\delta_0(f) = -\limsup_{x \to 0^+} x \log |\hat{f}(x)|. \tag{1-3}$$

Since $\hat{f}$ is a bounded analytic function on the interior of Σ, which is conformally equivalent to the unit disk, it has a canonical factorization into an inner and an outer factor (unless of course $\hat{f}$ vanishes everywhere). The inner factor (regarded as a function on the disk) is the product of a Blaschke product and a singular inner function, and the quantities $\delta_\infty(f)$ and $\delta_0(f)$ measure the atomic part of the positive Borel measure associated with the singular inner function at the points on the unit circle corresponding to $\frac{1}{2} + i\infty$ and 0, respectively; since $\hat{f}(s) = \hat{f}(1-s)$, this also applies to the reflected points $\frac{1}{2} - i\infty$ and 1. For a collection $\mathfrak{S}$ of functions in $L^1(G//K)$, we write

$$\delta_\infty(\mathfrak{S}) = \inf\{\delta_\infty(f) : f \in \mathfrak{S}\},$$
$$\delta_0(\mathfrak{S}) = \inf\{\delta_0(f) : f \in \mathfrak{S}\}.$$

Let $L^1_0(G//K)$ be the closed (convolution) ideal of $L^1(G//K)$ of all functions with integral 0. In other words, $L^1_0(G//K)$ consists of those functions in $L^1(G//K)$ whose Fourier transforms vanish at 0 and hence, by symmetry, at 1. We now formulate the main result, an analog of Wiener's tauberian theorem for $L^1(G//K)$.

Theorem 1.3. *Let* $\mathfrak{S}$ *be a family of functions in* $L^1(G//K)$, *and let* $I(\mathfrak{S})$ *be the smallest closed ideal in* $L^1(G//K)$ *containing* $\mathfrak{S}$.

(1) $I(\mathfrak{S}) = L^1(G//K)$ *holds if and only if the Fourier transforms of elements of* $\mathfrak{S}$ *have no common zeros in* Σ, *and* $\delta_\infty(\mathfrak{S}) = 0$.

(2) *$I(\mathfrak{S})=L^1_0(G//K)$ holds if and only if the only common zeros in Σ of the Fourier transforms of elements of $\mathfrak{S}$ are* 0 *and* 1, *and $\delta_\infty(\mathfrak{S})=\delta_0(\mathfrak{S})=0$.*

Remark. Part (1) was conjectured in [5]. The 1 there instead of π in the definition of δ_∞ here was due to an oversight in the calculation.

The usual proof of Wiener's theorem for $L^1(G)$ (with G a locally compact abelian group) uses localization. In our case the Fourier transforms are analytic functions, hence the Fourier image of $L^1(G//K)$ does not contain functions with compact support, and another approach is needed. We shall use the resolvent transform method. Here is a sketch of this method as it applies to our setup.

Let $L^1_\delta(G//K)$ be the unitization of $L^1(G//K)$; the unit is identified with δ, the Dirac point mass at the unit e of the group G. We shall prove in Section 4 that for each $\lambda\in\mathbf{C}\setminus\Sigma$ there exists $b_\lambda\in L^1(G//K)$, such that

$$\hat{b}_\lambda(z)=\frac{1}{z(1-z)-\lambda(1-\lambda)},\quad z\in\Sigma, \tag{1-4}$$

and that the set $\{b_\lambda:\lambda\in\mathbf{C}\setminus\Sigma\}$ spans a dense subspace of $L^1(G//K)$. The Laplace–Beltrami operator Δ acts on C^2 smooth functions in $L^1(G//K)$ (which form a dense subset of $L^1(G//K)$) in the following way,

$$\widehat{(\Delta\varphi)}(z)=z(1-z)\hat{\varphi}(z),\quad z\in\Sigma,$$

so that b_λ solves the equation

$$\Delta b_\lambda=\lambda(1-\lambda)b_\lambda+\delta.$$

Hence b_λ is an eigenfunction of Δ on $G\setminus\{e\}$. In Section 4 we shall show that $b_\lambda(\zeta)=2Q_{\lambda-1}(\cosh(2\zeta))$ for $\zeta>0$ and $\operatorname{Re}\lambda>1$, where $Q_{\lambda-1}$ is the Legendre function of the second kind. This function is holomorphic in λ for $\operatorname{Re}\lambda>1$. If we put $b_\lambda=b_{1-\lambda}$, as is consistent with (1-4), we see that b_λ is holomorphic on $\operatorname{Re}\lambda<0$ as well.

For a function g in $L^\infty(G//K)$, the dual Banach space to $L^1(G//K)$, we associate its *resolvent transform*

$$\mathfrak{R}[g](\lambda)=\langle b_\lambda,g\rangle,\quad \lambda\in\mathbf{C}\setminus\Sigma. \tag{1-5}$$

Fix a point $\xi\in\mathbf{C}\setminus\Sigma$. If we play around with (1-4), we get, for $\lambda\in\mathbf{C}\setminus\Sigma$,

$$\hat{b}_\lambda(z)=(1-\hat{b}_\xi(\lambda)^{-1}\hat{b}_\xi(z))^{-1}\,\hat{b}_\xi(z),\quad z\in\Sigma. \tag{1-6}$$

This formula will be useful in the sequel when we attempt to continue $\mathfrak{R}[g]$ analytically.

The maximal ideal space of $L^1_\delta(G//K)$ is identified with the one-point compactification $\Sigma\cup\{\infty\}$ of Σ. For each closed ideal I in $L^1_\delta(G//K)$, we identify the maximal ideal space of the quotient algebra $L^1_\delta(G//K)/I$ with the hull $Z_\infty(I)$ of I, in the standard way. Here,

$$Z_\infty(I)=\{z\in\Sigma\cup\{\infty\}:\hat{f}(z)=0 \text{ for all } f\in I\}.$$

Later on, we shall also need the notation

$$Z(f)=\{z\in\Sigma:\hat{f}(z)=0\}.$$

Let $\mathfrak{S}$ be a collection of functions in $L^1(G//K)$, and let $I(\mathfrak{S})$ denote the closed ideal in $L^1(G//K)$ generated by $\mathfrak{S}$.

Recall that ξ is a fixed point in $\mathbf{C}\backslash\Sigma$. Since $\hat{b}_\xi(\infty)=0$ and $\hat{b}_\xi(\lambda)=\hat{b}_\xi(s)$ if and only if $\lambda=s$ or $\lambda=1-s$, it follows that if $\lambda\in\mathbf{C}\backslash Z_\infty(I(\mathfrak{S}))$ then $\hat{\delta}-\hat{b}_\xi(\lambda)^{-1}\hat{b}_\xi$ does not vanish on $Z_\infty(I(\mathfrak{S}))$. Hence $\delta-\hat{b}_\xi(\lambda)^{-1}b_\xi+I(\mathfrak{S})$ is invertible in the quotient algebra $L^1_\delta(G//K)/I(\mathfrak{S})$. Put

$$B_\lambda=(\delta-(\lambda(1-\lambda)-\xi(1-\xi))b_\xi+I(\mathfrak{S}))^{*-1}*(b_\xi+I(\mathfrak{S})) \tag{1-7}$$

as an element of $L^1(G//K)/I(\mathfrak{S})$ (here, the $*$ is used to symbolize that the product and inversion are taken in convolution sense, though modulo the ideal). Taking Fourier transforms, and comparing with (1-6), we see that

$$B_\lambda=b_\lambda+I(\mathfrak{S}),\quad \lambda\in\mathbf{C}\backslash\Sigma. \tag{1-8}$$

In particular, B_λ does not depend on the point ξ that we have chosen. Let us return to the function $g\in L^\infty(G//K)$, and suppose it annihilates $I(\mathfrak{S})$. It follows that g may be considered as a bounded linear functional on $L^1(G//K)/I(\mathfrak{S})$. By (1-8), the resolvent transform $\mathfrak{R}[g]$ of g, defined by (1-5), can also be represented by the formula

$$\mathfrak{R}[g](\lambda)=\langle B_\lambda,g\rangle,\quad \lambda\in\mathbf{C}\backslash\Sigma.$$

By (1-7), B_λ is defined for all $\lambda\in\mathbf{C}\backslash Z_\infty(I(\mathfrak{S}))$ as an element in $L^1(G//K)/I(\mathfrak{S})$, and it clearly depends analytically on λ. Thus the formula

$$\mathfrak{R}[g](\lambda)=\langle B_\lambda,g\rangle,\quad \lambda\in\mathbf{C}\backslash Z_\infty(I(\mathfrak{S}))$$

gives a holomorphic extension of $\mathfrak{R}[g]$ to $\mathbf{C}\backslash Z_\infty(I(\mathfrak{S}))$.

From now on, we assume that the hull of $I(\mathfrak{S})$ is finite, say,

$$Z_\infty(I(\mathfrak{S}))=\{s_1,\dots,s_n,\infty\}.$$

(For our main result, Theorem 1.3, we have $Z_\infty(I(\mathfrak{S}))=\{\infty\}$ in (1) and $Z_\infty(I(\mathfrak{S}))=\{0,1,\infty\}$ in (2)). To prove Theorem 1.3, we shall later show, under appropriate conditions on $\mathfrak{S}$, that

(1) The functions b_λ, with $\lambda\in\mathbf{C}\setminus\Sigma$, span a dense subspace of $L^1(G//K)$.

(2) $\mathfrak{R}[g]$ is analytic at ∞ and it vanishes there.

(3) The singularities of $\mathfrak{R}[g](\lambda)$ at $s_1,\dots,s_n$ are simple poles.

Indeed, it follows from (2), (3) and the fact that $\mathfrak{R}[g](\lambda)=\mathfrak{R}[g](1-\lambda)$ that $\mathfrak{R}[g]$ has the form

$$\mathfrak{R}[g](\lambda)=\sum_{j=1}^{n}\frac{\alpha_j}{s_j(1-s_j)-\lambda(1-\lambda)},$$

for some complex numbers α_j. Let m_j be the complex homomorphism of $L^1(G//K)$ which corresponds to the point $s_j\in\Sigma$, and form the functional

$$m=\sum_{j=1}^{n}\alpha_j m_j.$$

Taking the resolvent transform of m, we see that $\mathfrak{R}[m]=\mathfrak{R}[g]$, and thus $m-g$ annihilates all the functions b_λ, with $\lambda\in\mathbf{C}\setminus\Sigma$. By (1), $g=m$. This shows that if $f\in L^1(G//K)$ and $\hat f(s_j)=0$ for all $j=1,\dots,n$, then it is annihilated by all $g\in L^\infty(G//K)$ which annihilate $I(\mathfrak{S})$. This finishes the proof of Theorem 1.3.

To implement this sketch, and show that $\mathfrak{R}[g]$ is indeed analytic at ∞ and has simple poles at the s_j's, the method also requires estimates. To this end we shall need an explicit expression for the function $\mathfrak{R}[g](\lambda)$. We achieve this by finding representatives in $L^1_\delta(G//K)$ for the cosets $B_\lambda\in L^1_\delta(G//K)/I(\mathfrak{S})$. Let Σ^o denote the interior of Σ. In Section 5, we will show that for every $f\in L^1(G//K)$ and $\lambda\in\Sigma^o$, there exists $T_\lambda f\in L^1(G//K)$ such that

$$\widehat{T_\lambda f}(z)=\frac{\hat f(\lambda)-\hat f(z)}{z(1-z)-\lambda(1-\lambda)},\quad z\in\Sigma\setminus\{\lambda\}. \tag{1-9}$$

Note the identity

$$\widehat{T_\lambda f}(z)(1-\hat b_\xi(\lambda)^{-1}\hat b_\xi(z))=\hat f(\lambda)\hat b_\xi(z)-\hat f(z)\hat b_\xi(z),$$

valid for $f\in L^1(G//K)$. Suppose $f\in I(\mathfrak{S})$, apply the inverse Fourier transform to the above identity, and mod out $I(\mathfrak{S})$, to get

$$(\hat b_\xi(\lambda)\,\delta-b_\xi+I(\mathfrak{S}))*(T_\lambda f+I(\mathfrak{S}))=\hat f(\lambda)\hat b_\xi(\lambda)(b_\xi+I(\mathfrak{S})).$$

Together with (1-7), this shows that for $f\in I(\mathfrak{S})$ and $\lambda\in\Sigma^o\setminus Z(f)$,

$$T_\lambda f/\hat{f}(\lambda)\in B_\lambda,$$

that is, $T_\lambda f/\hat{f}(\lambda)$ is a representative of the coset B_λ. It follows that

$$\mathfrak{R}[g](\lambda)=\frac{\langle T_\lambda f, g\rangle}{\hat{f}(\lambda)}, \quad \lambda\in\Sigma^o\setminus Z(f). \tag{1-10}$$

In Section 5 it will be shown that $T_\lambda f$ is explicitly given by

$$T_\lambda f(\tau)=Q_{\lambda-1}(\tau)\int_\tau^{+\infty} f(x)P_{\lambda-1}(x)\,dx-P_{\lambda-1}(\tau)\int_\tau^{+\infty} f(x)Q_{\lambda-1}(x)\,dx, \tag{1-11}$$

where $P_{\lambda-1}(z)$ and $Q_{\lambda-1}(z)$ are the Legendre functions of the first and second kind respectively, and $\tau=\cosh(2\zeta)\in[1,+\infty[$. The explicit formulas (1-10) and (1-11) will be used to derive the necessary estimates for $\mathfrak{R}[g]$.

We now indicate the organization of the article. In Section 2, we gather facts on Legendre functions, and in Section 3, we use these facts to find $b_\lambda\in L^1(G//K)$ such that (1-4) holds, and we prove that they span a dense subspace of $L^1(G//K)$. In Section 4, we find a concrete formula for the function $T_\lambda f\in L^1(G//K)$ appearing in (1-9), and we estimate its norm. In Section 5, we supply results from the theory of holomorphic functions, which are applied in Section 6 to the resolvent transform $\mathfrak{R}[g](\lambda)$. We thus obtain the announced Wiener-type completeness theorem, both for $L^1(G//K)$ and $L^1_0(G//K)$.

In Section 7 we follow [5], and use the completeness theorem to prove a generalization of Furstenberg's characterization of harmonic functions on the unit disk, and Agranovskiĭ's characterization of holomorphic functions on the unit disk.

We shall use the letter C to denote a positive constant (it may depend on quantities that are kept fixed), which may vary even within the same inequality.

2. Some facts on Legendre functions

In this section we list some facts on Legendre functions needed in the sequel. The standard references are [26], [28], [29], [18].

For complex numbers a, b, c, z, c not a negative integer or 0, the hypergeometric function of Gauss is given by

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k, \quad |z|<1, \tag{2-1}$$

where

$$(d)_0=1, \quad (d)_k=d(d+1)\cdot\ldots\cdot(d+k-1), \quad k=1,2,\ldots\,.$$

It has the integral representation

$$(2\text{-}2)\quad {}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \qquad \operatorname{Re} c>\operatorname{Re} b>0,\ |z|<1.$$

The ordinary differential equation

$$\frac{d}{dx}\left((1-x^2)\frac{du}{dx}\right)+\nu(\nu+1)u=0$$

has two linearly independent solutions P_ν and Q_ν, which may be expressed in terms of the hypergeometric function,

$$P_\nu(x)={}_2F_1\left(-\nu,\nu+1;1;\frac{1-x}{2}\right), \quad |x-1|<2,$$

$$Q_\nu(x)=\frac{\sqrt{\pi}\,\Gamma(\nu+1)}{2^{\nu+2}\Gamma\left(\nu+\frac{3}{2}\right)}x^{-\nu}\,{}_2F_1\left(\tfrac{1}{2}\nu+1,\tfrac{1}{2}(\nu+1);\nu+\tfrac{3}{2};x^{-2}\right),$$

where in the last formula $x\in\mathbf{C}\setminus\,]-\infty,1]$, $|x|>1$. The functions P_ν and Q_ν are called *Legendre functions of the first and second kind*, respectively. In the definition of Q_ν, $\nu+1$ and $\nu+\frac{3}{2}$ are assumed not to be negative integers, or 0. The function Q_ν extends analytically to $\mathbf{C}\setminus\,]-\infty,1]$, with a logarithmic branch point at 1. The function P_ν extends analytically to $\mathbf{C}\setminus\,]-\infty,-1]$, takes the value $P_\nu(1)=1$, and enjoys the symmetry property $P_{-\nu-1}(x)=P_\nu(x)$. In the following we shall concentrate on the functions $P_{\lambda-1}(x)$, $Q_{\lambda-1}(x)$, with particular interest in $x\in]1,+\infty[$ and $\lambda\in\Sigma$. A formula for $Q_{\lambda-1}$ which is sometimes handy is

$$(2\text{-}3)\qquad Q_{\lambda-1}(x)=\frac{\sqrt{\pi}\,\Gamma(\lambda)}{2^\lambda\Gamma\left(\lambda+\frac{1}{2}\right)}(1+x)^{-\lambda}\,{}_2F_1\left(\lambda,\lambda;2\lambda;\frac{2}{1+x}\right),$$

valid for $x\in\mathbf{C}\setminus\,]-\infty,1]$ with $|x+1|>2$, and $\lambda\in\mathbf{C}\setminus\{0,-1,-2,\ldots\}$. By (2-2) and (2-3), we have the integral formula

$$(2\text{-}4)\qquad \begin{aligned} Q_{\lambda-1}(x)&=2^{\lambda-1}(1+x)^{-\lambda}\int_0^1 t^{\lambda-1}(1-t)^{\lambda-1}\left(1-\frac{2t}{1+x}\right)^{-\lambda}dt\\ &=2^{\lambda-1}\int_0^1 t^{\lambda-1}(1-t)^{\lambda-1}(1+x-2t)^{-\lambda}\,dt, \qquad \operatorname{Re} x>1, \end{aligned}$$

for $\operatorname{Re}\lambda>0$, which immediately yields

$$(2\text{-}5)\qquad |Q_{\lambda-1}(x)|\le Q_{\operatorname{Re}\lambda-1}(x), \quad x\in]1,+\infty[,$$

for $\operatorname{Re}\lambda>0$. We need precise estimates of the function $Q_\lambda(x)$, for x near 1 and $+\infty$. To this end, we produce the following lemma.

Lemma 2.1. *For* $(\mu, x) \in]0,1] \times]1,+\infty[$,

$$1 \le {}_2F_1\left(\mu, \mu; 2\mu; \frac{2}{1+x}\right) \le 1 + \frac{2}{x-1}.$$

Proof. This follows directly from (2-1) by using the estimate

$$\frac{(\mu+j)^2}{2\mu+j} = \frac{\mu^2}{2\mu+j} + j \le j + \frac{1}{2},$$

for $j=0,1,2,3,\dots$. □

The desired estimate near $+\infty$ follows.

Lemma 2.2. *The estimate*

$$|Q_{\lambda-1}(x)| \le \frac{4}{\operatorname{Re}\lambda}(1+x)^{-\operatorname{Re}\lambda}, \quad x \in [2,+\infty[,$$

holds for $0 < \operatorname{Re}\lambda \le 1$.

Proof. The assertion follows from (2-5), (2-3), Lemma 2.1, and the observation

$$\frac{\sqrt{\pi}\,\mu\Gamma(\mu)}{2^{\mu}\Gamma\left(\mu+\frac{1}{2}\right)} \le \frac{4}{3}, \quad 0 < \mu \le 1.$$

The proof is complete. □

We also need an estimate near the point 1.

Lemma 2.3. *The estimate*

$$|Q_{\lambda-1}(x)| \le \frac{2}{\operatorname{Re}\lambda} + \log\frac{1}{x-1}, \quad x \in]1,2],$$

holds for $0 < \operatorname{Re}\lambda \le 1$.

Proof sketch. By (2-5) we can assume that λ is real, so that $0<\lambda\le 1$. By (2-4), we should estimate the (positive) integral

$$Q_{\lambda-1}(x) = 2^{\lambda-1}\int_0^1 t^{\lambda-1}(1-t)^{\lambda-1}(1+x-2t)^{-\lambda}\,dt.$$

After the change of variables $t=1-s$ and $x=1+2y$, this becomes

$$Q_{\lambda-1}(1+2y) = \frac{1}{2}\int_0^1 s^{\lambda-1}(1-s)^{\lambda-1}(s+y)^{-\lambda}\,ds, \quad y \in]0,+\infty[.$$

By estimating the integral separately on $]0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, the assertion follows. □

The Legendre functions of the first and second kind are related by the identities

$$P_{\lambda-1}(x)=\frac{\tan\lambda\pi}{\pi}(Q_{\lambda-1}(x)-Q_{-\lambda}(x)), \tag{2-6}$$

and

$$Q_{\lambda-1}(\tau)P_{\lambda-1}(x)-P_{\lambda-1}(\tau)Q_{\lambda-1}(x)=\frac{\tan\lambda\pi}{\pi}(Q_{-\lambda}(\tau)Q_{\lambda-1}(x)-Q_{\lambda-1}(\tau)Q_{-\lambda}(x)), \tag{2-7}$$

for $\lambda\in\mathbf{C}\setminus\frac{1}{2}\mathbf{Z}$ and $x\in\mathbf{C}\setminus]-\infty,1]$. We also have

$$Q_{\lambda-1}(x)P_{\lambda}(x)-P_{\lambda-1}(x)Q_{\lambda}(x)=\frac{1}{\lambda},\quad x\in\mathbf{C}\setminus]-\infty,1], \tag{2-8}$$

for those complex λ, for which the left hand side is well-defined.

We shall need the following formulæ concerning integration from 1 to x:

$$\begin{aligned}\int_1^x P_{\lambda-1}(t)P_{s-1}(t)\,dt=&(s(1-s)-\lambda(1-\lambda))^{-1}((1-\lambda)P_{s-1}(x)P_{\lambda-2}(x)\\&-(1-s)P_{\lambda-1}(x)P_{s-2}(x)+(\lambda-s)xP_{\lambda-1}(x)P_{s-1}(x)),\end{aligned} \tag{2-9}$$

and

$$\begin{aligned}\int_1^x Q_{\lambda-1}(t)P_{s-1}(t)\,dt=&(s(1-s)-\lambda(1-\lambda))^{-1}((1-\lambda)P_{s-1}(x)Q_{\lambda-2}(x)\\&-(1-s)Q_{\lambda-1}(x)P_{s-2}(x)+(\lambda-s)xQ_{\lambda-1}(x)P_{s-1}(x)+1).\end{aligned} \tag{2-10}$$

They are verified by checking that the two sides have the same derivatives, and take the same values at $x=1$ (or as $x\to1^+$). In the limit as $x\to+\infty$, (2-10) becomes [18, p. 795],

$$\begin{aligned}\int_1^{+\infty} Q_{\lambda-1}(t)P_{s-1}(t)\,dt&=\frac{1}{(\lambda-s)(\lambda+s-1)}\\&=\frac{1}{s(1-s)-\lambda(1-\lambda)},\quad \operatorname{Re}\lambda>1,\ s\in\Sigma.\end{aligned} \tag{2-11}$$

3. The functions b_λ

In Lemma 3.1, we find an element $b_\lambda\in L^1(G//K)$ satisfying (1-4), and in Lemma 3.2, we prove the density of the linear span of the set $\{b_\lambda:\lambda\in\mathbf{C}\setminus\Sigma\}$.

Lemma 3.1. *For each* λ *with* $\operatorname{Re}\lambda>1$, *define* $b_\lambda(\zeta)=\tilde{b}_\lambda(\tau)=2Q_{\lambda-1}(\tau)=2Q_{\lambda-1}(\cosh 2\zeta)$, *where* $\tau=\cosh 2\zeta$; *then*

(1) $$b_\lambda\in L^1(G//K),$$

(2) $$\hat{b}_\lambda(s)=\frac{1}{s(1-s)-\lambda(1-\lambda)},\quad s\in\Sigma,$$

(3) $$\|b_\lambda\|_{L^1}\le\frac{1}{(\operatorname{Re}\lambda-1)\operatorname{Re}\lambda}.$$

Proof. By (2-5), (2-4), and changing the order of integration, we get

$$\begin{aligned}\|b_\lambda\|_{L^1}&=2\int_1^{+\infty}|Q_{\lambda-1}(\tau)|\,\tfrac12 d\tau\le 2\int_1^{+\infty}Q_{\operatorname{Re}\lambda-1}(\tau)\,\tfrac12 d\tau\\&=2^{\operatorname{Re}\lambda}\int_1^{+\infty}\int_0^1 t^{\operatorname{Re}\lambda-1}(1-t)^{\operatorname{Re}\lambda-1}(1+\tau-2t)^{-\operatorname{Re}\lambda}\,dt\,\tfrac12 d\tau\\&=\frac{1}{\operatorname{Re}\lambda-1}\int_0^1 t^{\operatorname{Re}\lambda-1}\,dt=\frac{1}{(\operatorname{Re}\lambda-1)\operatorname{Re}\lambda}.\end{aligned}$$

This proves (1) and (3). Relation (2) follows from (1-1′) and (2-11). □

Lemma 3.2. *The functions* b_λ, $\lambda\in\mathbf{C}\setminus\Sigma$, *span a dense subspace of* $L^1(G//K)$.

Proof. Fix $\delta>0$. By the Paley–Wiener theorem [22], the functions f whose Fourier transforms extend analytically to $\Sigma_\delta=\{s\in\mathbf{C}:-\delta\le\operatorname{Re}s\le1+\delta\}$, and satisfy $|\hat{f}(s)|=O(|s|^{-3})$ as $|s|\to+\infty$ within Σ_δ, are dense in $L^1(G//K)$. Thus, it suffices to show that each such f is in the closed subspace spanned by the b_λ, $\lambda\in\mathbf{C}\setminus\Sigma$.

Fix $s\in\Sigma$. By Cauchy's formula and the conditions on $\hat{f}$, we see that

$$\hat{f}(s)=\frac{1}{2\pi i}\int_{\Gamma_1(\delta)+\Gamma_2(\delta)}\frac{\hat{f}(z)}{z-s}\,dz,$$

where $\Gamma_1(\delta)=1+\delta+i\mathbf{R}$ upward, and $\Gamma_2(\delta)=-\delta+i\mathbf{R}$ downward. Substituting $z\to 1-z$ in the second integral, recalling that $\hat{f}(1-z)=\hat{f}(z)$, and using the identity

$$\frac{1}{z-s}-\frac{1}{1-z-s}=\frac{1-2z}{z(1-z)-s(1-s)}=(2z-1)\hat{b}_z(s),$$

we see that

$$\hat{f}(s)=\frac{1}{2\pi i}\int_{\Gamma_1(\delta)}\hat{f}(z)\,\hat{b}_z(s)(2z-1)\,dz.$$

By Lemma 3.1 and the condition on $\hat{f}$, the $L^1(G//K)$ valued integral

$$\frac{1}{2\pi i}\int_{\Gamma_1(\delta)}\hat{f}(z)(2z-1)b_z\,dz$$

converges, and the identity above shows that it converges to f. Thus, its Riemann sums, which are linear combinations of the b_λ's, converge to f, as required. □

4. Representatives for the cosets $B_\lambda + I(\mathfrak{S})$

Toward the end of Section 1, we showed that for $f \in I(\mathfrak{S})$, $\lambda \in \Sigma^o$, $T_\lambda f/\hat{f}(\lambda)$ is a representative of the coset $B_\lambda + I(\mathfrak{S})$. This will later be important in estimating the extension of $\mathfrak{R}[g]$ which was constructed using $T_\lambda f$. In this section we prove that such $T_\lambda f \in L^1(G//K)$ exists by giving an explicit formula, and we estimate its L^1 norm.

Lemma 4.1. *Let $\lambda \in \Sigma^o$. For each $f \in L^1(G//K)$, put*

$$T_\lambda f(\tau) = Q_{\lambda-1}(\tau)\int_\tau^{+\infty} f(x)P_{\lambda-1}(x)\,dx - P_{\lambda-1}(\tau)\int_\tau^{+\infty} f(x)Q_{\lambda-1}(x)\,dx. \tag{4-1}$$

Then if we think of $\tau \in [1,+\infty[$ as a coordinatization of $G//K$ (τ is related to ζ via $\tau = \cosh(2\zeta)$), $T_\lambda f$ belongs to $L^1(G//K)$, and

(1) $\|T_\lambda f\|_{L^1} \le 250\|f\|_{L^1}\, d(\lambda, \partial\Sigma)^{-2}$, where $d(\lambda,\partial\Sigma)$ is the Euclidian distance of λ from the boundary $\partial\Sigma$ of Σ, and

(2) $\widehat{T_\lambda f}(s) = (\hat{f}(\lambda) - \hat{f}(s))(s(1-s) - \lambda(1-\lambda))^{-1}$, $s \in \Sigma \setminus \{\lambda\}$.

Proof. To obtain (1), we should estimate the integral of $|T_\lambda f(\tau)|$ on $[1,+\infty[$. We split the domain of integration into $[1,2[$ and $[2,+\infty[$.

We first estimate $\int_1^2 |T_\lambda f(\tau)|\,d\tau$. By (4-1),

$$\begin{aligned}
\int_1^2 |(T_\lambda f)(\tau)|\,d\tau &= \int_1^2 \biggl| Q_{\lambda-1}(\tau)\int_\tau^{+\infty} f(x)P_{\lambda-1}(x)\,dx \\
&\qquad - P_{\lambda-1}(\tau)\int_\tau^{+\infty} f(x)Q_{\lambda-1}(x)\,dx\biggr|\,d\tau \\
&\le \int_1^2 \biggl| P_{\lambda-1}(\tau)\int_\tau^2 f(x)Q_{\lambda-1}(x)\,dx\biggr|\,d\tau \\
&\quad + \int_1^2 \biggl| Q_{\lambda-1}(\tau)\int_\tau^2 f(x)P_{\lambda-1}(x)\,dx\biggr|\,d\tau \\
&\quad + \int_1^2 \biggl|\int_2^{+\infty} f(x)(Q_{\lambda-1}(\tau)P_{\lambda-1}(x) - P_{\lambda-1}(\tau)Q_{\lambda-1}(x))\,dx\biggr|\,d\tau.
\end{aligned}$$

We estimate the first integral using Lemmas 1.1 and 2.3:

$$\begin{aligned}
\int_1^2 \biggl| P_{\lambda-1}(\tau)\int_\tau^2 f(x)Q_{\lambda-1}(x)\,dx\biggr|\,d\tau &\le \frac{2}{\operatorname{Re}\lambda}\int_1^2\int_\tau^2 |f(x)|\,dx\,d\tau \\
&\quad + \int_1^2\int_\tau^2 |f(x)|\log\frac{1}{x-1}\,dx\,d\tau \\
&\le \frac{4}{\operatorname{Re}\lambda}\|f\|_{L^1} + \int_1^2 \log\frac{1}{\tau-1}\int_\tau^2 |f(x)|\,dx\,d\tau \\
&\le (4/\operatorname{Re}\lambda + 2)\|f\|_{L^1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\int_1^2 \left| Q_{\lambda-1}(\tau) \int_\tau^2 f(x) P_{\lambda-1}(x)\,dx \right| d\tau &\le 2\|f\|_{L^1} \int_1^2 |Q_{\lambda-1}(\tau)|\,d\tau \\ &\le \|f\|_{L^1} \int_1^2 \left(\frac{2}{\operatorname{Re}\lambda} + \log \frac{1}{\tau-1} \right) d\tau \\ &\le (4/\operatorname{Re}\lambda + 2)\|f\|_{L^1}.\end{aligned}$$

The third integral is estimated using the fact that $|P_{\lambda-1}(x)| \le 1$ on $[1,+\infty[$, and in a second step, Lemmas 2.2 and 2.3:

$$\begin{aligned}&\int_1^2 \left| \int_2^{+\infty} f(x)(Q_{\lambda-1}(\tau)P_{\lambda-1}(x) - P_{\lambda-1}(\tau)Q_{\lambda-1}(x))\,dx \right| d\tau \\ &\quad\le \int_1^2 \int_2^{+\infty} |f(x)|(|Q_{\lambda-1}(x)| + |Q_{\lambda-1}(\tau)|)\,dx\,d\tau \\ &\quad= \int_2^{+\infty} |f(x)|(|Q_{\lambda-1}(x)| + 1 + 2/\operatorname{Re}\lambda)\,dx \\ &\quad\le \int_2^{+\infty} |f(x)|(1 + 6/\operatorname{Re}\lambda)\,dx \\ &\quad= 2(1+6/\operatorname{Re}\lambda)\|f\|_{L^1} \le \frac{14}{\operatorname{Re}\lambda}\|f\|_{L^1}.\end{aligned}$$

In conclusion, we have

$$\int_1^2 |T_\lambda f(\tau)|\, \tfrac{1}{2} d\tau \le \frac{7}{\operatorname{Re}\lambda}\|f\|_{L^1}. \tag{4-2}$$

We now estimate

$$\int_2^{+\infty} |T_\lambda f(\tau)|\,d\tau.$$

By (2-7), for $\lambda \in \Sigma^o \setminus \{\frac{1}{2}\}$, $T_\lambda f(\tau)$ may be written as

$$T_\lambda f(\tau) = \frac{\tan\lambda\pi}{\pi} \left(Q_{-\lambda}(\tau) \int_\tau^{+\infty} f(x) Q_{\lambda-1}(x)\,dx - Q_{\lambda-1}(\tau) \int_\tau^{+\infty} f(x) Q_{-\lambda}(x)\,dx \right),$$

so that

$$\begin{aligned}|T_\lambda f(\tau)| \le \frac{|\tan\lambda\pi|}{\pi} \bigg(&|Q_{\lambda-1}(\tau)| \int_\tau^{+\infty} |f(x) Q_{-\lambda}(x)|\,dx \\ &+ |Q_{-\lambda}(\tau)| \int_\tau^{+\infty} |f(x) Q_{\lambda-1}(x)|\,dx \bigg).\end{aligned} \tag{4-3}$$

By (2-5), we can assume that in the integrals on the right hand side of (4-3), λ is real, that is, $\lambda\in]0,1[\setminus\{\frac{1}{2}\}$. To estimate the first term inside the brackets on the right hand side of (4-3), note that by Lemma 2.2, for $\tau\in[2,+\infty[$,

$$|Q_{\lambda-1}(\tau)|\int_\tau^{+\infty}|f(x)Q_{-\lambda}(x)|\,dx\le\frac{16}{\lambda(1-\lambda)}(1+\tau)^{-\lambda}\int_\tau^{+\infty}|f(x)|(1+x)^{\lambda-1}\,dx.$$

Integrating with respect to τ, and changing the order of integration, we get

$$\begin{aligned}\int_2^{+\infty}&|Q_{\lambda-1}(\tau)|\int_\tau^{+\infty}|f(x)Q_{-\lambda}(x)|\,dx\,d\tau\\ &\le\frac{16}{\lambda(1-\lambda)}\int_2^{+\infty}(1+\tau)^{-\lambda}\int_\tau^{+\infty}|f(x)|(1+x)^{\lambda-1}\,dx\,d\tau\\ &=\frac{16}{\lambda(1-\lambda)}\int_2^{+\infty}|f(x)|(1+x)^{\lambda-1}\int_2^{x}(1+\tau)^{-\lambda}\,d\tau\,dx\\ &\le\frac{16}{\lambda(1-\lambda)^2}\int_2^{+\infty}|f(x)|\,dx\le\frac{32}{\lambda(1-\lambda)^2}\|f\|_{L^1}.\end{aligned}$$

Replacing λ by $1-\lambda$, we get

$$\int_2^{+\infty}|Q_{-\lambda}(\tau)|\int_\tau^{+\infty}|f(x)Q_{\lambda-1}(x)|\,dx\,d\tau\le\frac{32}{\lambda^2(1-\lambda)}\|f\|_{L^1}.$$

In conclusion, we get, for a general complex λ, $\lambda\in\Sigma^o\setminus\{\frac{1}{2}\}$,

$$\int_2^{+\infty}|T_\lambda f(\tau)|\,\tfrac{1}{2}d\tau\le\frac{16|\tan\lambda\pi|}{\pi(\operatorname{Re}\lambda)^2(1-\operatorname{Re}\lambda)^2}\|f\|_{L^1}.$$

Taking into account (4-2), we get

$$\int_1^{+\infty}|T_\lambda f(\tau)|\,\tfrac{1}{2}d\tau\le\frac{16}{\pi}\frac{|\tan\lambda\pi|+\pi/15}{(\operatorname{Re}\lambda)^2(1-\operatorname{Re}\lambda)^2}\|f\|_{L^1}.$$

One shows that

$$\frac{16}{\pi}\frac{|\tan\lambda\pi|+\pi/15}{(\operatorname{Re}\lambda)^2(1-\operatorname{Re}\lambda)^2}\le15\frac{1+|\lambda-1/2|^{-1}}{d(\lambda,\partial\Sigma)^2},\quad\lambda\in\Sigma^o\setminus\{\tfrac{1}{2}\},$$

and hence

$$\text{(4-4)}\qquad\|T_\lambda f\|_{L^1}=\int_1^{+\infty}|T_\lambda f(\tau)|\,\tfrac{1}{2}d\tau\le15\frac{1+|\lambda-1/2|^{-1}}{d(\lambda,\partial\Sigma)^2}\|f\|_{L^1},\quad\lambda\in\Sigma^o\setminus\{\tfrac{1}{2}\}.$$

For fixed $\tau\in]1,+\infty[$, the function $T_\lambda f(\tau)$ depends holomorphically on $\lambda\in\Sigma^o$, by inspection of (4-1) (and if needed, (2-5)). It follows that $|T_\lambda f(\tau)|$ is subharmonic in $\lambda\in\Sigma^o$, whence

$$\|T_\lambda f\|_{L^1}=\int_2^{+\infty}|T_\lambda f(\tau)|\,\tfrac{1}{2}d\tau,\quad\lambda\in\Sigma^o,$$

is subharmonic. If we use the maximum principle for subharmonic functions, we can improve the estimate (4-4) near the point $\lambda=\frac{1}{2}$ (apply the principle to the disk $|\lambda-\frac{1}{2}|<\frac{1}{6}$),

$$\|T_\lambda f\|_{L^1}\leq 250\|f\|_{L^1}\, d(\lambda,\partial\Sigma)^{-2},\quad \lambda\in\Sigma^o;$$

part (1) of the lemma follows.

For the proof of part (2), we use (4-1) and (1-1′):

$$\widehat{T_\lambda f}(s)=\int_1^{+\infty} P_{s-1}(\tau)\int_\tau^{+\infty} f(x)P_{\lambda-1}(x)Q_{\lambda-1}(\tau)\,dx\,\tfrac{1}{2}d\tau \\ -\int_1^{+\infty} P_{s-1}(\tau)\int_\tau^{+\infty} f(x)Q_{\lambda-1}(x)P_{\lambda-1}(\tau)\,dx\,\tfrac{1}{2}d\tau.$$

By changing the order of integration, we get

$$\widehat{T_\lambda f}(s)=\int_1^{+\infty} f(x)P_{\lambda-1}(x)\int_1^{x} Q_{\lambda-1}(\tau)P_{s-1}(\tau)\,\tfrac{1}{2}d\tau\,dx \\ -\int_1^{+\infty} f(x)Q_{\lambda-1}(x)\int_1^{x} P_{\lambda-1}(\tau)P_{s-1}(\tau)\,\tfrac{1}{2}d\tau\,dx.$$

By (2-9), (2-10), and (2-8), we have

$$\widehat{T_\lambda f}(s)=(s(1-s)-\lambda(1-\lambda))^{-1}\left(\int_1^{+\infty} f(x)P_{\lambda-1}(x)\,\tfrac{1}{2}dx-\int_1^{+\infty} f(x)P_{s-1}(x)\,\tfrac{1}{2}dx\right),$$

which proves (2). □

We now summarize the properties of the resolvent transform that we indicated in Section 1.

Theorem 4.2. *Assume $g\in L^\infty(G//K)$ annihilates the closed ideal I in $L^1(G//K)$, and fix any $f\in I$.*

(1) *The resolvent transform $\mathfrak{R}[g](\lambda)$ of g is defined for $\lambda\in\mathbf{C}\setminus Z_\infty(I)$, and there, it is holomorphic in λ. It is given by the formula*

$$\mathfrak{R}[g](\lambda)=\begin{cases}\langle b_\lambda,g\rangle, & \lambda\in\mathbf{C}\setminus\Sigma,\\ \langle T_\lambda f,g\rangle/\hat{f}(\lambda), & \lambda\in\Sigma^o\setminus Z(f).\end{cases}$$

(2) *$\mathfrak{R}[g](\lambda)=\mathfrak{R}[g](1-\lambda)$, for all $\lambda\in\mathbf{C}\setminus Z_\infty(I)$.*

(3) *$|\mathfrak{R}[g](\lambda)|\leq\|g\|_{L^\infty}/d(\lambda,\partial\Sigma)$, $\lambda\in\mathbf{C}\setminus\Sigma$.*

(4) *$|\langle T_\lambda f,g\rangle|\leq 250\|f\|_{L^1}\|g\|_{L^\infty}\, d(\lambda,\partial\Sigma)^{-2}$, $\lambda\in\Sigma^o$.*

Proof. Properties (1), (2), and (4) follow from Lemma 4.1 and the discussion in Section 1. The estimate (3) is a direct consequence of (1-5) and Lemma 3.1. □

5. Results from the theory of holomorphic functions

The reader may skip this section, and use it later as reference only. The following result is classical, and commonly referred to as the log log theorem. It has its roots in the work of Carleman, Levinson, Sjöberg, and Beurling. The variant we use here can be found in Beurling's paper [7].

Theorem 5.1. *Let Q be the rectangle $]-1,1[\times]-1,1[$, regarded as a subset of the complex plane. Let $M:]0,1]\to[e,+\infty[$ be a continuous decreasing function, and let $A(Q,M)$ be the set of holomorphic functions f on Q that satisfy*

$$|f(x+iy)|\le M(|y|), \quad x+iy\in Q\setminus\mathbf{R}.$$

Then $A(Q,M)$ is a normal family on Q if and only if

$$\int_0^1 \log\log M(y)\,dy<+\infty.$$

Based on Theorem 5.1, one can obtain the following statement about functions that have properties shared by certain resolvent transforms of functions in $L^\infty(G//K)$ (Theorem 4.2). Some notation is needed: given a nonidentically vanishing bounded holomorphic function f on the interior Σ^o of Σ, we write, in analogy with (1-2) and (1-3),

$$\begin{aligned}\delta_\infty^+(f)&=-\limsup_{t\to+\infty} e^{-\pi t}\log\left|f\left(\tfrac12+it\right)\right|,\\ \delta_\infty^-(f)&=-\limsup_{t\to+\infty} e^{-\pi t}\log\left|f\left(\tfrac12-it\right)\right|,\\ \delta_0(f)&=-\limsup_{x\to0^+} x\log|f(x)|.\end{aligned}$$

Extend the notation to collections $\mathfrak{S}$ of functions in $H^\infty(\Sigma^o)$ by taking infima, as in Section 1. For $z\in\mathbf{C}$, $d(z,\partial\Sigma)$ is the Euclidian distance from z to the boundary $\partial\Sigma$.

Theorem 5.2. *Let $M:]0,1]\to[e,+\infty[$ be a continuous decreasing function with*

$$\int_0^1 \log\log M(t)\,dt<+\infty.$$

Let G be a holomorphic function on $\mathbf{C}\setminus Z$, where Z is a finite subset of Σ. Suppose G satisfies, for some nonidentically vanishing function $f\in H^\infty(\Sigma^o)$, the estimates

$$\begin{aligned}|G(z)|&\le M(d(z,\partial\Sigma)), && z\in\mathbf{C}\setminus\Sigma,\\ |f(z)\,G(z)|&\le M(d(z,\partial\Sigma)), && z\in\Sigma^o\setminus Z.\end{aligned}$$

Fix a bounded open neighborhood U of Z. Then, for each ε, $0<\varepsilon$, there is a positive constant $C(\varepsilon)$, such that

$$|G(z)| \leq \begin{cases} C(\varepsilon)\exp((\varepsilon+\delta_\infty^+(f))e^{\pi \operatorname{Im} z}), & z\in \mathbf{C}_+\setminus U, \\ C(\varepsilon)\exp((\varepsilon+\delta_\infty^-(f))e^{-\pi \operatorname{Im} z}), & z\in \mathbf{C}_-\setminus U, \end{cases}$$

where $\mathbf{C}_+$ and $\mathbf{C}_-$ are the open upper and lower half planes, respectively.

The details of the proof can be found in [20], and they involve, in addition to the log-log theorem, the Paley–Wiener [24], [8] and Ahlfors–Heins theorems [2], as well as the Phragmén–Lindelöf principle [8].

The following result is more to the point as regards what we need to prove the completeness theorem. Again, the proof is more or less carried out in [20]; the essential ingredient is the Ahlfors distortion theorem, which is applied to a bottle-type domain around the strip Σ. For the reader who wishes to look at the details, it should be mentioned that it may be necessary to use instead of M a comparable, but smoother, function $\widetilde{M}$, at the relevant places of the proof in [20].

Theorem 5.3. *Let M, Z, U, G, and f be as in Theorem 5.2, but now the function f may be any element of a collection $\mathfrak{S}$ of functions in $H^\infty(\Sigma^o)$. Suppose that $\delta_\infty^+(\mathfrak{S})=\delta_\infty^-(\mathfrak{S})=0$. Then G is bounded on $\mathbf{C}\setminus U$, and hence holomorphic on $\mathbf{C}\cup\{\infty\}\setminus Z$.*

The next result treats the behavior near finite points.

Lemma 5.4. *Let M, Z, G, and f be as in Theorem 5.2, and suppose $0\in Z$. We also require that M satisfies $\log M(t)=o(t^{-1})$ as $t\to 0^+$. Fix a small bounded open neighborhood U of 0, not containing any other point of Z. Then, for each ε, $0<\varepsilon$, there is a positive constant $C(\varepsilon)$, such that*

$$|G(z)| \leq C(\varepsilon)\exp((\varepsilon+\delta_0(f))|z|^{-1}), \quad z\in U.$$

Proof sketch. Consider the function $H(z)=G(1/z)$, which is holomorphic on a punctured neighborhood of ∞. Let F be the entire function which approximates H near ∞, in the sense that $F(z)-H(z)\to 0$ as $z\to\infty$. You get F by considering the Laurent series expansion of H. Lemma 4.4 in [19] (which in many ways is similar to Theorem 5.2 here) applies to F after a rotation of the complex plane, and when the result is converted back into information about G, the assertion follows. □

Lemma 5.5. *Let M, Z, G, and f be as in Lemma 5.4, but now the function f may be any element of a collection $\mathfrak{S}$ of functions in $H^\infty(\Sigma^o)$, with $\delta_0(\mathfrak{S})=0$.*

Suppose G also has the estimate

$$|G(z)| \le \frac{C}{d(z,\partial\Sigma)}, \quad z \in \mathbf{C}\setminus\Sigma,$$

for some positive constant C. Then G has a simple pole at 0.

Proof sketch. By Lemma 5.4, the function $H(z)=G(1/z)$, which is holomorphic in a punctured neighborhood of ∞, increases subexponentially near ∞. The additional estimate on G in $\mathbf{C}\setminus\Sigma$, together with the Phragmén–Lindelöf principle, shows that H increases at most polynomially near ∞. Hence G has a pole at 0, which must be simple, again because of the estimate on G in $\mathbf{C}\setminus\Sigma$. □

6. The proof of the completeness theorem

We now prove Theorem 1.3. Recall the setting, as presented in Section 1: $g\in L^\infty(G//K)$ annihilates $I(\mathfrak{S})$, and

$$Z_\infty(I(\mathfrak{S})) = \{s_1, s_2, \dots, s_n, \infty\}.$$

It is assumed that the points $s_1, \dots, s_n$ are on the boundary $\partial\Sigma$, and that

$$\delta_\infty(\mathfrak{S}) = \delta_{s_1}(\mathfrak{S}) = \dots = \delta_{s_n}(\mathfrak{S}) = 0. \tag{6-1}$$

The quantity $\delta_{s_j}(\mathfrak{S})$ is obtained from $\delta_{s_j}(f)$ by taking the infimum over all $f\in\mathfrak{S}$, and

$$\delta_{s_j}(f) = -\limsup_{t\to 0^+} t \log|\hat f(s_j+t)|,$$

for s_j on the imaginary axis (for the other part of $\partial\Sigma$, $1+i\mathbf{R}$, the definition is analogous). By the symmetry $\hat f(s)=\hat f(1-s)$, we get $\delta^+_\infty(\mathfrak{S})=\delta^-_\infty(\mathfrak{S})=0$ (notation as in Section 5). The symmetry also forces the points $s_1, \dots, s_n$ to lie symmetrically with respect to the mapping $z\mapsto 1-z$.

Condition (6-1) is necessary in order to have $I(\mathfrak{S})$ coincide with the closed convolution ideal of functions in $L^1(G//K)$ whose Fourier transforms vanish on $\{s_1, \dots, s_n\}$, as is seen by an argument based on the Beurling–Rudin theorem characterizing the closed ideals in the disk algebra (for details, see [20], [5]). The difficult part of Theorem 1.3 is the sufficiency of (6-1).

By Theorem 4.2, the resolvent transform $\mathfrak{R}[g](z)$ is holomorphic throughout $\mathbf{C}\setminus\{s_1, \dots, s_n\}$, and so the function $H(z)=\hat f(z)\mathfrak{R}[g](z)$ is analytic throughout Σ^o, and has

$$|H(z)| \le C\, d(z,\partial\Sigma)^{-2}, \quad z\in\Sigma^o.$$

By Theorem 5.3, $\mathfrak{R}[g]$ is bounded in a neighborhood of ∞, and by Theorem 4.2, $\mathfrak{R}[g](\infty)=0$. Moreover, by Theorem 4.2 and Lemma 5.5, $\mathfrak{R}[g]$ has simple poles at the points $s_1, \dots, s_n$. Modulo the discussion in Section 1, the proof is complete. □

7. Characterization of harmonic and holomorphic functions in the unit disk

Let f be a harmonic function on the unit disk $\mathbf{D}$. For any $g\in G$, the composition $f\circ g$ is also harmonic, and its value at zero is the average of its values on any circle centered at 0. Identifying points in $\mathbf{D}$ with elements of G/K and radial Borel measures on $\mathbf{D}$ with K-bi-invariant Borel measures on G, we obtain that if μ is a radial Borel measure on $\mathbf{D}$ with $\mu(\mathbf{D})=1$, then

$$\int_G f(gh)\,d\mu(h) = f(g) \quad \text{for all } g\in G, \tag{7-1}$$

whenever $f\in L^\infty(G/K)$ is harmonic on $\mathbf{D}$; one says that f is μ-harmonic if (7-1) holds [16]. Conversely, one may ask under what conditions on a radial measure μ the only μ-harmonic functions $f\in L^\infty(G/K)$ are harmonic.

This question was treated by Furstenberg for measures and functions on G/K where G is a semisimple Lie group with finite center, and K is a maximal compact subgroup of G. He showed [17, Theorem 5, p. 370] that when μ is an absolutely continuous radial probability measure on G/K, every $f\in L^\infty(G/K)$ satisfying (7-1) is necessarily harmonic. As Furstenberg uses probabilistic methods, the assumption that μ is a probability measure is essential to his method.

The results of the previous section allow us to give some easy answers to these converse problems.

Since μ is radial, $d\mu(g^{-1})=d\mu(g)$, so that equation (7-1) becomes $f*\mu=f$.

We need to specify how we compute the Fourier transform of a radial measure μ. If $\tau\in[1,+\infty[$ is the parameter used earlier in this paper for $G//K$ (related to the more standard $\zeta\in[0,+\infty[$ via $\tau=\cosh 2\zeta$), we may think of μ as a Borel measure on $[1,+\infty[$, so that for continuous rotation invariant bounded functions f on G,

$$\int_G f(g)\,d\mu(g) = \int_{[1,+\infty[} \tilde f(\tau)\,d\mu(\tau),$$

where $\tilde f(\tau)=f(\zeta)$, as in Section 1. The Fourier transform of μ is

$$\hat\mu(s) = \int_{[1,+\infty[} P_{s-1}(\tau)\,d\mu(\tau), \quad s\in\Sigma.$$

For instance, the Fourier transform of the Dirac measure δ at the origin (which corresponds to the Dirac measure at the unit $e\in G$, and the unit point mass at $\tau=1$) is $\hat\delta(s)=1$.

Theorem 7.1. *Let μ be a finite complex-valued radial Borel measure on* $\mathbf{D}$ *such that* $\mu(\mathbf{D})=1$, $\mu(\{0\})\neq 1$, $\hat{\mu}(s)\neq 1$ *for* $s\in\Sigma\setminus\{0,1\}$, *and* $\limsup_{t\to 0+} t\log|1-\hat{\mu}(x)|=0$. *Then every bounded μ-harmonic function is harmonic.*

Proof. As in the proof of Theorem 3.1 in [5], it suffices to show that if $f\in L^\infty(G/K)$ is a radial solution of the equation $f*\mu=f$, then f is constant.

Let I be the closed ideal in $L^1_0(G//K)$ generated by $\mathfrak{S}=(\mu-\delta)*L^1(G//K)$. Since $\hat{\mu}(s)\neq 1$ for $s\in\Sigma\setminus\{0,1\}$, the hull of this ideal is $Z_\infty(I)=\{0,1,\infty\}$. By Lemma 1.2, $\hat{\mu}(\frac{1}{2}+it)\to\mu(\{0\})$ as $|t|\to+\infty$, so that by the assumption $\mu(\{0\})\neq 1$, $\mathfrak{S}$ contains functions whose Fourier transforms decay arbitrarily slowly at ∞, so that $\delta_\infty(\mathfrak{S})=0$. The condition of the theorem implies that $\delta_0(\mathfrak{S})=0$. We thus conclude by Theorem 1.3 that $I=L^1_0(G//K)$. But if f solves the convolution equation, then $f*(\mu-\delta)*L^1(G//K)=\{0\}$; thus $f*L^1_0(G//K)=\{0\}$, and f is constant. □

Corollary 7.2. *If μ is a radial regular probability measure on* $\mathbf{D}$ *with $\mu(\{0\})<1$, then every bounded μ-harmonic function is harmonic.*

Proof. The formula

$$P_{s-1}(\tau)=\frac{1}{2\pi}\int_0^{2\pi}\left(\tau+\sqrt{\tau^2-1}\,\cos\theta\right)^{-s}d\theta,\quad \tau\in[1,+\infty[,$$

together with Lemma 1.1, shows that $0\le P_{s-1}(\tau)\le 1$ for $s\in[0,1]$. It also follows that

$$|P_{s-1}(\tau)|\le P_{\operatorname{Re}s-1}(\tau)\le 1,\quad s\in\Sigma,$$

where the first inequality is strict unless $\tau=1$ or $s\in[0,1]$, and the second unless $\operatorname{Re}s$ is 0 or 1. Since not all the mass of μ is concentrated to the origin, it follows that

$$|\hat{\mu}(s)|\le\int_{[1,+\infty[}|P_{s-1}(\tau)|\,d\mu(\tau)<1,\quad s\in\Sigma\setminus\{0,1\}.$$

For $s=0$, $P_{s-1}(\tau)=P_{-1}(\tau)=1$; let us see how $P_{s-1}(\tau)$ deviates from 1 for $s\in\,]0,1]$ close to 0. We obtain

(7-2)
$$\begin{aligned}\frac{d}{ds}P_{s-1}(\tau)\Big|_{s=0}&=-\frac{1}{2\pi}\int_0^{2\pi}\log\left(\tau+\sqrt{\tau^2-1}\,\cos\theta\right)d\theta\\&=-2\log\left(\frac{1}{2}\left(\sqrt{\tau+\sqrt{\tau^2-1}}+\sqrt{\tau-\sqrt{\tau^2-1}}\right)\right)\le 0,\quad \tau\in[1,+\infty[.\end{aligned}$$

Fix a closed interval $[a,b]$ in $]1,+\infty[$. Write

$$L(\tau)=2\log\left(\frac{1}{2}\left(\sqrt{\tau+\sqrt{\tau^2-1}}+\sqrt{\tau-\sqrt{\tau^2-1}}\right)\right),\quad \tau\in[1,+\infty[,$$

and notice that $L(\tau)\geq 0$, with strict inequality for $\tau\in]1,+\infty[$, and $1-P_{s-1}(\tau)=L(\tau)s+O(s^2)$ as $s\to 0^+$, uniformly in $\tau\in[a,b]$. It follows that

$$1-\hat{\mu}(s)=\int_{[1,+\infty[}(1-P_{s-1}(\tau))\,d\mu(\tau)\geq\int_{[a,b]}(1-P_{s-1}(\tau))\,d\mu(\tau)$$
$$=s\int_{[a,b]}L(\tau)\,d\mu(\tau)+O(s^2),\quad \text{as } s\to 0^+.$$

Again since the probability measure μ is not concentrated to the origin, we may pick $[a,b]$ such that $\int_{[a,b]}L(\tau)\,d\tau>0$. We find that $s\log(1-\hat{\mu}(s))\to 0$ as $s\to+\infty$, so that the assertion follows from Theorem 7.1. □

Remark. A related result was obtained in the context of the whole complex plane, or more generally, $\mathbf{R}^n$, by Choquet and Deny [12].

The following "two circles" theorem was announced in [1]. The proof there is not complete, and a correct proof was given in [5] under certain additional assumptions. By using Theorem 1.3 we are now able to give a proof of the theorem in its full generality. For this we fix two circles γ_1 and γ_2 centered at the origin in $\mathbf{D}$, with radii r_1 and r_2, respectively, $0<r_1,r_2<1$. For $j=1,2$, let χ_j be the characteristic function of the disk $|z|<r_j$, which we may regard as a function on $G//K$. The Fourier transform of χ_j is

$$\widehat{\chi}_j(s)=\int_1^{\tau_j}P_{s-1}(\tau)\,\tfrac{1}{2}d\tau=\tfrac{1}{2}\,\mathfrak{P}_{s-1}(\tau_j),\quad s\in\Sigma,$$

where $\tau_j=(1+r_j^2)/(1-r_j^2)$, and $\mathfrak{P}_{s-1}$ is the antiderivative of P_{s-1}, with $\mathfrak{P}_{s-1}(1)=0$. The function $\mathfrak{P}_{s-1}$ can be expressed in terms of the associated Legendre function P_{s-1}^{-1}. By Lemma 1.1, the functions $\widehat{\chi}_1$ and $\widehat{\chi}_2$ extend to entire functions of finite exponential type.

Theorem 7.3. *Let f be a Borel measurable function on $\mathbf{D}$ with $|f(z)|\leq C(1-|z|^2)^{-1}$, $z\in\mathbf{D}$, such that for $j=1,2$*

$$\int_{g(\gamma_j)}f(z)\,dz=0,\quad \textit{for almost all } g\in G.$$

Suppose the functions $\widehat{\chi}_1$ and $\widehat{\chi}_2$ have no common zero in the strip Σ. Then f coincides almost everywhere in $\mathbf{D}$ with a holomorphic function.

This result should be compared with the two circle characterization of holomorphic functions in [6]. That result does not assume any growth conditions on the function, but has a strong assumption on the possible values of the radii of the circles. The classical theory of mean periodic functions due to Laurent Schwartz is sufficient under these assumptions. In our Theorem 7.3, we impose growth conditions, but make weaker assumptions on the radii, which necessitates different spectral analysis tools. The Wiener type theorem developed here provides the required tool.

Proof. We use the same notation as in [1], [5]. Since the entire function $\widehat{\chi}_j$ has finite exponential type, it cannot decay arbitrarily rapidly at ∞; in particular, $\delta_\infty(\chi_j)=0$ $(j=1,2)$. Also, by assumption, $\widehat{\chi}_1$ and $\widehat{\chi}_2$ lack common zeros in Σ. By Theorem 1.3, the closed convolution ideal generated by χ_1 and χ_2 is all of $L^1(G//K)$. As in the proof of Theorem 4.3 in [5], we consider the mollification $R_\phi f$ of f, which has $\bar{\partial} R_\phi f \in L^\infty(\mathbf{D})$ by the growth condition on f; here, $\bar{\partial}=(1-|z|^2)^2\,\partial/\partial\bar{z}$. Moreover,

$$\bar{\partial} R_\phi f * \chi_j = 0, \quad j=1,2,$$

holds, whence $\bar{\partial} R_\phi f * \psi=0$ for all functions ψ in the closed ideal generated by χ_1 and χ_2, that is,

$$\bar{\partial} R_\phi f * \psi = 0 \quad \text{for all } \psi \in L^1(G//K).$$

This implies that $\bar{\partial} R_\phi f \equiv 0$, that is, $R_\phi f$ is holomorphic. By letting ϕ run through a smooth approximate convolution identity, $R_\phi f$ converges to f in $L^1_{\mathrm{loc}}(\mathbf{D})$, and the conclusion that f is analytic on $\mathbf{D}$ follows. □

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