A STRUCTURE THEORY FOR A CLASS OF LATTICE-ORDERED RINGS

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The foundations of a systematic general theory of lattice-ordered rings were laid by Birkhoff and Pierce in [3]. They introduced, as an object for further study, the class of f-rings. This special class of lattice-ordered rings displays a rich structure: it can be characterized as the class of all subdirect unions of ordered rings. Birkhoff and Pierce obtained many properties of f-rings, basing their structure theory on the l-radical of an f-ring. In [20], Pierce obtained an important decomposition theorem for f-rings with zero l-radical. This paper continues the study of the structure of f-rings.

In Chapter I, we present the necessary background material and obtain a characterization of the l-radical of an f-ring that yields a new proof of the decomposition theorem of Pierce.

In Chapter II, we present a structure theory for f-rings based on an f-ring analogue of the Jacobson radical for abstract rings. In Section 1, the J-radical of an f-ring A is defined to be the intersection of the maximal modular right l-ideals of A. In Section 2, the J-radical is characterized in terms of the notion of l-quasi-regularity, and this characterization is used to obtain certain properties of the J-radical.

In Section 3, we consider a representation theory for f-rings. We show that every f-ring that has a faithful, irreducible, l-representation is a totally ordered ring with identity that contains no non-zero proper one-sided l-ideals. In Section 4, the notions of l-primitive f-ring

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and l-primitive l-ideal are introduced. The central result is: an f-ring is l-primitive if and only if it has a faithful, irreducible, l-representation. As a consequence of this we have: every maximal modular right l-ideal is an l-primitive (two-sided) l-ideal; hence the J-radical of an f-ring A is the intersection of the l-primitive l-ideals of A. This yields the main decomposition theorem for f-rings with zero J-radical: they are precisely the subdirect unions of ordered rings with identity that contain no non-zero proper one-sided l-ideals.

In Section 5 of Chapter II, we characterize the subdirectly irreducible f-rings and consider the special results obtainable for f-rings that satisfy the descending chain condition for l-ideals. Section 6 treats some special questions that arise during the course of the earlier part of the chapter.

In Chapter III, we consider the problems of imbedding an f-ring as a right l-ideal (respectively, as a sub-f-ring) of an f-ring with identity. In general, neither type of imbedding is possible. With each f-ring A, we associate a ring extension A_1 of A with identity, and we define a partial ordering of A_1 (the "strong order") that extends the partial order on A and makes A_1 into a partially ordered ring. The main results are: 1) A can be imbedded as a sub-f-ring of an f-ring with identity if and only if A_1, with the strong order, can be imbedded in an f-ring whose identity element is that of A_1; 2) A can be imbedded as a right l-ideal in an f-ring with identity if and only if A is what is called an "f_D-ring" and the strong order makes A_1 into an f-ring.

In Chapter IV, we consider the existence in f-rings of one-sided l-ideals that are not two-sided. In Section 1, we present an example of an ordered ring without non-zero divisors of zero that contains a one-sided l-ideal that is not two-sided, and we show that every such ordered ring contains a subring isomorphic to this example. In Section 2, we briefly consider the problem for larger classes of f-rings.

Chapter I. Background material

1. Lattice-ordered groups

In this section, we present those definitions and results in the theory of lattice-ordered groups that will be needed in what follows. Standard references on lattice-ordered groups are [2] Chapter XIV, [4], and [14].

In later sections, we will be concerned only with lattice-ordered groups that are commutative. Hence, whenever it would be inconvenient to do otherwise, the results in this section will be stated for abelian groups. For this reason, we will write all groups additively. Partial order relations will be denoted by $\leqslant$; the symbol $<$ will be used in totally ordered systems only.

DEFINITION 1.1. A *partially ordered group* is a group G which is partially ordered, and in which $g_1 \leqslant g_2$ implies $a + g_1 + b \leqslant a + g_2 + b$ for all a, $b \in G$. If G is a lattice under this partial order, then G is called a *lattice-ordered group.* If G is totally ordered, then G is called an *ordered group.*(1)

Let G be a partially ordered group. An element b of G is said to be *positive* if $b \geqslant 0$. The set of all positive elements of G is denoted by G^+. If G is a lattice-ordered group and $a \in G$, then the *absolute value* of a is $|a| = a \vee (-a)$, the *positive part* of a is $a^+ = a \vee 0$, and the *negative part* of a is $a^- = (-a) \vee 0$.(2)

PROPOSITION 1.2. *A partially ordered group G is a lattice-ordered group if and only if $a^+ = a \vee 0$ exists for every $a \in G$* ([2], p. 215, Theorem 2).

The interaction of the group operation and the lattice operations in a lattice-ordered group creates many interesting relations and identities. Those that we will use are collected in the following proposition; their proofs may be found in [2] and [4].

PROPOSITION 1.3. *If G is an abelian lattice-ordered group, and if a, b, $c \in G$, then:*

i) $a + (b \vee c) = (a + b) \vee (a + c)$ *and* $a + (b \wedge c) = (a + b) \wedge (a + c)$.

ii) $(-a) \wedge (-b) = -(a \vee b)$.

iii) $a + b = (a \vee b) + (a \wedge b)$.

iv) $|a + b| \leqslant |a| + |b|$ *and* $|a - b| \geqslant ||a| - |b||$.

v) $a = a^+ - a^-$.

vi) $a^+ \wedge a^- = 0$.

vii) *If b, $c \in G^+$ and $a = b - c$, then $b = a^+ + x$ and $c = a^- + x$, where $x = b \wedge c$.*

viii) *If $a \wedge b = 0$ and $a \wedge c = 0$, then $a \wedge (b + c) = 0$.*

ix) *For any non-negative integer n, $n(a \wedge b) = na \wedge nb$ and $n(a \vee b) = na \vee nb$. In particular, $(na)^+ = na^+$, $(na)^- = na^-$, and $n|a| = |na|$.*

x) *If n is a positive integer and $na \geqslant 0$, then $a \geqslant 0$.*

xi) $|a| = a^+ + a^-$.

xii) *Every element of G has infinite order.*

xiii) *If a, b, $c \in G^+$ and $a \leqslant b + c$, then there are elements b', $c' \in G^+$ with $b' \leqslant b$, $c' \leqslant c$, and $a = b' + c'$.*

Throughout this paper, the word "homomorphism" ("isomorphism") will, unless qualified, denote a mapping that is both an algebraic homomorphism (isomorphism) and

(1) The meaning of "ordered group" used here is found in [2]. In [4], the term "ordered group" is used to designate what we have here called "partially ordered group".

(2) In [2], the term "negative part of a", and the notation a^-, are used to designate the element $a \wedge 0 = -[(-a) \vee 0]$.

a lattice homomorphism (isomorphism). Thus, for lattice-ordered groups, we have ([3], p. 52, Lemma 1):

PROPOSITION 1.4. *A group homomorphism θ of a lattice-ordered group G into a lattice-ordered group G' is a homomorphism if and only if one of the following conditions is satisfied for $a, b \in G$:*

i) $(a \vee b)\theta = a\theta \vee b\theta$;

ii) $(a \wedge b)\theta = a\theta \wedge b\theta$;

iii) $|a|\theta = |a\theta|$;

iv) $a^+\theta = (a\theta)^+$;

v) $a \wedge b = 0$ *implies* $a\theta \wedge b\theta = 0$.

If θ is a homomorphism of a lattice-ordered group G into a lattice-ordered group G', then the kernel of θ is an l-subgroup of G in the sense of:

DEFINITION 1.5. An *l-subgroup* of a lattice-ordered group G is a normal subgroup H of G that satisfies:

$$a \in H \text{ and } |b| \leqslant |a| \text{ imply } b \in H.$$

If H is any l-subgroup of G, then the difference group $G - H$ can be made into a lattice-ordered group by defining $a + H \in (G - H)^+$ if and only if $a^- \in H$. Then we have:

THEOREM 1.6. *There is a one-to-one correspondence $\theta \leftrightarrow H_\theta$ between the homomorphisms defined on a lattice-ordered group G and the l-subgroups of G; such that, if θ maps G onto G_θ, then G_θ and $G - H_\theta$ are isomorphic under the correspondence $a\theta \leftrightarrow a + H_\theta$.*

THEOREM 1.7. *If θ is a homomorphism of a lattice-ordered group G onto a lattice-ordered group G', and if $\mathcal{H}$ denotes the family of all l-subgroups of G that contain the kernel of θ, then $H \to H\theta$ is a one-to-one mapping of $\mathcal{H}$ onto the family of all l-subgroups of G'* ([2], p. ix, Ex. 2(a)).

If S is any non-empty subset of an abelian lattice-ordered group G, then the smallest l-subgroup of G that contains S will be denoted $\langle S \rangle$:

$$\langle S \rangle = \{a \in G\colon |a| \leqslant n|b| + m|c|;\ b, c \in S,\ n, m \text{ positive integers}\}.$$

If H and K are l-subgroups of G, then the *join* of H and K is

$$\langle H + K \rangle = \{a \in G\colon |a| \leqslant |h| + |k|;\ h \in H,\ k \in K\}.$$

By Proposition 1.3, xiii), this is just $H + K = \{h + k\colon h \in H,\ k \in K\}$. It is clear that the intersection of any collection of l-subgroups of G is an l-subgroup of G.

The following observation, although an obvious consequence of the definition of l-subgroup, is very important, and will be used often.

PROPOSITION 1.8. *If G is an ordered group, then the l-subgroups of G form a chain.*

Many of the partially ordered groups that are important in applications are Archimedean in the sense of the following definition.

DEFINITION 1.9. A partially ordered group G is said to be *Archimedean* if for every pair a, b of elements of G, with $a \neq 0$, there is an integer n such that $na \nleq b$.

THEOREM 1.10 (Hölder). *Any Archimedean ordered group is isomorphic to a subgroup of the additive group of all real numbers, and so is commutative* ([2], p. 226, Theorem 15).

Thus, if G is an Archimedean ordered group, then every non-zero positive element of G is a strong order unit in the sense of:

DEFINITION 1.11. An element e of a lattice-ordered group G is called a *strong order unit* of G if for every $a \in G$ there is a positive integer n such that $ne \geqslant a$. A *weak order unit* of G is a positive element e of G which satisfies: $e \wedge a = 0$ if and only if $a = 0$.

2. Lattice-ordered rings

In Sections 2 and 3, we present those results from [3] that we will need.

Throughout this paper, all rings are assumed to be associative, and a ring identity element, when it exists, will be denoted by 1. As usual, if I and J are any two subsets of a ring A, then IJ denotes the set $\left\{\sum_{i=1}^{n} a_i b_i \colon a_i \in I,\ b_i \in J\right\}$.

DEFINITION 2.1. A *partially ordered ring* is a ring A which is partially ordered and in which:

i) $a \geqslant b$ implies $a + c \geqslant b + c$ for each $c \in A$, and

ii) $a \geqslant 0$ and $b \geqslant 0$ imply $ab \geqslant 0$.

If A is a lattice, then A is called a *lattice-ordered ring*; if A is totally ordered, then A is called an *ordered ring*.

A partially ordered ring is said to be *Archimedean* if its partially ordered additive group is Archimedean.

PROPOSITION 2.2. *The set A^+ of positive elements of a partially ordered ring A satisfies:*

I) $0 \in A^+$;

II) $A^+ \cap (-A^+) = \{0\}$;

III) $A^+ + A^+ \subseteq A^+$;

IV) $A^+ \cdot A^+ \subseteq A^+$;

V) *$a \geqslant b$ if and only if $a - b \in A^+$.*

Conversely, if A^+ is a subset of a ring A which satisfies conditions I)–IV), *then* V) *defines a partial order in* A, *under which A is a partially ordered ring.*

If A is a lattice-ordered ring, then the additive group of A is a lattice-ordered group. Thus, we may speak of the absolute value, positive part, and negative part of any $a \in A$, and the relations in Proposition 1.3 are valid in A. We also have:

PROPOSITION 2.3. *In any lattice-ordered ring A,*

i) $a \geqslant 0$ *implies* $a(b \vee c) \geqslant ab \vee ac$, $a(b \wedge c) \leqslant ab \wedge ac$, $(b \vee c)a \geqslant ba \vee ca$, *and* $(b \wedge c)a \leqslant ba \wedge ca$;

ii) $|ab| \leqslant |a|\,|b|$.

DEFINITION 2.4. A *homomorphism* of a lattice-ordered ring A into a lattice-ordered ring A' is a ring homomorphism θ of A into A' that satisfies $(a \vee b)\theta = a\theta \vee b\theta$ for every pair a, b of elements of A (cf. Proposition 1.4).

DEFINITION 2.5. A subset I of a lattice-ordered ring A is an *l-ideal* of A if:

i) I is a ring ideal of A, and

ii) $a \in I$, $b \in A$, and $|b| \leqslant |a|$ imply $b \in I$.

If I is merely a right (left) ring ideal of A, then I is called a *right* (*left*) *l-ideal* of A.

Every l-ideal in a lattice-ordered ring is the kernel of a homomorphism, and we have (cf. Theorems 1.6 and 1.7):

THEOREM 2.6. *There is a one-to-one correspondence $\theta \leftrightarrow I_\theta$ between the homomorphisms defined on a lattice-ordered ring A and the l-ideals of A; such that, if θ maps A onto A_θ, then A_θ and A/I_θ are isomorphic under the correspondence $a\theta \leftrightarrow a + I_\theta$.*

THEOREM 2.7. *If θ is a homomorphism of a lattice-ordered ring A onto a lattice-ordered ring A', and if $\mathfrak{J}$ denotes the family of all* (*right, left, two-sided*) *l-ideals of A that contain the kernel of θ, then $I \to I\theta$ is a one-to-one mapping of $\mathfrak{J}$ onto the family of all* (*right, left, two-sided*) *l-ideals of A'.*

A (right, left, two-sided) l-ideal I of A is said to be *proper* if $I \neq A$. If I is such that it is contained in no other proper (right, left, two-sided) l-ideal, then I is said to be a *maximal* (right, left, two-sided) l-ideal.

DEFINITION 2.8. A lattice-ordered ring A is said to be *l-simple* if $A^2 \neq \{0\}$ and if A contains no non-zero proper l-ideals.

If S is any subset of a lattice-ordered ring A, then $\langle S \rangle$ denotes the smallest l-ideal of A containing S; $\langle S \rangle_r (\langle S \rangle_l)$ denotes the smallest right (left) l-ideal of A containing S.

It is clear that the intersection of any collection of (right, left, two-sided) l-ideals of a lattice-ordered ring A is a (right, left, two-sided) l-ideal of A. The *join* of two (right, left, two-sided) l-ideals I, J of A is the (right, left, two-sided) l-ideal

$$I+J=\{a+b:\ a\in I,\ b\in J\}$$

(cf. the discussion following Theorem 1.7). The product of two (right, left, two-sided) l-ideals I, J is the (right, left, two-sided) l-ideal

$$\langle IJ\rangle=\{c\in A:\ |c|\leqslant \sum_{i=1}^{n} a_i b_i;\ a_i\in I,\ b_i\in J\}.$$

Since $c\in\langle IJ\rangle$ implies $|c|\leqslant \sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n}|a_i|\right)\left(\sum_{j=1}^{n}|b_j|\right)$ for suitable $a_i\in I$, $b_j\in J$, we may write

$$\langle IJ\rangle=\{c\in A:\ |c|\leqslant ab;\ a\in I,\ b\in J\}.$$

An immediate consequence of Proposition 1.8 is:

PROPOSITION 2.9. *If A is an ordered ring, then the (right, left, two-sided) l-ideals of A form a chain.*

f-rings

The class of lattice-ordered rings admits a disquieting amount of pathology. For example, Birkhoff and Pierce have given an example of a commutative lattice-ordered ring with identity element 1 in which 1 is not a positive element (even though a square). For this reason, they have suggested the study of a special class of lattice-ordered rings: those that are f-rings in the sense of the following definition.

DEFINITION 3.1. An *f-ring* is a lattice-ordered ring in which $a\wedge b=0$ and $c\geqslant 0$ imply $ca\wedge b=ac\wedge b=0$.

It is to this class of rings that we will restrict our attention.

Every ordered ring is an f-ring, since, in an ordered ring, $a\wedge b=0$ implies either $a=0$ or $b=0$. Any abelian lattice-ordered group G can be made into an f-ring by defining $ab=0$ for all a, $b\in G$. The following, more interesting, example of an f-ring serves to motivate the choice of terminology.

EXAMPLE 3.2 (cf. [10]). Let X be a Hausdorff space, $C(X)$ the set of all continuous real-valued functions on X. If all operations are defined pointwise, then $C(X)$ becomes an Archimedean f-ring: for f, $g\in C(X)$ and each $x\in X$,

$$(f+g)(x)=f(x)+g(x),\ \ fg(x)=f(x)\cdot g(x)$$
$$(f\wedge g)(x)=f(x)\wedge g(x).$$

With each point x of X there are associated two l-ideals of special interest, defined by:

$$O_x = \{f \in C(X) : f[U] = \{0\} \text{ for some neighborhood } U \text{ of } x\}, \text{ and}$$
$$M_x = \{f \in C(X) : f(x) = 0\}.$$

In general, O_x is properly contained in M_x. For each x, the l-ideal M_x is maximal in $C(X)$, and $C(X)/M_x$ is isomorphic to the ordered field of real numbers.

If A is an f-ring, we will call a subring of the ring A a *sub-f-ring* if it is also a sublattice of the lattice A.

PROPOSITION 3.3. *If A is an f-ring, then:*

i) *every sub-f-ring of A is an f-ring;*
ii) *every homomorphic image of A is an f-ring;*
iii) *if A has an identity, then* 1 *is a weak order unit in A.*

We also have the following partial characterizations of f-rings as lattice-ordered rings:

PROPOSITION 3.4. *If A is a lattice-ordered ring with positive identity element* 1, *then A is an f-ring if:*

i) 1 *is a strong order unit in A; or*
ii) 1 *is a weak order unit in A and A contains no non-zero positive nilpotent elements.*

Now, let $\{A_\alpha : \alpha \in \Gamma\}$ be a non-empty family of f-rings, and consider the set A of all functions $a : \Gamma \to \bigcup \{A_\alpha : \alpha \in \Gamma\}$ such that $a_\alpha = a(\alpha) \in A_\alpha$ for each $\alpha \in \Gamma$. In A, define addition and multiplication by

$$(a + b)_\alpha = a_\alpha + b_\alpha \text{ and } (ab)_\alpha = a_\alpha b_\alpha \text{ for each } \alpha \in \Gamma.$$

Then, as is well known, A is a ring. Moreover, if we define a partial order in A by

$$a \geqslant b \text{ if and only if } a_\alpha \geqslant b_\alpha \text{ for each } \alpha \in \Gamma,$$

then A is a lattice-ordered ring, where

$$(a \wedge b)_\alpha = a_\alpha \wedge b_\alpha \text{ and } (a \vee b)_\alpha = a_\alpha \vee b_\alpha \text{ for each } \alpha \in \Gamma.$$

It is clear that A is an f-ring.

DEFINITION 3.5. The f-ring A described above is called the *complete direct union* of the family $\{A_\alpha : \alpha \in \Gamma\}$ of f-rings.

For each $\alpha \in \Gamma$, the mapping $a \to a_\alpha$ is a homomorphism of A onto A_α. If B is any sub-f-ring of A, then this mapping, restricted to B, is a homomorphism of B into A_α.

DEFINITION 3.6. A sub-f-ring of the complete direct union of the family $\{A_\alpha : \alpha \in \Gamma\}$ of f-rings is said to be a *subdirect union* of that family if the homomorphism $a \to a_\alpha$ maps B onto A_α, for each $\alpha \in \Gamma$.

We have the following important characterization ([2], p. 92, Theorem 9):

THEOREM 3.7. *An f-ring A is isomorphic to a subdirect union of the family $\{A_\alpha : \alpha \in \Gamma\}$ of f-rings if and only if there is a collection $\{I_\alpha : \alpha \in \Gamma\}$ of l-ideals in A such that A/I_α is isomorphic to A_α for each $\alpha \in \Gamma$, and $\bigcap \{I_\alpha : \alpha \in \Gamma\} = \{0\}$.*

We will denote a subdirect union of the family $\{A_\alpha : \alpha \in \Gamma\}$ of f-rings by $\Sigma_s \{A_\alpha : \alpha \in \Gamma\}$. Note that, in general, this does not denote a unique f-ring. For example, if Γ is infinite and if each A_α contains an identity, then the complete direct union of the family $\{A_\alpha : \alpha \in \Gamma\}$ is a subdirect union of this family that contains an identity element. A subdirect union of the same family that does not contain an identity element is the discrete direct union in the sense of the following definition.

DEFINITION 3.8. The *discrete direct union* of the family $\{A_\alpha : \alpha \in \Gamma\}$ of f-rings is the sub-f-ring of their complete direct union consisting of those elements a with $a_\alpha = 0$ except for a finite number of $\alpha \in \Gamma$.

If an f-ring A is the complete direct union of a finite family $\{A_i : i = 1, 2, \ldots, n\}$ of f-rings, it will be called merely the *direct union* of this family.

An f-ring will be called subdirectly irreducible in case all of its subdirect union representations are trivial. Stated precisely,

DEFINITION 3.9. An f-ring A is said to be *subdirectly irreducible* if $A \neq \{0\}$ and every isomorphism θ of A onto a subdirect union $\Sigma_s \{A_\alpha : \alpha \in \Gamma\}$ of f-rings is such that the mapping $a \to (a\theta)_\alpha$ is an isomorphism for at least one $\alpha \in \Gamma$.

COROLLARY 3.10. *An f-ring A is subdirectly irreducible if and only if the intersection of all of the non-zero l-ideals of A is not $\{0\}$* (Theorem 3.7).

Thus, A is subdirectly irreducible if and only if A contains a unique smallest non-zero l-ideal, given by

$$\bigcap \{I : I \text{ is a non-zero } l\text{-ideal of } A\}.$$

The following lemma and its corollary give insight into the nature of subdirectly irreducible f-rings.

LEMMA 3.11. *If a and b are elements of an f-ring A such that $a \wedge b = 0$, then $\langle a \rangle \cap \langle b \rangle = \{0\}$.*

Now, if an f-ring A is not an ordered ring, then there are incomparable elements x, y in A. Then $a = (x - y)^+$ and $b = (x - y)^-$ are non-zero elements of A with $a \wedge b = 0$ (Proposition 1.3, vi)). By the lemma, $\langle a \rangle$ and $\langle b \rangle$ are non-zero l-ideals in A with zero intersection, so A is not subdirectly irreducible, by Corollary 3.10. Thus, we have:

COROLLARY 3.12. *Every subdirectly irreducible f-ring is an ordered ring.*

It was remarked in [3] that every ordered ring is a subdirectly irreducible f-ring. However, the following example shows that this is not the case. In Chapter II, we will characterize those f-rings that are subdirectly irreducible.

Example 3.13. Let $Q[\lambda]$ denote the ring of polynomials in one (commuting) indeterminate λ over the ordered field Q of rational numbers. Order $Q[\lambda]$ lexicographically, with the constant term dominating: $a_0 + a_1\lambda + \cdots + a_n\lambda^n \geqslant 0$ if and only if $a_0 > 0$, or $a_0 = 0$ and $a_1 > 0$, or $\ldots$, or $a_0 = a_1 = \cdots = a_{n-1} = 0$ and $a_n \geqslant 0$. Under this order, $Q[\lambda]$ is a commutative ordered ring; it is clearly non-Archimedean, since $n\lambda \leqslant 1$ for every integer n.

The proper l-ideals of $Q[\lambda]$ are just the ring ideals $\langle \lambda^n \rangle = \{a_n\lambda^n + a_{n+1}\lambda^{n+1} + \cdots + a_{n+k}\lambda^{n+k}\}$ for each positive integer n. Thus, by Corollary 3.10, $Q[\lambda]$ is not subdirectly irreducible, since $\bigcap \{\langle \lambda^n \rangle : n = 1, 2, \ldots\} = \{0\}$.

Now, every ordered ring is an f-ring, so every subdirect union of ordered rings is also an f-ring. Conversely, the well-known theorem of Birkhoff ([2], p. 92, Theorem 10) states, in this case, that every f-ring is isomorphic to a subdirect union of subdirectly irreducible f-rings. Thus:

THEOREM 3.14. *A lattice-ordered ring is an f-ring if and only if it is isomorphic to a subdirect union of ordered rings.*

It is now an easy matter to determine some of the special properties of f-rings. For, any property that holds in every ordered ring and that is preserved under subdirect union is enjoyed by all f-rings. The following properties are of this sort:

THEOREM 3.15. *If A is an f-ring, then, for $a, b, c \in A$:*

i) *If $a \geqslant 0$, then $a(b \vee c) = ab \vee ac$, $a(b \wedge c) = ab \wedge ac$, $(b \vee c)a = ba \vee ca$, and $(b \wedge c)a = ba \wedge ca$.*

ii) $|ab| = |a| \cdot |b|$.

iii) *If $a \wedge b = 0$, then $ab = 0$.*

iv) $a^2 \geqslant 0$.

v) *If $a, b \in A^+$ and n is any positive integer, then $n|ab - ba| \leqslant a^2 + b^2$.*

COROLLARY 3.16. *An f-ring without non-zero divisors of zero is an ordered ring.*

COROLLARY 3.17. *Every Archimedean f-ring is commutative.*

There are, however, non-commutative f-rings. Examples of non-commutative ordered rings will be given in Chapters II and IV. Any ring constructed from these by subdirect union will be a non-commutative f-ring.

In any f-ring A, the set $Z_n(A) = \{a \in A : a^n = 0\}$ is an l-ideal of A for each positive integer n.

DEFINITION 3.18.(1) If A is an f-ring, then the *l-radical* of A, denoted $N(A)$, is the set of all nilpotent elements of A:

$$N(A) = \bigcup\{Z_n(A) : n = 1, 2, \ldots\}.$$

It is easily seen that $N(A)$ is an l-ideal of A and that $A/N(A)$ has zero l-radical. Now, for each n, $Z_n(A)$ is a nilpotent l-ideal; that is, $[Z_n(A)]^n = \{0\}$. However, $N(A)$ may not be nilpotent, as is shown by the following example.

Example 3.19. Let $Q[\lambda]$ be the ring of polynomials with rational coefficients, ordered lexicographically as in Example 3.13. For each positive integer n, let A_n denote the ordered ring $Q[\lambda]/\langle\lambda^n\rangle$. Then, in A_n, the element $\bar{\lambda} = \lambda + \langle\lambda^n\rangle$ satisfies $\bar{\lambda}^{n-1} \neq 0$ and $\bar{\lambda}^n = 0$. If A denotes the discrete direct union of the family $\{A_n : n = 1, 2, \ldots\}$, then it is clear that $N(A)$ is not nilpotent.

DEFINITION 3.20. An f-ring A is said to satisfy the *descending chain condition* for l-ideals if every properly descending chain $I_1 \supset I_2 \supset I_3 \supset \ldots$ of l-ideals of A is finite.

THEOREM 3.21. *If A is an f-ring that satisfies the descending chain condition for l-ideals, then $N(A)$ is a nilpotent l-ideal.*

4. Ordered rings and prime l-ideals

The results presented in this section are, as is the case with all of this chapter, not entirely new. However, as presented here, each has some claim to originality.

Recall (e.g., [23], p. 221) that if A is a commutative ordered ring without non-zero divisors of zero, then there is exactly one way that the field F of quotients of A may be ordered so that A is an ordered subring of F. This order on F is given by

$$a/b \geqslant 0 \quad \text{if and only if } ab \geqslant 0.$$

It is easily seen that F is Archimedean if and only if A is.

We now present alternate proofs of two results of Hion ([11], [12]).

(1) Birkhoff and Pierce have considered the l-radical of an arbitrary lattice-ordered ring. The definition given here is merely the special form that the l-radical assumes in f-rings.

THEOREM 4.1. *If an ordered ring A contains non-zero divisors of zero, then it contains a non-zero nilpotent l-ideal.*

Proof. If A contains non-zero divisors of zero, then there are non-zero positive elements a, b of A with $ab = 0$. Thus, $0 = ab \geqslant (a \wedge b)^2 \geqslant 0$. Since A is an ordered ring, this means that either $a^2 = 0$ or $b^2 = 0$. Thus $Z_2(A)$ is a non-zero nilpotent l-ideal of A (cf. the discussion following Definition 3.18).

THEOREM 4.2. *If A is an Archimedean ordered ring, then either* i) *A is isomorphic to an ordered subring of the ordered field of real numbers, or* ii) *$A^2 = \{0\}$ and the ordered additive group of A is isomorphic to a subgroup of the ordered additive group of real numbers.*

Proof. If A contains a non-zero divisor of zero, then A contains a non-zero l-ideal I with $I^2 = \{0\}$, by Theorem 4.1. Since A is Archimedean, its additive group contains no non-zero l-subgroups (Theorem 1.10), so $I = A$. Thus $A^2 = \{0\}$, and ii) now follows by Theorem 1.10.

If A contains no non-zero divisors of zero, then, since A is commutative by Corollary 3.17, we may imbed A in its field F of quotients. As remarked above, F is an Archimedean ordered field. By a well-known theorem, F is isomorphic to a subfield of the ordered field of real numbers, so i) holds.

Recall that, in any abstract ring A, a proper (ring) ideal P of A is said to be *prime* in case $I_1 I_2 \subseteq P$ for any two (ring) ideals I_1, I_2 of A implies $I_1 \subseteq P$ or $I_2 \subseteq P$ (cf. [19]). Also, a ring A is said to be prime if and only if $\{0\}$ is a prime ideal in A. Hence, P is a prime ideal in a ring A if and only if A/P is a prime ring. Prime l-ideals, that is, l-ideals that are prime ring ideals, play an important role in the structure theory for f-rings.

THEOREM 4.3. *A proper l-ideal P in an f-ring A is prime if and only if $I_1 I_2 \subseteq P$ for any two l-ideals I_1, I_2 of A implies $I_1 \subseteq P$ or $I_2 \subseteq P$.*

Proof. Suppose that $I_1 I_2 \subseteq P$ for any two l-ideals I_1, I_2 of A implies $I_1 \subseteq P$ or $I_2 \subseteq P$. If I_1 and I_2 are any two (ring) ideals of A with $I_1 I_2 \subseteq P$, then $\langle I_1 I_2 \rangle \subseteq P$. That $I_1 \subseteq P$ or $I_2 \subseteq P$ will be shown by proving that $\langle I_1 \rangle \langle I_2 \rangle \subseteq \langle I_1 I_2 \rangle$. If $a \in \langle I_1 \rangle$ and $b \in \langle I_2 \rangle$, then $|a| \leqslant \sum_i |a_i|$ and $|b| \leqslant \sum_j |b_j|$ for $a_i \in I_1$ and $b_j \in I_2$. Then, $|ab| = |a| \cdot |b| \leqslant (\sum_i |a_i|)(\sum_j |b_j|) = \sum_{i,j} |a_i b_j|$, whence $ab \in \langle I_1 I_2 \rangle$. Thus, $\langle I_1 \rangle \langle I_2 \rangle \subseteq \langle I_1 I_2 \rangle \subseteq P$.

The converse is obvious.

THEOREM 4.4. *An f-ring A is prime if and only if $A \neq \{0\}$ and A is an ordered ring without non-zero divisors of zero.*

Proof. If A is a prime f-ring, then $A \neq \{0\}$ by definition. Moreover, A is an ordered ring, for we have seen (cf. the discussion following Lemma 3.11) that any f-ring that contains incomparable elements contains non-zero l-ideals I_1, I_2 with $I_1 \cap I_2 = \{0\}$, and hence $I_1 I_2 \subseteq I_1 \cap I_2 = \{0\}$. If A contained non-zero divisors of zero, then it would contain a non-zero l-ideal I with $I^2 = \{0\}$, by Theorem 4.1, contrary to the definition of prime ring. Thus, A is an ordered ring without non-zero divisors of zero.

The converse is obvious.

COROLLARY 4.5. *If A is an ordered ring, and if $A \neq N(A)$, then $N(A)$ is a prime l-ideal.*

Proof. $A/N(A) \neq \{0\}$, and $A/N(A)$ is an ordered ring without non-zero divisors of zero, by Theorem 4.1 and the remark following Definition 3.18.

COROLLARY 4.6. *A proper l-ideal P in an f-ring A is prime if and only if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in A$.*

Proof. If P is prime, then A/P is a prime f-ring, so it contains no non-zero divisors of zero.

Conversely, suppose $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in A$. It is clear that A/P contains no non-zero divisors of zero. Hence, by Corollary 3.16, A/P is an ordered ring without non-zero divisors of zero. Since $A/P \neq \{0\}$, it is a prime f-ring by Theorem 4.4. Thus, P is a prime l-ideal.

Corollary 4.6 is the first appearance of a phenomenon that will present itself again in Chapter II. We may describe this, roughly, as follows: in considering the l-ideal structure of f-rings, the absence of commutativity seems to have less effect than in the ideal structure of abstract rings. In the present instance, the equivalence in Corollary 4.6 is true for (ring) ideals only in commutative abstract rings.

A subset M of an f-ring A is called a *multiplicative system* if $a, b \in M$ implies $ab \in M$. By Corollary 4.6, a proper l-ideal P of A is prime if and only if the complementary set of P in A (i.e., $\{a \in A : a \notin P\}$) is a multiplicative system. Conversely, we have the following result. The proof, which is omitted, is a standard argument (e.g., [18], p. 105).

THEOREM 4.7. *If I is an l-ideal in an f-ring A, and M a multiplicative system in A that does not meet I, then there is a prime l-ideal P of A that contains I and does not meet M.*

We can now prove the following important theorem ([20]).

THEOREM 4.8. *The l-radical $N(A)$ of an f-ring A is the intersection of the prime l-ideals of A. Hence, $N(A) = \{0\}$ if and only if A is isomorphic to a subdirect union of ordered rings without non-zero divisors of zero.*

Proof. Since $N(A)$ consists of the nilpotent elements of A, it is contained in every prime l-ideal of A, by Corollary 4.6. Conversely, if $a \notin N(A)$, then $M = \{a, a^2, a^3, \dots\}$ is a multiplicative system that does not meet $N(A)$. Hence, there is a prime l-ideal P that contains $N(A)$ but does not meet M, by Theorem 4.7.

Thus, $N(A) = \{0\}$ if and only if there is a family $\{P_\alpha: \alpha \in \Gamma\}$ of prime l-ideals in A with $\bigcap\{P_\alpha: \alpha \in \Gamma\} = \{0\}$, hence if and only if A is isomorphic to a subdirect union of the family $\{A/P_\alpha: \alpha \in \Gamma\}$ of prime f-rings (Theorem 3.7). By Theorem 4.4, each of these has the required form.

In [19], McCoy defined a notion of the radical of an abstract ring which coincides, in the presence of the descending chain condition for right ideals, with the classical radical. One of his important results was the exact analogue of Theorem 4.8. Thus, we may view the l-radical of an f-ring as the analogue of the McCoy radical for abstract rings.

According to Proposition 2.9, the l-ideals of an ordered ring form a chain. Hence, by Theorem 2.7, we have:

PROPOSITION 4.9. *If an l-ideal I of an f-ring A is such that A/I is an ordered ring, then the l-ideals of A that contain I form a chain.*

Thus, any such l-ideal I is contained in at most one maximal l-ideal. It is easily shown (just as in abstract rings) that every l-ideal in an f-ring with identity is contained in a maximal l-ideal. Hence:

PROPOSITION 4.10. *If A is an f-ring with identity, then every l-ideal I in A such that A/I is an ordered ring is contained in a unique maximal l-ideal.*

In particular, every prime l-ideal in an f-ring A with identity is contained in a unique maximal l-ideal. This result is, in a certain sense, a generalization of similar results that have appeared in less general contexts (e.g., [5], [10]).

CHAPTER II. THE J-RADICAL AND THE STRUCTURE OF J-SEMISIMPLE f-RINGS

In this chapter, we consider a structure theory for f-rings which is modelled after the Jacobson theory for abstract rings. Most of the definitions are exact analogues of the definitions of corresponding notions in the Jacobson theory; many of the theorems can be viewed similarly.

1. The J-radical of an f-ring

DEFINITION 1.1. A right l-ideal I of an f-ring A is said to be *modular* if there exists an $e \in A$ such that $x - ex \in I$ for each $x \in A$. The element e is said to be a *left identity modulo I*.

PROPOSITION 1.2. *Let A be an f-ring, I a modular right l-ideal in A.*

i) *There exists an $e \in A^+$ such that e is a left identity modulo I.*

ii) *If I' is a right l-ideal of A containing I, then I' is modular (with the same left identity).*

iii) *If e is a left identity modulo I, then I is proper if and only if $e \notin I$.*

iv) *If I is proper, then I can be imbedded in a maximal (modular) right l-ideal.*

Proof. i) Suppose e is a left identity modulo I. Then, for each $x \in A$, we have $|x| - |e|\,|x| \in I$, since $x - ex \in I$ and $|x - ex| \geqslant \big||x| - |ex|\big|$. Thus, for each $x \in A$, we have $x^+ - |e|x^+ \in I$ and $x^- - |e|x^- \in I$, so $x - |e|x = (x^+ - |e|x^+) - (x^- - |e|x^-) \in I$.

ii) and iii) are obvious, and iv) is an easy application of Zorn's lemma, using iii).

DEFINITION 1.3. Let A be any f-ring. The *J-radical* of A, which is denoted $J(A)$, is the intersection of all the maximal modular right l-ideals of A. If $A \neq \{0\}$ and $J(A) = \{0\}$, then A is said to be *J-semisimple*. If $J(A) = A$, that is, if A contains no maximal modular right l-ideals, then A is said to be a *J-radical ring*.

The examples that follow demonstrate the fact that the J-radical of an f-ring is distinct from the l-radical; however, we will see (Corollary 2.6) that the J-radical always contains the l-radical.

For any f-ring A, let $R(A)$ denote the Perlis-Jacobson radical of A considered as an abstract ring; that is, $R(A)$ is the intersection of the maximal modular right (ring) ideals of A. Since $R(A)$ contains all nil one-sided (ring) ideals of A ([13], p. 9), we have $N(A) \subseteq R(A)$. However, the following examples show that $J(A)$ and $R(A)$ bear little relationship to each other.

Example 1.4. Let A denote the ring $Q[\lambda]$ of polynomials with rational coefficients, ordered lexicographically as in Example I.3.13. A is an ordered ring without non-zero nilpotent elements, so $N(A) = \{0\}$. Also, $R(A) = \{0\}$ ([13], p. 22). The unique maximal (modular) right l-ideal of A is $\langle\lambda\rangle$, whence $J(A) = \langle\lambda\rangle$. We have:

$$\{0\} = N(A) = R(A) \subset J(A) = \langle\lambda\rangle \neq \{0\}.$$

Example 1.5. Let B denote the ordered subring $\{p/q\colon (q, 2) = 1\}$ of the ordered field Q of rational numbers, and consider the direct union C of B and the ring A of Example 1.4. Then C consists of all ordered pairs (x, y), where $x \in B$ and $y \in A$. Since B contains no non-zero proper l-ideals, it is readily seen that $J(C)$ consists of all elements of the form $(0, y)$, where $y \in \langle\lambda\rangle$. Now (cf. [13], p. 21, and p. 10, Theorem 1), $R(C)$ consists of all elements of the form $(x, 0)$, where x is an element of the ring ideal (2) of B. Thus, in this case, $N(C) = \{0\}$, $R(C) \neq \{0\}$, $J(C) \neq \{0\}$, and $R(C) \cap J(C) = \{0\}$.

Example 1.6. Let A denote the ring of even integers with the usual order. Then we have $N(A) = R(A) = \{0\}$. However, A is an l-simple commutative f-ring without identity, so A is clearly a J-radical ring.

It is interesting to note that the analogue of one of the outstanding unsolved problems in the Jacobson theory for abstract rings has its solution in the preceding example: there do exist l-simple J-radical rings.

2. The notion of l-quasi-regularity

If a is any element of an f-ring A, let $(1-a)A$ denote the set $\{x - ax : x \in A\}$. The right l-ideal $\langle(1-a)A\rangle_r$ generated by this set is the smallest modular right l-ideal with a as a left identity.

DEFINITION 2.1. An element a of an f-ring A is said to be *right l-quasi-regular* (right l-QR) if $\langle(1-a)A\rangle_r = A$.

PROPOSITION 2.2. *Let A be an f-ring, $a \in A$.*

i) *a is right l-QR if and only if there are a finite number of elements $x_1, \ldots, x_n$ in A such that $|a| \leqslant \sum_{i=1}^{n} |x_i - ax_i|$.*

ii) *If $|a|$ is right l-QR, then a is also.*

iii) *If a is nilpotent, then a is right l-QR.*

iv) *If a is a non-zero idempotent, then a is not right l-QR.*

Proof.

i) If a is right l-QR, then $a \in \langle(1-a)A\rangle_r = A$. But this means that there are elements $z_1, \ldots, z_n \in (1-a)A$ such that $|a| \leqslant |z_1| + |z_2| + \cdots + |z_n|$. Each z_i is of the form $x_i - ax_i$ for some $x_i \in A$.

Conversely, if there are elements $x_1, \ldots, x_n \in A$, with $|a| \leqslant \sum_{i=1}^{n} |x_i - ax_i|$, then it is clear that $a \in \langle(1-a)A\rangle_r$. But, since $\langle(1-a)A\rangle_r$ is a modular right l-ideal with left identity a, this implies (Proposition 1.2) that $\langle(1-a)A\rangle_r = A$. Thus, a is right l-QR.

ii) If $|a|$ is right l-QR, then there are elements $x_1, \ldots, x_n$ of A with

$$|a| \leqslant \sum_{i=1}^{n} |x_i - |a|\, x_i| \leqslant \sum_{i=1}^{n} |x_i^+ - |a|\, x_i^+| + \sum_{i=1}^{n} |x_i^- - |a|\, x_i^-|$$

$$\leqslant \sum_{i=1}^{n} |x_i^+ - a x_i^+| + \sum_{i=1}^{n} |x_i^- - a x_i^-|.$$

Thus, by i), a is right l-QR.

iii) Suppose $a^n = |a|^n = 0$. Set $x = |a| + |a|^2 + \cdots + |a|^{n-1}$. Then $|a| = x - |a|x$, so $|a|$ is right l-QR. By ii), a is right l-QR.

iv) Suppose a is right l-QR, say $|a| \leqslant \sum_{i=1}^{n} |x_i - ax_i|$ for $x_1, \ldots, x_n \in A$. If a is also idempotent, then

$$0 \leqslant a = a^2 \leqslant a \sum_{i=1}^{n} |x_i - a x_i| = \sum_{i=1}^{n} |a x_i - a^2 x_i| = 0.$$

Definition 2.3. A right l-ideal I in an f-ring A is said to be l-quasi-regular (l-QR) in case every element of I is a right l-QR element of A.

Note that we do not have any analogue of the notion of the quasi-inverse of an element that occurs in the Jacobson theory. However, the following proposition is the analogue of a result concerning quasi-inverses in abstract rings ([13], p. 8, Proposition 1).

Proposition 2.4. *If I is an l-QR right l-ideal in an f-ring A, then each element of I is a right l-QR element of the f-ring I; that is, if $a \in I$, then there are elements $x_1, \ldots, x_n \in I$ such that $|a| \leqslant \sum_{i=1}^{n} |x_i - ax_i|$.*

Proof. If $a \in I$, then a is a right l-QR element of A, hence there are elements $y_1, \ldots, y_n \in A$ such that $|a| \leqslant \sum_{i=1}^{n} |y_i - ay_i|$ (Proposition 2.2). Then $a^2 = |a|^2 \leqslant |a| \sum_{i=1}^{n} |y_i - ay_i| = \sum_{i=1}^{n} |ay_i - a(ay_i)|$. If we let I_a denote the right l-ideal in the f-ring I generated by $(1-a)I$, then we have shown that $a^2 \in I_a$. But, a is a left identity modulo I_a (in I), so $a^2 \in I_a$ implies $a \in I_a$, whence a is right l-QR in the f-ring I.

The following theorem, the central result of this section, will be improved in Section 4, where it will be shown that $J(A)$ is a (two-sided) l-ideal in A.

Theorem 2.5. *The J-radical of an f-ring A is an l-QR right l-ideal of A that contains every l-QR right l-ideal of A.*

Proof. If a is not right l-QR, then $\langle (1-a)A \rangle_r$ is a proper modular right l-ideal in A with left identity a. By Proposition 1.2, $\langle (1-a)A \rangle_r$ can be imbedded in a maximal modular right l-ideal M which also has left identity $a \notin M$. But then, $a \notin J(A)$.

Now, suppose that I is an l-QR right l-ideal in A and that $I \not\subseteq J(A)$. Then there is a maximal modular right l-ideal M of A such that $I \not\subseteq M$. Let e be a left identity modulo M. By Proposition 1.2, we may assume that e is positive. Since M is maximal, $I + M = A$. Hence, by Proposition I.1.3, xiii), there are positive elements $z \in M$ and $b \in I$ such that $e = z + b$. Then $e - z = b \in I$ is right l-QR. Thus, by Proposition 2.2, there are elements $x_1, \ldots, x_n \in A$ such that $0 \leqslant e - z \leqslant \sum_{i=1}^{n} |x_i - (e-z)x_i| = \sum_{i=1}^{n} |(x_i - ex_i) + zx_i|$. But M is a modular right l-ideal with left identity e, and $z \in M$, whence $x_i - ex_i$ and zx_i are elements of M for each i, so the right hand member of the above inequality is an element of M. Then $e - z \in M$, whence $e \in M$, since $z \in M$. This is a contradiction.

COROLLARY 2.6. *$J(A)$ contains all nil right l-ideals of A; hence $J(A)$ contains the l-radical $N(A)$* (Proposition 2.2).

Since the l-radical contains all of the nilpotent elements of A (Definition I.3.18), Corollary 2.6 yields the following result, which is in contrast with the situation in the Jacobson theory.

COROLLARY 2.7. *If A is J-semisimple, then A contains no non-zero nilpotent elements.*

We have already seen (Example 1.4) that the converse of Corollary 2.7 is not true; there are f-rings without non-zero nilpotent elements which are not J-semisimple.

COROLLARY 2.8. *$J(A)$ contains no non-zero idempotent elements* (Proposition 2.2).

COROLLARY 2.9. *If φ is a homomorphism of A onto an f-ring A', then $\varphi[J(A)] \subseteq J(A')$.*

Proof. The image of an l-QR right l-ideal in A is an l-QR right l-ideal in A'.

As in the case of its analogue in the Jacobson theory, the inequality of Corollary 2.9 cannot, in general, be strengthened; for we have the following example of a homomorphic image of a J-semisimple f-ring that is not J-semisimple.

Example 2.10. Let $A = C(R)$, the f-ring of all continuous real-valued functions on the space R of real numbers with the usual topology (cf. Example I.3.2). It is easily seen that A is J-semisimple (e.g. Theorem 2.11). Let $p \in R$. There are functions in A that vanish at p but not on any neighborhood of p (e.g. the function f defined by $f(x) = (x - p)$, so that $O_p \neq M_p$. Thus, O_p is an l-ideal in A which is properly contained in exactly one maximal (modular) right l-ideal. Hence, A/O_p contains precisely one non-zero maximal (modular) l-ideal M_p/O_p, which must be $J(A/O_p)$.

THEOREM 2.11. *If A is an Archimedean f-ring with identity, then A is J-semisimple.*

Proof. We can write A as a subdirect union of ordered rings A_α, for α in some index set Γ (Theorem I.3.14). For each $\alpha \in \Gamma$, let h_α denote the homomorphism of A onto A_α, and let I_α denote the kernel of h_α. Now, $\bigcap \{I_\alpha : \alpha \in \Gamma\} = \{0\}$, and, for each α, A_α contains an identity $h_\alpha(1)$, so that $J(A_\alpha)$ is a proper right l-ideal in A_α by Corollary 2.8. Now suppose that a is a positive element in $J(A)$. For each $\alpha \in \Gamma$, $h_\alpha(a)$ is a positive element of $J(A_\alpha)$, by Corollary 2.9. Since each A_α is an ordered ring and $J(A_\alpha)$ is a proper right l-ideal in A_α, we have $nh_\alpha(a) \leqslant h_\alpha(1)$ for all positive integers n. Thus, we have, for each positive integer n and each $\alpha \in \Gamma$, $h_\alpha(1 - na) \geqslant 0$. But then $h_\alpha[(1 - na) \wedge 0] = 0$ for each $\alpha \in \Gamma$ and each positive integer n;

whence $(1-na)\wedge 0 \in \bigcap\{I_\alpha : \alpha \in \Gamma\} = \{0\}$, for each positive integer n. Since A is Archimedean, this means that $a = 0$, so $J(A) = \{0\}$.

COROLLARY 2.12 (Birkhoff–Pierce). *An Archimedean f-ring with identity contains no non-zero nilpotent elements* (Corollary 2.7).

Neither Theorem 2.11 nor its corollary can be strengthened to a statement concerning arbitrary Archimedean f-rings (without identity). For, an example of an Archimedean ordered ring A in which $ab = 0$ for all a, $b \in A$ is easily constructed.

3. *l*-representations and *l*-modules

As in the Jacobson theory, a fundamental role in the structure theory for f-rings is played by a representation theory. We will restrict our attention to l-representations in the sense of the following definition.

DEFINITION 3.1 (Birkhoff–Pierce). A homomorphism θ of an f-ring A onto an f-ring $\bar{A}$ is said to be an *l-representation* of A in the abelian lattice-ordered group G if the elements of $\bar{A}$ are (group) endomorphisms of G, and the operations and order in $\bar{A}$ are defined by:

i) for every $g \in G$ and $\bar{a}, \bar{b} \in \bar{A}$,

$$g(\bar{a}+\bar{b}) = g\bar{a} + g\bar{b} \quad \text{and} \quad g(\bar{a}\bar{b}) = (g\bar{a})\bar{b};$$

ii) $\bar{a} \in \bar{A}^+$ if and only if $g\bar{a} \in G^+$ for every $g \in G^+$.

The l-representation is said to be *faithful* if θ is an isomorphism.

Birkhoff and Pierce have shown ([3], p. 57, Corollary 3), using the familiar (right) regular representation, that every f-ring has a faithful l-representation. This section is devoted to the study of those f-rings that have faithful, irreducible, l-representations (Definition 3.7). These f-rings will be used to obtain an important characterization of the J-radical in Section 4.

It is often more convenient, in considering a given l-representation of an f-ring A in an abelian lattice-ordered group G, to focus attention on G and upon the interplay between the operations in A and those in G. In such cases, we view G as an A-l-module in the sense of the following definition.

DEFINITION 3.2. If A is an f-ring and G is an abelian lattice-ordered group, then G is said to be an *A-l-module* if a law of composition is defined on $G \times A$ into G which, for g_1, $g_2 \in G$ and a, $b \in A$, satisfies:

i) $(g_1+g_2)a = g_1a + g_2a$,

ii) $g_1(a+b) = g_1a + g_1b$,

iii) $g_1(ab) = (g_1a)b$, and

iv) $ga \in G^+$ for every $g \in G^+$ if and only if $ga^- = 0$ for every $g \in G$.

THEOREM 3.3. *There is a one-to-one correspondence $\theta \leftrightarrow G_\theta$ between the l-representations of an f-ring A and the A-l-modules, such that θ is an l-representation of A in G_θ and $g(a\theta) = ga$ for each $g \in G_\theta$.*

Proof. Suppose that $a \to \bar{a}$ is an l-representation of A in G. We may define a law of composition on $G \times A$ into G by setting $ga = g\bar{a}$. Now $g\bar{a} \in G^+$ for every $g \in G^+$ if and only if $\bar{a} \geqslant 0$, hence if and only if $\bar{a}^- = 0$. Since $a \to \bar{a}$ is a homomorphism, $\bar{a}^- = \overline{a^-}$, so we have $ga \in G^+$ for every $g \in G^+$ if and only if $ga^- = 0$ for every $g \in G$. Thus, since conditions i)–iii) of Definition 3.2 are clearly satisfied, this law of composition makes G into an A-l-module.

Conversely, if G is an A-l-module, then, for each $a \in A$, define the mapping $\bar{a}$ of G into itself by setting $g\bar{a} = ga$. By condition i) of Definition 3.2, $\bar{a}$ is a (group) endomorphism of G. Conditions ii) and iii) show that the mapping $a \to \bar{a}$ is a (ring) homomorphism of A onto the ring $\bar{A} = \{\bar{a} : a \in A\}$, where the operations in $\bar{A}$ are as in i) of Definition 3.1.

Now let I denote the kernel of this ring homomorphism: $I = \{a \in A : ga = 0$ for each $g \in G\}$. Then I is a (ring) ideal of A and, moreover, if $a \in I$, then $|a| \in I$, since $0 = ga \in G^+$ for every $g \in G^+$ implies $ga^- = 0$ for each $g \in G$, whence $g|a| = g(a + 2a^-) = 0$ for each $g \in G$. If $a \in I$ and $|b| \leqslant |a|$, then $|a| - |b| \in A^+$, so $g(|a| - |b|) = -g|b| \in G^+$ for every $g \in G^+$. But $g|b| \in G^+$ for each $g \in G^+$, so $g|b| = 0$ for every $g \in G^+$. Hence, if $g \in G$, then $g|b| = g^+|b| - g^-|b| = 0$, so $|b| \in I$. Finally, b^+, $b^- \in I$, and, since I is a ring ideal of A, we have shown that $b = b^+ - b^- \in I$ whenever $|b| \leqslant |a|$ and $a \in I$. Thus, I is an l-ideal in A.

Hence, we may use the ring homomorphism $a \to \bar{a}$ to transfer the lattice structure of A to $\bar{A}$. In this way, $\bar{A}$ becomes an f-ring isomorphic to A/I. The partial order on $\bar{A}$ is given by $\bar{a} \in \bar{A}^+$ if and only if $a^- \in I$, hence if and only if $ga^- = 0$ for every $g \in G$. By condition iv) of Definition 3.2, we have $\bar{a} \in \bar{A}^+$ if and only if $g\bar{a} \in G^+$ for every $g \in G^+$. Thus, the partial ordering of A is as required by Definition 3.1: $a \to \bar{a}$ is an l-representation of A in G.

Thus, to every A-l-module G corresponds an l-representation of A in G (the l-representation "associated" with the A-l-module G), and conversely. We will use the notions of l-module and l-representation interchangeably, as convenience dictates.

We say that an A-l-module is faithful if and only if the associated l-representation is faithful. Note that an A-l-module G is faithful if and only if $ga = 0$ for all $g \in G$ implies $a = 0$.

LEMMA 3.4. *If G is an A-l-module, then $|ga| \leqslant |g|\,|a|$ for every $g \in G$ and $a \in A$. Conversely, if $\bar{A}$ is an f-ring of (group) endomorphisms of an abelian lattice-ordered group G which satisfies $|g\bar{a}| = |g|\,|\bar{a}|$ for every $g \in G$ and every $\bar{a} \in \bar{A}$, then $\bar{a} \in \bar{A}^+$ if and only if $g\bar{a} \in G^+$ for every $g \in G^+$.*

Proof. First, if G is an A-l-module, $g \in G$, and $a \in A$, then

$$|ga| = |(g^+ - g^-)(a^+ - a^-)| \leqslant |g^+a^+| + |g^+a^-| + |g^-a^+| + |g^-a^-|$$
$$= g^+a^+ + g^+a^- + g^-a^+ + g^-a^- = (g^+ + g^-)(a^+ + a^-) = |g|\,|a|.$$

If $\bar{A}$ is an f-ring of (group)endomorphisms of the abelian lattice-ordered group G in which $|g\bar{a}| = |g|\,|\bar{a}|$ for every $g \in G$ and every $\bar{a} \in \bar{A}$, then $\bar{a} \in \bar{A}^+$ if and only if $\bar{a} = |\bar{a}|$, whence if and only if $g\bar{a} = |g|\,|\bar{a}| = |g\bar{a}| \in G^+$ for every $g \in G^+$.

We may re-state the second part of Lemma 3.4 as follows: If a law of composition is defined on $G \times A$ into G which satisfies conditions i)–iii) of Definition 3.2 and $|ga| = |g|\,|a|$ for every $g \in G$ and $a \in A$, then G is an A-l-module. The following example shows that this condition is not satisfied by every A-l-module.

Example 3.5. Let G be the lattice-ordered group consisting of all ordered pairs (m, n) of integers, with addition and lattice operations defined coordinatewise:

$(m, n) + (m', n') = (m + m', n + n')$

$(m, n) \in G^+$ if and only if $m \geqslant 0$ and $n \geqslant 0$.

Let A denote the ring of integers with the usual order.

Define a law of composition on $G \times A$ into G by setting $(m, n)p = (0, p(m + n))$. It is easily seen that this makes G into an A-l-module. However, the condition above is not satisfied: $|(1, -1)2| = |(0, 2-2)| = (0,0) \neq (0, 4) = |(1, -1)|\,|2|$.

Definition 3.6. If G is an A-l-module, then H is said to be an *A-l-submodule* of G if:

i) H is an l-subgroup of the lattice-ordered group G;

ii) for every $h \in H$ and every $a \in A$, $ha \in H$.

Definition 3.7. An A-l-module G is said to be *irreducible* (and the associated l-representation of A is also said to be irreducible) if:

i) the only A-l-submodules of G are $\{0\}$ and G;

ii) there is a non-zero element $e \in G^+$ with $eA = G$.

Theorem 3.8. *If an f-ring A has a faithful, irreducible, l-representation, then A is a prime f-ring.*

Proof. Suppose G is a faithful, irreducible, A-l-module. Let I_1, I_2 be non-zero l-ideals in A. Then, since G is faithful, there are elements $g_1, g_2 \in G$ such that $g_1 I_1 \neq \{0\}$ and $g_2 I_2 \neq \{0\}$. The l-subgroup $\langle g_1 I_1 \rangle = \{g \in G\colon |g| \leqslant |g_1 a|,\ a \in I_1\}$ is clearly an A-l-submodule of G. Since G is irreducible and $g_1 I_1 \neq \{0\}$, we have $\langle g_1 I_1 \rangle = G$.

Choose $b \in I_2$ such that $g_2 b \neq 0$. Since $g_2 \in \langle g_1 I_1 \rangle$, there is an element $a \in I_1$ such that $|g_2| \leqslant |g_1 a|$. Then $0 \neq |g_2 b| \leqslant |g_2|\,|b| \leqslant |g_1 a|\,|b| \leqslant |g_1|\,|a|\,|b| = |g_1|\,|ab|$; so $0 \neq ab \in I_1 I_2$. Thus, A is a prime f-ring.

COROLLARY 3.9. *If an f-ring A has a faithful, irreducible, l-representation, then A is an ordered ring without non-zero divisors of zero* (Theorem I.4.4).

PROPOSITION 3.10. *If G is a faithful, irreducible, A-l-module, and $e \in G^+$ satisfies $eA = G$, then the mapping $a \to ea$ is a homomorphism of the ordered additive group of A onto G. The kernel of this homomorphism, $I_e = \{a \in A: ea = 0\}$, is a maximal modular right l-ideal in A.*

Proof. Note that, if $a \in A^+$, then $ea \in G^+$, since $e \in G^+$. Now, since A is totally ordered, $a \wedge b = 0$ implies either $a = 0$ or $b = 0$. Hence, $a \wedge b = 0$ implies $ea \wedge eb = 0$. Thus, since the mapping $a \to ea$ is clearly a group homomorphism onto G, $a \to ea$ is, by Proposition I.1.4, a homomorphism of the ordered additive group of A onto G.

Then, by Theorem I.1.6, the kernel I_e of this homomorphism is an l-subgroup of the ordered additive group of A which is clearly a right (ring) ideal of A. It is easily seen that the irreducibility of G forces I_e to be a maximal right l-ideal of A.

Since $eA = G$, there is a $u \in A$ such that $eu = e$. If $a \in A$ then $e(a - ua) = ea - eua = 0$, so u is a left identity modulo I_e.

COROLLARY 3.11. *If G is a faithful, irreducible, A-l-module, then G is an ordered group, and $|ga| = |g|\,|a|$ for every $g \in G$ and every $a \in A$.*

Proof. G is an ordered group, since it is a homomorphic image of the ordered additive group of A. The second statement now follows. For, either $|g|\,|a| = ga = |ga|$ or $|g|\,|a| = -ga = |ga|$, since A and G are both totally ordered.

LEMMA 3.12. *If G is a faithful, irreducible, A-l-module and if $0 \neq g_1 \in G^+$, then $I_{g_1} = \{a \in A : g_1 a = 0\} = I_e$.*

Proof. It is clear that I_{g_1} is a right (ring) ideal in A. Also, if $a \in I_{g_1}$ and $|b| \leqslant |a|$, then $0 \leqslant |g_1 b| = |g_1|\,|b| \leqslant |g_1|\,|a| = |g_1 a| = 0$. Thus I_{g_1} is a right l-ideal in A. By Proposition 3.10, I_e is a maximal right l-ideal in the ordered ring A. Hence, $I_{g_1} \subseteq I_e$, since the right l-ideals of A form a chain.

If $I_{g_1} \neq I_e$, then $g_1 I_e \neq \{0\}$. It is clear that the l-subgroup $\langle g_1 I_e \rangle = \{g \in G: |g| \leqslant |g_1 a|, a \in I_e\}$ is an A-l-submodule of G. Since $g_1 I_e \neq \{0\}$, we have $\langle g_1 I_e \rangle = G$, by irreducibility. Let $u \in A^+$ be a left identity modulo I_e (Proposition 3.10). Then $u \notin I_e$ and $u \geqslant a$ for every $a \in I_e$, since A is an ordered ring. Since $\langle g_1 I_e \rangle = G$, there is an $a \in I_e$ such that $g_1 a \geqslant g_1 u \neq 0$. Then $g_1(a - u) \geqslant 0$, and this means that $g_1(a - u) = 0$, since $a \leqslant u$. Thus, $a - u \in I_{g_1} \subseteq I_e$. Since $a \in I_e$, this implies that $u \in I_e$, a contradiction.

THEOREM 3.13. *If an f-ring A has a faithful, irreducible, l-representation, then A contains no non-zero proper right l-ideals.*

Proof. Let G be a faithful, irreducible, A-l-module. Now, by Corollary 3.9 and Proposition 3.10, A is an ordered ring containing the (unique) maximal modular right l-ideal I_e. By Lemma 3.12, if $a \in I_e$, then $g_1 a = 0$ for every $g_1 \in G^+$. But then $ga = 0$ for every $g \in G$. Since G is a faithful A-l-module, this means $a = 0$. Thus $\{0\}$ is a maximal right l-ideal in A.

COROLLARY 3.14. *If an f-ring A has a faithful, irreducible, l-representation, then A has an identity.*

Proof. By Proposition 3.10 and Theorem 3.13, $\{0\}$ is a maximal modular right l-ideal in A. Thus, there is an $e \in A$ such that $x - ex \in \{0\}$ for every $x \in A$; that is, e is a left identity in A. By Corollary 3.9, A contains no non-zero divisors of zero. Thus e is a left identity in a ring without non-zero divisors of zero, so e is, in fact, a two-sided identity in A.

COROLLARY 3.15. *If an f-ring A has a faithful, irreducible, l-representation, then A is a J-semisimple ordered ring with identity.*

4. l-primitivity

We have seen (Corollary 3.15) that every f-ring that has a faithful, irreducible, l-representation is a J-semisimple ordered ring with identity. In this section we will see, conversely, that these are the only J-semisimple ordered rings and that they are the components in a subdirect union representation for J-semisimple f-rings.

DEFINITION 4.1. If I is any right l-ideal in an f-ring A, then the *quotient l-ideal* $(I : A)$ is the set $\{a \in A : Aa \subseteq I\}$.

LEMMA 4.2. *Let A be an f-ring, I a right l-ideal in A. Then:*

i) *$(I : A)$ is an l-ideal in A.*

ii) *If I is modular, then $(I : A)$ is the largest l-ideal of A contained in I.*

Proof. It is well known ([13], p. 7), and also easily seen, that $(I : A)$ is a (ring) ideal in A and that if I is modular, then $(I : A)$ is the largest (ring) ideal of A contained in I. Moreover, $(I : A)$ is an l-ideal. For, if $a \in (I : A)$, $|b| \leqslant |a|$, and $x \in A$, then $|xb| = |x|\,|b| \leqslant |x|\,|a| = |xa|$, whence $xb \in I$, since $xa \in I$. Thus, $b \in (I : A)$.

DEFINITION 4.3. An f-ring A is said to be *l-primitive* in case there is a maximal modular right l-ideal M of A with $(M : A) = \{0\}$. An l-ideal P of A is an *l-primitive l-ideal* in case A/P is an l-primitive f-ring.

It is clear that an l-ideal P of A is l-primitive if and only if there is a maximal modular right l-ideal M of A with $P = (M : A)$.

Definition 4.3 is the exact analogue of the definition of primitive ring and primitive ideal in the Jacobson theory. Now, every primitive commutative (abstract) ring is a field ([13], p. 7, Theorem 1), so every l-ideal in a commutative f-ring that is also a primitive (ring) ideal is l-primitive. However, the following examples show that, in general, the notions of primitivity and l-primitivity do not coincide.

Example 4.4. Let A denote the ring of integers with the usual order. Then A is a commutative, l-simple, ordered ring with identity, whence it is easily seen to be l-primitive. However, A is not a primitive ring, since it is not a field.

Example 4.5. Let F be an ordered field, σ an (order-preserving) automorphism of F such that no power of σ is the identity automorphism. (For an example of such F, σ, cf. [1], p. 46.) If $a \in F$, then we will denote the image of a under σ by a^σ. Let $F[\lambda, \sigma]$ denote the set of polynomials in one indeterminate λ over F. Addition in $F[\lambda, \sigma]$ is defined as usual; the product of two elements in $F[\lambda, \sigma]$ is obtained by first multiplying, formally, termwise

$$(\sum_i a_i \lambda^i)(\sum_j b_j \lambda^j) = \sum_{i,j} a_i \lambda^i b_j \lambda^j,$$

and then simplifying by means of the rule $\lambda a = a^\sigma \lambda$, $a \in F$.

Then (cf. [13], p. 22, Example 3), $F[\lambda, \sigma]$ is a primitive ring in which the only (two-sided) ideals are the principal ideals (λ^n). Now, $F[\lambda, \sigma]$ can be made into an ordered ring by ordering lexicographically: $a_0 + a_1\lambda + \cdots + a_n\lambda^n \geqslant 0$ if and only if $a_0 > 0$, or $a_0 = 0$ and $a_1 > 0$, or ..., or $a_0 = a_1 = \cdots = a_{n-1} = 0$ and $a_n \geqslant 0$. The unique maximal modular l-ideal of $F[\lambda, \sigma]$ is $\langle\lambda\rangle$. Hence $\{0\}$ is an l-ideal of $F[\lambda, \sigma]$ that is a primitive (ring) ideal, but is not l-primitive.

THEOREM 4.6. *An f-ring A is l-primitive if and only if A has a faithful, irreducible, l-representation.*

Proof. If A has a faithful, irreducible, l-representation, then we have already seen (Theorem 3.13 and Corollary 3.14) that $\{0\}$ is the (unique) maximal modular right l-ideal in A. Thus A is clearly an l-primitive f-ring.

Conversely, suppose A is an l-primitive f-ring. Then A contains a maximal modular right l-ideal M with $(M : A) = \{0\}$. Let G denote the additive abelian lattice-ordered group $A - M$. Define a law of composition on $G \times A$ into G by $(x + M)a = xa + M$ for each $x + M \in G$ and each $a \in A$.

Note that the law of composition that we have defined satisfies $|x + M|\,|a| = (|x| + M)|a| = |xa| + M = |(x + M)a|$ for every $x + M \in G$ and every $a \in A$. Since conditions i)–iii) of Definition 3.2 are obviously satisfied, G is an A-l-module by the remark following Lemma 3.4. Moreover, G is a faithful A-l-module. For, if $(x+M)a=0$ for every $x+M\in G$, then $xa\in M$ for every $x\in A$, whence $a\in(M:A)=\{0\}$.

It remains to show that G is an irreducible A-l-module (Definition 3.7). If $e\in A^+$ is a left identity modulo M, then $e+M$ is a non-zero positive element of G such that, if $x\in A$, then $(e+M)x=ex+M=x+M$. Thus, $(e+M)A=G$. If H is any l-subgroup of G, then H is of the form $M'-M$, where M' is an l-subgroup of the lattice-ordered additive group of A that contains M (Theorem I.1.7). If, moreover, H is an A-l-submodule of G, then M' is clearly a right l-ideal of A, whence $M'=M$ or $M'=A$, since M is a maximal right l-ideal. But this means that $H=\{0\}$ or $H=G$, so G contains no non-zero proper A-l-submodules. Thus, G is a faithful, irreducible, A-l-module.

Corollary 4.7 *In an f-ring A, every maximal modular right l-ideal is a two-sided l-ideal. Hence, $J(A)$ is a two-sided l-ideal of A.*

Proof. Let M be a maximal modular right l-ideal in A. Then $(M:A)=P$ is the largest l-ideal of A contained in M, and A/P is l-primitive. But A/P has a faithful, irreducible, l-representation, so A/P contains no non-zero proper right l-ideals by Theorem 3.13. Since M/P is a proper right l-ideal in A/P, we have $M/P=\{0\}$; that is, $M=P$.

Corollary 4.8. *An f-ring A is l-primitive if and only if it is an l-simple ordered ring with identity* (Corollary 3.14).

Thus, every l-primitive l-ideal is maximal. It is not true that every maximal l-ideal is l-primitive (for example, the l-ideal $\{0\}$ in the ring of even integers). However, if A has an identity, then every maximal l-ideal is modular, and we have:

Corollary 4.9 *If A is an f-ring with identity, then an l-ideal of A is l-primitive if and only if it is maximal.*

Corollary 4.10. *An Archimedean f-ring is l-primitive if and only if it is isomorphic to a subring of the ordered field of real numbers that contains the identity* (Theorem I.4.2).

Corollary 4.11. *An ordered ring is J-semisimple if and only if it is l-primitive.*

Proof. If A is an ordered ring, then there is at most one maximal modular right l-ideal M in A, since the right l-ideals of A form a chain (Proposition I.2.9). If M exists, then it must coincide with $J(A)$, so $J(A)=\{0\}$ if and only if $\{0\}$ is a maximal modular right l-ideal in A, hence if and only if A is l-primitive.

Since, by Corollary 4.7, every maximal modular right l-ideal is a two-sided l-ideal (hence an l-primitive l-ideal), $J(A)$ is the intersection of l-primitive l-ideals. This leads to the main structure theorem for J-semisimple f-rings:

THEOREM 4.12. *An f-ring A is J-semisimple if and only if A is isomorphic to a subdirect union of l-simple ordered rings with identity.*

Proof. A is J-semisimple if and only if the family $\{P_\alpha : \alpha \in \Gamma\}$ of l-primitive l-ideals of A satisfies $\bigcap \{P_\alpha : \alpha \in \Gamma\} = \{0\}$, hence if and only if A is isomorphic to a subdirect union of the l-primitive f-rings A/P_α, by Theorem I.3.7. By Corollary 4.8, each of the A/P_α has the desired form.

We now know (Corollary 4.7) that the J-radical of an f-ring is a two-sided l-ideal. With this, we can now complete the description of $J(A)$ that was begun in Section 2.

We may define the concepts "left J-radical", "left l-primitivity", and "left l-quasi-regular" in a manner similar to that above. Corollary 4.7 and its analogue in this "left theory" then show that the J-radical and the left J-radical coincide. These facts are summarized in the following:

THEOREM 4.13. *Let A be an f-ring. The J-radical $J(A)$ is an l-QR l-ideal, and*

i) *$J(A)$ is the intersection of all maximal modular left l-ideals of A (and each of these is actually a two-sided l-ideal);*

ii) *$J(A)$ is the join of the l-QR left l-ideals of A.*

It is interesting to note that the analogue of another of the open questions in the Jacobson theory is readily solved here: l-primitivity and left l-primitivity do coincide for f-rings.

COROLLARY 4.14. *Every l-QR right l-ideal in an f-ring A consists entirely of left l-QR elements of A.*

Theorem 4.12 provides a description of J-semisimple f-rings. For f-rings that are not J-semisimple, we have:

THEOREM 4.15. *If A is an f-ring that is not a J-radical ring, then $A/J(A)$ is J-semisimple.*

Proof. Let $\mathfrak{P}$ denote the collection of maximal modular right l-ideals of A. If $P \in \mathfrak{P}$, then $P \subseteq J(A)$, so $P/J(A)$ is clearly a maximal modular right l-ideal of $A/J(A)$. Since $\bigcap\{P : P \in \mathfrak{P}\} = J(A)$, we have $J(A/J(A)) \subseteq \bigcap\{P/J(A) : P \in \mathfrak{P}\} = \{0\}$. Thus, if $J(A) \neq A$, then $A/J(A)$ is J-semisimple.

The following theorem is an analogue of a result in the Jacobson theory ([13], p. 10, Theorem 1). However, it occurs here in much stronger form: the analogous result for abstract rings is true only for two-sided ideals.

THEOREM 4.16. *If I is any right l-ideal in an f-ring A, then $J(I) = J(A) \cap I$.*

Proof. Let B denote the right l-ideal of I generated by $J(I)\cdot I$. Clearly, B is a right l-ideal in A. Moreover, $B \subseteq J(I)$, so each of its elements is right l-QR. Hence, $B \subseteq J(A)$, so $a \in J(I)$ implies $a^2 \in J(A)$. But $A/J(A)$ is $\{0\}$ or J-semisimple, hence it contains no non-zero nilpotent elements (Corollary 2.7), so $a \in J(I)$ implies $a \in J(A)$.

Conversely, $J(A) \cap I$ is an l-QR right l-ideal in A, so it is an l-QR right l-ideal in I by Proposition 2.4. Hence, $J(A) \cap I \subseteq J(I)$.

5. *f*-rings with descending chain condition for *l*-ideals

In this section, we will be concerned primarily with f-rings that satisfy the descending chain condition for l-ideals (Definition I.3.20). As in the structure theory for abstract rings, these f-rings exhibit several special properties.

If A is an ordered ring that satisfies the descending chain condition for l-ideals, then A must contain a smallest non-zero l-ideal, since the l-ideals of A form a chain. Hence, by Corollary I.3.10, A is subdirectly irreducible. Thus, we are led first to consider subdirectly irreducible f-rings.

LEMMA 5.1. *If A is a subdirectly irreducible f-ring with zero l-radical, then A is l-simple.*

Proof. By definition, $A \neq \{0\}$. Now, A is an ordered ring (Corollary I.3.12), and A contains no non-zero nilpotent elements, so A contains no non-zero divisors of zero, by Theorem I.4.1. Thus, $A^2 \neq \{0\}$.

Suppose A is not l-simple. Then, since A is subdirectly irreducible, A contains a smallest non-zero l-ideal I. We show first that, if b, c are any two non-zero positive elements of I, then $bc < b \wedge c$. For suppose that $bc \geqslant b$. Then, if x is any positive element of A not in I, we have $bcx \geqslant bx$, whence $b(cx - x) \geqslant 0$. Since $b > 0$ and A contains no non-zero divisors of zero, $cx \geqslant x$, whence $x \in I$. This contradiction shows that $bc < b$. Similarly, $bc < c$, so $bc < b \wedge c$.

Now let $a > 0$ be a fixed element in I, and set $J = \{d \in A: |d| \leqslant bac \text{ for some } b, c \in I\}$. Then J is an l-ideal in A: if d, $d' \in J$, then there are elements $b, b', cc' \in I$ with $|d| \leqslant bac$ and $|d'| \leqslant b'ac'$. Since A is ordered, we may assume that $bac \geqslant b'ac'$. Then, $|d - d'| \leqslant |d| + |d'| \leqslant (2b)ac$, and $|dx| = |d|\,|x| \leqslant ba(c|x|)$ for all $x \in A$, whence $(d - d')$, dx, and (similarly) xd are in J. Since $a \neq 0$ and $N(A) = \{0\}$, J is a non-zero l-ideal of A, so $J \supseteq I$, since I is the smallest non-zero l-ideal of A. Then $a \in J$, so there are positive elements b, $c \in I$ with $a \leqslant bac$. But, by the above, $bac < ba \wedge c < b \wedge a \wedge c \leqslant a$. This contradiction yields the lemma.

The following theorem characterizes those f-rings that are subdirectly irreducible. This is another instance in which the structure theory for f-rings differs markedly from that for abstract rings. The analogous problem is much more difficult for commutative abstract rings (cf. [8] and [17]), and very little has been done in the non-commutative case.

THEOREM 5.2. *An f-ring A is subdirectly irreducible if and only if either*

i) *A is l-simple; or*

ii) *A is an ordered ring, and there is a non-zero element a in A^+ such that $a^2 = 0$ and, for any non-zero element b in A such that $b^2 = 0$, we have $\langle b\rangle \supseteq \langle a\rangle$. In this case, $I = \langle a\rangle$ is the smallest non-zero l-ideal in A.*

Proof. Suppose first that A is a subdirectly irreducible f-ring. Then, by Corollary I.3.12, A is an ordered ring. If A is not l-simple, then $N(A) \neq \{0\}$, by Lemma 5.1. In particular, $Z_2(A) = \{a \in A : a^2 = 0\} \neq \{0\}$. Now, A may fail to be l-simple in either of two ways. In the first case, $A = Z_2(A)$, and A contains no non-zero proper l-ideals. It is clear that, in this case, condition ii) is satisfied, with $I = \langle a\rangle = A$ for any non-zero element a of A. In the second case, A contains proper non-zero l-ideals, so it contains a smallest such l-ideal I (Corollary I.3.10). Since I is contained in every non-zero l-ideal of A, it is contained in $Z_2(A)$, and if a is any non-zero element in I, then $I = \langle a\rangle$. Hence, condition ii) holds in this case.

Conversely, every l-simple f-ring is trivially subdirectly irreducible. Suppose A satisfies condition ii), and let B be any non-zero l-ideal in A. If b is any non-zero element in B, then either $|b| > a$ or $a \geqslant |b|$. If $a \geqslant |b|$, then $b^2 = |b|^2 \leqslant |a|^2 = 0$, so $B \supseteq \langle b\rangle \supseteq \langle a\rangle = I$, by condition ii). If $|b| > a$, then, clearly, $B \supseteq I$. Thus, I is contained in every non-zero l-ideal in A, so A is subdirectly irreducible.

For the remainder of this section, we will be concerned only with f-rings that satisfy the descending chain condition for l-ideals. As remarked earlier, every ordered ring with descending chain condition is subdirectly irreducible. Hence, by Lemma 5.1, we obtain:

COROLLARY 5.3 (Birkhoff–Pierce). *An ordered ring $A \neq \{0\}$, with zero l-radical, that satisfies the descending chain condition for l-ideals is l-simple.*

The proof of this result given by Birkhoff and Pierce actually proves the stronger statement in Lemma 5.1. It is their proof that we have used above.

If A is an f-ring that satisfies the descending chain condition for l-ideals, and if P is a prime l-ideal in A, then, by Theorem I.4.4, A/P is an ordered ring without non-zero divisors of zero that clearly satisfies the descending chain condition for l-ideals. By Corollary 5.3, A/P is an l-simple f-ring, so we have:

COROLLARY 5.4. *In an f-ring satisfying the descending chain condition for l-ideals, every prime l-ideal is a maximal l-ideal.*

In the Jacobson theory for abstract rings, the radical of any ring which satisfies the descending chain condition for right ideals is a nilpotent ideal. The analogue of this result does not hold for the J-radical, for we have seen, in Example 1.6, a commutative J-radical ring that is l-simple (and hence contains no non-zero nilpotents). However, we do have the result in question for f-rings with identity.

THEOREM 5.5. *If A is an f-ring with identity that satisfies the descending chain condition for l-ideals, then $J(A)$ is nilpotent (hence $J(A) = N(A)$).*

Proof. We show that $J(A) = N(A)$; the theorem then follows by Theorem I.3.21. By Corollary 5.4, every prime l-ideal of A is a maximal l-ideal. Since A has an identity, every maximal l-ideal is l-primitive by Corollary 4.9. Hence $N(A)$, the intersection of the prime l-ideals of A (Theorem I.4.8), is the intersection of the l-primitive l-ideals of A; that is, $N(A) = J(A)$.

COROLLARY 5.6. *Let A be an ordered ring that satisfies the descending chain condition for l-ideals.*

i) *Every proper l-ideal in A is nilpotent.*

ii) *If A has an identity, then every proper one-sided l-ideal in A is nilpotent.*

Proof. By Theorem I.3.21, $N(A)$ is nilpotent. Thus, if $N(A) = A$, then A is nilpotent. If $N(A) \neq A$, then $N(A)$ is a prime l-ideal of A, by Corollary I.4.5. So $N(A)$ is the unique maximal l-ideal in the ordered ring A, by Corollary 5.4. Thus, i) follows in either case.

If A has an identity, then $J(A)$ is the unique maximal right (left) l-ideal of A. Hence, ii) follows from Theorem 5.5.

We close this section with a decomposition theorem for J-semisimple f-rings satisfying the descending chain condition for l-ideals, the analogue of the classical Wedderburn–Artin theorem. The proof given here is a simplification of the proof in [3].

LEMMA 5.7 (Birkhoff–Pierce). *If $A \neq \{0\}$ is an f-ring that satisfies the descending chain condition for l-ideals, then A has zero l-radical if and only if A is isomorphic to a (finite) direct union of l-simple ordered rings.*

Proof. If A has zero l-radical, then it contains a collection $\{P_\alpha\}$ of prime l-ideals with $\cap P_\alpha = \{0\}$. Since A satisfies the descending chain condition for l-ideals, we may choose a finite number of these, say $P_1, \ldots, P_n$, with $\bigcap_{i=1}^{n} P_i = \{0\}$. We may also assume that this finite collection is minimal, in the sense that no proper sub-collection has intersection $\{0\}$. By Corollary 5.4, each P_i is a maximal l-ideal, hence

$$(P_1 \cap \ldots \cap P_{j-1}) + P_j = A \text{ for } j = 2, 3, \ldots, n.$$

Thus, by [2] (p. 87, Theorem 4), A is isomorphic to the direct union of the l-simple ordered rings A/P_i, $i = 1, 2, \ldots, n$.

The converse is obvious.

If, in the proof above, we replace the word "prime" by the word "l-primitive" each time it occurs, we obtain a proof of the following theorem, where this time the converse is immediate by Theorem 4.12.

THEOREM 5.8. *If A is an f-ring that satisfies the descending chain condition for l-ideals, then A is J-semisimple if and only if A is isomorphic to a (finite) direct union of l-simple ordered rings with identity.*

COROLLARY 5.9. *If A is a J-semisimple f-ring that satisfies the descending chain condition for l-ideals, then A has an identity.*

COROLLARY 5.10. *If A is an Archimedean f-ring that satisfies the descending chain condition for l-ideals, then the following statements are equivalent.*

i) *A is J-semisimple.*

ii) *A contains an identity.*

iii) *A is isomorphic to a (finite) direct union of subrings of the ordered field of real numbers, each of which contains the identity.*

Proof. Now, i) and ii) are equivalent by Corollary 5.9 and Theorem 2.11; and iii) implies ii) trivially. By Theorem 5.8, if A is J-semisimple, then A is isomorphic to a finite direct union of l-simple ordered rings $A_1, \ldots, A_n$, each of which contains an identity. Thus, A contains an isomorphic copy of each A_i, and, since A is Archimedean, each A_i must be Archimedean also. By Theorem I.4.2, the A_i have the required form.

It can be seen very easily that the decomposition given by Theorem 5.8 for a J-semisimple f-ring A that satisfies the descending chain condition for l-ideals is unique in the following sense: if $A = \sum_{i=1}^{n} A_i$ and $A = \sum_{i=1}^{m} B_i$ are two such decompositions, then $m = n$ and, for some arrangement of $B_1, \ldots, B_m$, A_i and B_i are isomorphic, for $i = 1, \ldots, n$. The analogous statement for abstract rings ([13], p. 42, Theorem 1) depends only upon the fact that $A^2 = A$ (Corollary 5.9) and upon the notion of indecomposability, which has the same meaning in f-rings, so the argument need not be repeated here.

6. Remarks

In this section, several asides to the main stream of this chapter are considered. They consist mostly of questions that arose naturally during the course of this study.

1. It was remarked in Section 3 that an l-primitive f-ring is, in a certain sense, the f-ring analogue of a division ring: the l-primitive f-rings are precisely those f-rings with identity that contain no non-zero proper right (left) l-ideals. Since an l-primitive f-ring is an ordered ring without non-zero divisors of zero, it nevertheless seems natural to ask whether an l-primitive f-ring can always be imbedded in an (ordered) division ring. The following example gives a negative answer to this question.

Example 6.1. Chehata ([7]) and Vinogradov ([22]) have independently given the same example of an ordered cancellative semigroup S with least element which cannot be imbedded in a group. The semigroup S is the famous example of Malcev ([16]): it consists of a countable number of elements which are discretely ordered; thus we may denote the elements of A by $x_1, x_2, \ldots$, where $i < j$ implies $x_i < x_j$. We may adjoin an identity element to S as follows: let $S' = S \cup \{x_0\}$, where, for $i = 1, 2, \ldots$, $x_i x_0 = x_0 x_i = x_i$ and $x_0 < x_i$, and $x_0^2 = x_0$.

Now let A denote the semigroup ring of S' over the (ordered) field of rational numbers: A consists of all finite sums $\sum_{i=0}^{n} a_i x_i$, where each a_i is a rational number. Addition in A is defined coordinatewise; the rule for multiplication is given by

$$\left(\sum_{i=0}^{n} a_i x_i\right)\left(\sum_{j=0}^{m} b_j x_j\right) = \sum_{k} \Big(\sum_{x_i x_j = x_k} a_i b_j \Big) x_k.$$

Order A lexicographically: $\sum_{i=0}^{n} a_i x_i > 0$, where $a_n \neq 0$, if and only if $a_n > 0$. Clearly, this makes A into an ordered ring with identity x_0. Every positive element of A that is less than x_0 in this order is of the form $a x_0$, $0 \leqslant a < 1$, so A is l-simple. Hence, A is an l-primitive f-ring. A cannot be imbedded in a division ring. For, if it were, then the semigroup S, a sub-semigroup of the multiplicative semigroup of A, would be imbedded in a group, contrary to the result stated above.

2. By Corollary 4.7, every maximal modular right l-ideal in any f-ring is a two-sided ideal. It seems natural to ask if there exist f-rings that contain right l-ideals that are not two-sided. The following example answers this question affirmatively.

Example 6.2. Let A be the algebra over an ordered field F generated by two elements e, z, where $e^2 = e$, $ez = z$, and $ze = z^2 = 0$. The elements of A are of the form $ae + bz$, where $a, b \in F$. Order A lexicographically with e dominating z: $ae + bz \geqslant 0$ if and only if $a > 0$ or $a = 0$ and $b \geqslant 0$. It is easily seen that, under this order, A is an ordered ring.

Now let F be a non-Archimedean field, Φ a non-zero proper l-subgroup of the ordered additive group of F. Then $\Phi z = \{bz: b \in \Phi\}$ is a right l-ideal in A, since $\Phi z \cdot A = \{0\}$. But Φz is not two-sided: $A \cdot \Phi z = Fz \nsubseteq \Phi z$.

In this example, the existence of a right l-ideal that is not two-sided is intimately connected with the presence of a left annihilator of the ring. This leads us to ask if there exist ordered rings without non-zero divisors of zero which contain one-sided l-ideals that are not two-sided. The affirmative answer to this question will be presented in Chapter IV.

3. In the Jacobson theory for abstract rings, the analogue of our definition of right l-quasi-regularity (Definition 2.1) has an equivalent formulation in terms of the circle operation ([13], p. 7). This use of the circle operation yields a more conveniently applied condition on the elements of the radical: the existence of a right quasi-inverse. The question considered here is the following: does there exist an f-ring analogue of the existence of a right quasi-inverse that is equivalent to right l-quasi-regularity?

If A is an f-ring with identity, and if $a \in A$ is right l-QR, then $1 \in \langle (1-a)A \rangle_r = A$. Hence, there are elements $x_1, \ldots, x_n$ in A such that $1 \leqslant \sum_{i=1}^{n} |x_i - ax_i| = |1-a| \sum_{i=1}^{n} |x_i|$. Then $1 = 1^2 \leqslant [(1-a) \sum_{i=1}^{n} |x_i|]^2 = (1-a)y$, where $y = \sum_{i=1}^{n} |x_i| (1-a) \sum_{i=1}^{n} |x_i|$. If we write $y = 1 - x$, then $1 \leqslant (1-a)(1-x)$, so $a + x - ax \leqslant 0$. Conversely, if $a + x - ax \leqslant 0$, then, with $y = 1 - x$, it is easily seen that $1 \leqslant y - ay$, whence $\langle (1-a)A \rangle_r = A$.

Thus, the proper f-ring analogue of the existence of a right quasi-inverse of an element a seems to be the existence of an element x such that $a \circ x = a + x - ax \leqslant 0$. We have thus proved the following result:

PROPOSITION 6.3. *If A is an f-ring with identity, then $a \in A$ is right l-QR if and only if there exists an $x \in A$ such that $a \circ x \leqslant 0$.*

We also have the following result, the proof of which is omitted:

PROPOSITION 6.4. *If A is an ordered ring, then $a \in A$ is right l-QR if and only if there is an $x \in A$ such that $a \circ x \leqslant 0$.*

It is clear that we may, by applying Proposition 6.4 in each coordinate, extend this result to every complete direct union and every discrete direct union of ordered rings (Definitions I.3.5 and I.3.8). However, the status of this question for arbitrary subdirect unions of ordered rings is not known.

CHAPTER III. IMBEDDING AN f-RING IN AN f-RING WITH IDENTITY

The problem that gives rise to this chapter is the following: given an f-ring A without identity, is it possible to imbed A as a sub-f-ring of an f-ring A^* with identity? That is, we hope to find an f-ring A^* with identity and an isomorphism of A into A^*.

It is more natural to ask if a given f-ring A can be imbedded as a right l-ideal of an f-ring A^* with identity. It will be seen that both questions have, in general, a negative answer. Hence, the central problem of the chapter becomes the characterization of those f-rings that can be imbedded as right l-ideals (respectively, sub-f-rings) of f-rings with identity.

1. Preliminary concepts

In this section, we present the notation and terminology that is to be used throughout this chapter. The results stated in this section are due to Brown and McCoy ([6]) and are valid for arbitrary rings. *In this section only, the terms ring, homomorphism, ideal, imbedding, etc. will have their usual (ring theoretic) meaning.*

Let A be any ring. If a fixed element $a \in A$ satisfies $ab = ba = nb$ for some fixed integer n and all $b \in A$, then a is said to be an *n-fier* of A, and n is said to *have an n-fier* a in A. Note that the zero of A is always a 0-fier of A, and that there is a 1-fier e of A if and only if e is the identity element in A.

The set K of all integers n which have n-fiers in A is an ideal in the ring of integers. The ideal K is called the *modal ideal* of A; its non-negative generator k is called the *mode* of A.

It is well known that every ring can be imbedded as an ideal in a ring with identity. The usual construction, which was apparently first published by Dorroh ([9]), follows.

Throughout this chapter, for any ring A, we will let A' denote the ring of ordered pairs (n, a), n an integer and $a \in A$, where

$$(n, a) = (m, b) \text{ if and only if } n = m \text{ and } a = b,$$

$$(n, a) + (m, b) = (n + m, a + b),$$

$$(n, a)(m, b) = (nm, ma + nb + ab).$$

Then A' is a ring with identity in which $\bar{A} = \{(0, a): a \in A\}$ is an ideal isomorphic with A. When no confusion will result, we will suppose the identification $a \leftrightarrow (0, a)$ has been made, so that A is considered as an ideal in A'.

If a is an n-fier of A, for $n \neq 0$, then the set $\varrho(n, a)$ of all integral multiples of $(n, -a) \in A'$ is a two-sided ideal in A'. Denote the ring $A'/\varrho(n, a)$ by $A(n, a)$, and let $(m, b) \to \overline{(m, b)}$ denote the natural homomorphism of A' onto $A(n, a)$. Then the correspondence $b \to \overline{(0, b)}$ from A onto the ideal of $A(n, a)$ consisting of all elements of the form $\overline{(0, b)}$ is an isomorphism.

As usual, if no confusion will result, we will assume that the identification $b \leftrightarrow \overline{(0, b)}$ has been made, so that A is considered as an ideal in $A(n, a)$.

Let $\mathfrak{S}$ denote the collection of rings consisting of A' and all $A(n, a)$, for $0 \neq n \in K$ and a an n-fier of A. Every member of $\mathfrak{S}$ is a ring with identity that contains A. A central result in [6] is the "completeness" of this set of extensions of A in the sense of the following:

THEOREM 1.1 (Brown–McCoy). *If B is any ring with identity that contains A, then there is a subring B_1 of B that contains A and a ring $S \in \mathfrak{S}$ such that B_1 is isomorphic with S under an isomorphism that leaves A elementwise fixed.*

The main results of this chapter are, in a certain sense, the analogues of this result (Theorems 2.5 and 4.4, Corollary 2.6).

2. The strong order

In this section, we will consider the special form that the results stated in Section 1 take for f-rings. Throughout the rest of this chapter, we will let A denote an f-ring without identity, k the mode of A, and x a k-fier of A. If $k \neq 0$, then x is unique:

LEMMA 2.1. *If A is an f-ring with mode $k > 0$, and if x is a k-fier of A, then:*

i) $x \geqslant 0$;

ii) *A contains no non-zero annihilator on either side;*

iii) *If a is an n-fier of A, then $a = tx$, where $t = n/k$.*

The simple proof of this lemma is omitted. For any f-ring A, we will let A_1 denote the ring $A'/\varrho(k, x)$, where, for $k \neq 0$, x is the (unique) k-fier of A, and $x = 0$ if $k = 0$. (Hence, A_1 is isomorphic to A' if $k = 0$.)

We will see that the ring extension A_1 of A can always be partially ordered by a partial ordering that extends the order on A and makes A_1 into a partially ordered ring. To define this partial order on A_1, we first consider the subset $P(A')$ of the ring extension A' defined by: $(n, a) \in P(A')$ if and only if either

i) $n = 0$ and $a \in A^+$, or

ii) $n \neq 0$ and $nd + da \in A^+$, $nd + ad \in A^+$ for all $d \in A^+$. Since, in A', $(n, a)(0, d) = (0, nd + ad)$ and $(0, d)(n, a) = (0, nd + da)$, condition ii) merely states that (n, a) $(0, d) \in P(A')$ and $(0, d)(n, a) \in P(A')$ for all $(0, d) \in P(A')$. The following example shows that the conditions "$nd + da \in A^+$ for all $d \in A^+$" and "$nd + ad \in A^+$ for all $d \in A^+$" are independent.

Example 2.2 (cf. Example II.6.2). Let A be the algebra over the ordered field Q of rational numbers generated by two elements e, z, where $e^2 = e$, $z^2 = ze = 0$, and $ez = z$. Order A lexicographically with e dominating: $ae + bz \geqslant 0$ if and only if $a > 0$ or $a = 0$ and

$b \geqslant 0$. In A', consider the element $(1, -e-z)$. For any element $ae+bz \in A^+$, $1\cdot(ae+bz)+(-e-z)(ae+bz)=ae+bz-(ae+bz)=0 \in A^+$. However, $e \in A^+$, and $1e+e(-e-z)=-z \notin A^+$.

We will show that $P(A')$ has the following properties: I) $(0,0) \in P(A')$, II) $P(A') \cap (-P(A'))=\varrho(k,x)$, III) $P(A')+P(A') \subseteq P(A')$, IV) $P(A')\cdot P(A') \subseteq P(A')$. Then, by II) and III), $(n,a) \in P(A')$ if and only if $(n,a)+\varrho(k,x) \subseteq P(A')$. Hence, the subset A_1^+ of A_1 in the following theorem is well-defined.

THEOREM 2.3. *Let A be any f-ring, and A_1 as above. The subset A_1^+ of A_1* defined by

$$\overline{(n,a)} \in A_1^+ \text{ if and only if } (n,a) \in P(A')$$

is the set of positive elements for a partial ordering of A_1 that extends the order of A and under which A_1 is a partially ordered ring.

Proof. By Proposition I.2.2 and the remarks made above, it is sufficient to prove properties I)–IV) stated above. Property I) clearly holds.

II) Suppose $(n,a) \in P(A') \cap (-P(A'))$. If $n=0$, then $a \in A^+ \cap (-A^+)$, so $a=0$, whence $(n,a)=(0,0) \in \varrho(k,x)$. If $n \neq 0$, then it can easily be shown that $-a$ is an n-fier of A, so $n=tk$ and $-a=tx$ for some integer t, by Lemma 2.1, iii). That is, $(n,a)=t(k,-x) \in \varrho(k,x)$.

Conversely, if $(n,a)=t(k,-x) \in \varrho(k,x)$, then $nd+ad=tkd-txd=0=nd+da \in A^+ \cap (-A^+)$ for every $d \in A^+$, so $(n,a) \in P(A') \cap (-P(A'))$.

III) Suppose (n,a), $(m,b) \in P(A')$. For every $d \in A^+$, $nd+ad$, $nd+da$, $md+bd$, $md+db$ are positive elements of A, so $(n+m)d+(a+b)d=(nd+ad)+(md+bd) \in A^+$ and, similarly, $(n+m)d+d(a+b) \in A^+$. Thus, if $n+m \neq 0$, it is clear that $(n,a)+(m,b) \in P(A')$. If $n+m=0$, then there are two cases:

i) $n=m=0$. Then a, $b \in A^+$, so $a+b \in A^+$, whence $(n,a)+(m,b)=(0,a+b) \in P(A')$.

ii) $n=-m \neq 0$. Suppose $n>0$. Now, since $(-n,b) \in P(A')$, we note that $b \in A^+$, since $-nb^-+bb^-=-nb^- -b^{-2} \in A^+$ implies that $b^-=0$. Also, if $0 \neq d \in A^+$, then $-nd+bd \in A^+$, so $bd \neq 0$. As noted above, $(n+m)d+d(a+b)=d(a+b) \in A^+$ for every $d \in A^+$. Hence, in particular, $b(a+b) \in A^+$, so $[b(a+b)]^-=b(a+b)^-=0$. Thus, $(a+b)^-=0$, so we have $(n,a)+(m,b)=(0,a+b) \in P(A')$.

IV) Suppose (n,a), $(m,b) \in P(A')$. We wish to show that $(n,a)(m,b)=(nm, nb+ma+ab) \in P(A')$. There are three cases:

i) $n=m=0$. In this case, a, $b \in A^+$, so $ab \in A^+$, whence $(n,a)(m,b)=(0,ab) \in P(A')$.

ii) $n \neq 0$, $m=0$ or $n=0$, $m \neq 0$. We suppose $n \neq 0$, $m=0$; the other case is similar.

Since $(m, b) = (0, b) \in P(A')$, we have $b \in A^+$; since $(n, a) \in P(A')$, we have $nb + ab \in A^+$. Thus, $(n, a)(m, b) = (0, nb + ab) \in P(A')$.

iii) $nm \neq 0$. Then, for every $d \in A^+$, we have $nmd + (nb + ma + ab)d = n(md + bd) + a(md + bd) \in A^+$, since (n, a), $(m, b) \in P(A')$. Similarly, $nmd + d(nb + ma + ab) \in A^+$ for every $d \in A^+$. Hence, $(n, a)(m, b) \in P(A')$.

Definition 2.4. The partial order for A_1 defined by the set A_1^+ of Theorem 2.3 is called the *strong order* for A_1.

If A_1 is made into a partially ordered ring under a partial ordering $\leqslant$ that extends the order on A, then any $\overline{(n, a)} \geqslant 0$ is positive in the strong order for A_1. For, if $d \in A^+$, then $\overline{(0, d)} \geqslant 0$, since $\leqslant$ extends the order on A. Then, since A_1 is a partially ordered ring under $\leqslant$, $\overline{(n, a)}\,\overline{(0, d)} = \overline{(0, nd + ad)} \geqslant 0$ and $\overline{(0, d)}\,\overline{(n, a)} = \overline{(0, nd + da)} \geqslant 0$, whence $nd + ad$, $nd + da \in A^+$. Thus, $\overline{(n, a)}$ is positive in the strong order for A_1. In the usual terminology, we have shown that the strong order is "stronger" than $\leqslant$. Thus, the strong order is the strongest possible partial ordering of A_1 that extends the order on A and makes A_1 into a partially ordered ring.

Theorem 2.5. *Let A be an f-ring without identity.*

i) *If A is imbedded as a sub-f-ring of an f-ring B with identity, then there is a subring C of B containing A such that C is ring isomorphic to A_1 under a correspondence that leaves A elementwise fixed. Moreover, the partial order induced in A_1 by that of B restricted to C via this ring isomorphism is the strong order for A_1.*

ii) *A can be imbedded as a sub-f-ring of an f-ring with identity if and only if A_1, with the strong order, can be imbedded as a subring of an f-ring whose identity element is the identity element of A_1.*

iii) *A_1 can be made into an f-ring by a partial ordering that extends the partial order on A if and only if the strong order for A_1 makes A_1 into an f-ring.*

The proof of this theorem will be accomplished by establishing a series of intermediate results, which will be designated by 1), 2), etc. We first remark that the subring C of part i) of the theorem need not be a sub-f-ring of B, as will be shown later by an example.

In the construction of the subring C, we will, at times, need to differentiate between the following three cases:

I) A has mode $k > 1$; we will, as usual, let x denote the k-fier of A.

II) A has mode $k = 0$, and there is an element $y \in A^+$ such that $y|a| \wedge |a|y \geqslant |a|$ for every $a \in A$.

III) A has mode $k = 0$, and A does not satisfy II). Hence, for each $a \in A^+$, there is an element $z_a \in A^+$ such that $az_a \wedge z_a a \not\geqslant z_a$.

Now, since B is an f-ring, it is isomorphic to a subdirect union of ordered rings (Theorem I.3.14). Thus, we may choose a collection $\{I_\alpha : \alpha \in \Gamma\}$ of l-ideals in B with zero intersection such that each $B_\alpha = B/I_\alpha$ is an ordered ring. For each $\alpha \in \Gamma$, let h_α denote the natural homomorphism of B onto B_α.

In B, define the element $\bar{1}$ as follows: in cases I) and II), $\bar{1} = 1 \wedge 2y$, where $y \in A^+$ is such that $y|a| \wedge |a|y \geqslant |a|$ for each $a \in A$; in case III), $\bar{1} = 1$, the identity of B. Then,

1) For each $\alpha \in \Gamma$, we have $h_\alpha(\bar{1}) = h_\alpha(1)$ or $h_\alpha(\bar{1}) = 0$.

For, if $h_\alpha(\bar{1}) \neq h_\alpha(1)$, then $\bar{1} = 1 \wedge 2y \neq 1$, and $h_\alpha(\bar{1}) = h_\alpha(1 \wedge 2y) = h_\alpha(1) \wedge h_\alpha(2y) = h_\alpha(2y)$, since B_α is an ordered ring. Then, if we assume that $h_\alpha(y) \neq 0$, we have $h_\alpha(y) = h_\alpha(y) \cdot h(1) > h_\alpha(y) \cdot h_\alpha(2y) = 2h_\alpha(y^2)$. But, since $y^2 \geqslant y$, this yields $h_\alpha(y) > 2h_\alpha(y) > 0$, a contradiction. Hence, $h_\alpha(y) = 0$, so $h_\alpha(\bar{1}) = h_\alpha(1) \wedge 2h_\alpha(y) = 0$. An immediate consequence of 1) is

2) $\bar{1}^2 = \bar{1}$, and $b\bar{1} = \bar{1}b$ for each $b \in B$.

Now, if $\bar{1} \neq 1$, then $\bar{1} = 1 \wedge 2y$, and $a\bar{1} = a(1 \wedge 2y) = a \wedge 2ay = a = \bar{1}a$ for each $a \in A^+$, since $ay \geqslant a$ and $ya \geqslant a$. Thus, since we may write each element of A as the difference of positive elements, we have

3) $\bar{1}a = a\bar{1} = a$ for each $a \in A$.

If B is an ordered ring, then we have the following result in case I):

4) If B is an ordered ring, and y is an n-fier of A, for $n > 0$, then $n \cdot 1 = y$.

For, $y > 0$ by Lemma 2.1, whence $(y-1)y = (n-1)y > 0$. Hence, $y > 1$, so we have $0 = |y^2 - ny| = y|y - n1| \geqslant |y - n1| \geqslant 0$.

We use 4) to prove:

5) In case I), $k \cdot \bar{1} = x$.

By 4), whenever $h_\alpha(\bar{1}) = h_\alpha(1)$, we have $kh_\alpha(\bar{1}) = h_\alpha(x)$, since $h_\alpha(x)$ is *a* k-fier of the subring $h_\alpha(A)$ of the ordered ring B_α with identity. If $h_\alpha(\bar{1}) = 0$, then $h_\alpha(x) = h_\alpha(\bar{1} \cdot x) = 0$. Thus, $h_\alpha(k\bar{1} - x) = kh_\alpha(\bar{1}) - h_\alpha(x) = 0$ for each $\alpha \in \Gamma$, so $k\bar{1} - x \in \bigcap\{I_\alpha : \alpha \in \Gamma\} = \{0\}$.

We now have:

6) The subring C of B generated by $\bar{1}$ and A is (ring) isomorphic with A_1 under the correspondence $n\bar{1} + a \leftrightarrow \overline{(n, a)}$.

For, by 2) and 3), $\bar{1}$ is the identity element for C. In cases II) and III), it is clear that this correspondence is one-to-one. In case I), we have $\overline{(n, a)} = 0$ if and only if $(n, a) \in \varrho(k, x)$, whence if and only if $n = tk$ and $a = -tx$ for some integer t. Thus, under the correspondence $n\bar{1} + a \leftrightarrow \overline{(n, a)}$, $n\bar{1} + a \to 0$ if and only if $n\bar{1} + a = tk\bar{1} - tx = 0$, by 5). Hence, in this case, also, the correspondence is one-to-one. Since $n\bar{1} + a \to \overline{(n, a)}$ is clearly a (ring) homomorphism, this proves 6).

We will call the "strong order for C" that partial order on C that is induced, via this isomorphism, by the strong order for A_1.

7) The partial ordering of B, restricted to C, is the strong order for C.

The partial ordering of B, restricted to C, extends the order on A and makes C into a partially ordered ring. As we have already noted (cf. the discussion following Definition 2.4), the strong order is the strongest such partial order on C; that is $n\bar{1} + a \in B^+$ implies that $n\bar{1} + a$ is positive in the strong order.

Conversely, suppose $n\bar{1} + a$ is positive in the strong order for C, but that $n\bar{1} + a \notin B^+$. We will show that, in each of the three cases, there is an element $d \in A^+$ such that $nd + da \notin A^+$, thus contradicting the assumption that $n\bar{1} + a$ is positive in the strong order for C

I) Since $n\bar{1} + a \notin B^+$, we have $k(n\bar{1} + a) \notin B^+$, by Proposition I.1.5, x). But, $k(n\bar{1} + a) = n \cdot k\bar{1} + ka = nx + xa$, by 5), and $x \in A^+$ by Lemma 2.1.

II) View the elements of B as elements of the complete direct union of the B_α. Then, since $n\bar{1} + a \notin B^+$, there is an $\alpha \in \Gamma$ such that $h_\alpha(n\bar{1} + a) < 0$. Now, $h_\alpha(\bar{1}) = h_\alpha(1)$, since otherwise, by 1), $h_\alpha(\bar{1}) = 0$ and, by 2) and 3), $h_\alpha(n\bar{1} + a) = h_\alpha(\bar{1}) \cdot h_\alpha(n\bar{1} + a) \not< 0$. Thus, since $2y \geqslant 2y \wedge 1 = \bar{1}$, we have $h_\alpha(2y) \geqslant h_\alpha(\bar{1})$, and $h_\alpha(n(2y) + (2y)a) = h_\alpha(2y) \cdot h_\alpha(n\bar{1} + a) \leqslant h_\alpha(n\bar{1} + a) < 0$, whence $n(2y) + (2y)a \notin A^+$.

III) We first note that, since $n\bar{1} + a$ is positive in the strong order for C, we must have $n \geqslant 0$. For if $n < 0$, then $0 \leqslant nz + az \leqslant -z + az$ and, similarly, $0 \leqslant -z + za$ for every $z \in A^+$. But this means that $|az| = |a|\,|z| \geqslant a|z| \geqslant |z|$ and $|za| \geqslant |z|$ for every $z \in A$, contrary to the hypothesis of case III). Thus, $n \geqslant 0$.

Now, since we have assumed that $n\bar{1} + a \notin B^+$, we must have $n > 0$, since $n\bar{1} + a$ is positive in the strong order for C, and the strong order coincides, in A, with the partial order on B.

As before, there is an $\alpha \in \Gamma$ with $h_\alpha(n\bar{1} + a) < 0$ and $h_\alpha \bar{1} = h_\alpha(1) \neq 0$ is the identity element of B_α. Thus, we have $h_\alpha(-a) > nh_\alpha(\bar{1}) = nh_\alpha(1) > 0$, so $h_\alpha(|a|) = |h_\alpha(-a)| = h_\alpha(-a) \geqslant h_\alpha(1)$. Hence, $h_\alpha(n|a| + |a|a) = h_\alpha(|a|) \cdot h_\alpha(n\bar{1} + a) \leqslant h_\alpha(n\bar{1} + a) < 0$, which implies that $n|a| + |a|a \not\geqslant 0$.

Now, 6) and 7) prove part i) of Theorem 2.5. To prove part ii), we have:

8) $B^* = \bar{1}B = \{\bar{1}b : b \in B\}$ is an f-ring with identity $\bar{1}$.

For, by 2), the mapping $b \to \bar{1}b$ is a ring homomorphism of B onto B^*. Moreover, since $\bar{1} \in B^+$, we have $(\bar{1}b)^+ = \bar{1}b^+$ for each $b \in B$, so, by Proposition I.1.4, this mapping is also a homomorphism of the lattice structure. Since $\bar{1}$ is the identity element in C, the elements of C are fixed under this homomorphism. It is clear that $\bar{1}$ is the identity element of B^*, so part ii) of Theorem 2.5 is proved.

Now, if $B = A_1$, then $b \to \bar{1} b$ is an isomorphism, since B is generated as a ring by 1 and A. This proves part iii) of Theorem 2.5.

Thus, the proof of Theorem 2.5 is now complete. An immediate consequence of part i) of the theorem is the following corollary:

Corollary 2.6. *An ordered ring A without identity can be imbedded as a subring of an ordered ring with identity if and only if A_1 is an ordered ring under the strong order.*

The following example shows that the subring C of B in part i) of Theorem 2.5 need not be a sublattice of B.

Example 2.7. Let A denote the f-ring of all continuous real-valued functions on the interval $[0, 1]$ which satisfy $f(0) = 0$ and $f(1) = 2n$ for some integer n, where the algebraic and lattice operations are as in Example I.3.2. Let B denote the f-ring of all continuous real-valued functions on $[0, 1]$. Then A is an f-ring without identity that is imbedded as a sub-f-ring of the f-ring B with identity. The subring C of B consists of all $n\bar{1} + f$, where $\bar{1}$ is the identity in B (i.e. $\bar{1}(x) = 1$) and $f \in A$; hence $g \in C$ if and only if $g(0) = m$ and $g(1) = 2n + m$ for integers m, n.

Now, C is not a sublattice of B since, if f denotes the element of A defined by $f(x) = 2x$, then $g = \bar{1} \wedge f$ satisfies $g(0) = 0$ and $g(1) = 1$, whence $g \notin C$.

The next theorem describes a special class of f-rings which can be imbedded as l-ideals in f-rings with identity. The statement concerning ordered rings will be strengthened considerably in Section 4.

Theorem 2.8. *Let A be an f-ring in which $|ab| \leqslant |a| \wedge |b|$ for every pair a, b of elements of A.*

i) *Under the strong order, $A_1 = A'$ is an f-ring and A is an l-ideal in A_1.*

ii) *If A is an ordered ring, then, under the strong order, A_1 is also.*

Proof. Since $|ab| \leqslant |a| \wedge |b|$ for every pair a, b of elements of A, it is clear that the mode of A is 0, hence that $A_1 = A'$. Since, for every $d \in A^+$, and every integer $n \geqslant 1$, we have $nd \geqslant d = |d| \geqslant |ad| \geqslant -ad$ and, similarly, $nd \geqslant -da$, for every $a \in A$, the strong order for A_1 is just the lexicographic order:

$$(n, a) \in A_1^+ \text{ if and only if } n > 0 \text{ or } n = 0 \text{ and } a \geqslant 0.$$

For each $(n, a) \in A_1$, it is clear that $(n, a)^+ = (n, a) \vee 0$ exists and is given by:

$$(n, a) \vee 0 = \begin{cases} (n, a) & \text{if } n > 0, \\ (0, a^+) & \text{if } n = 0, \\ (0, 0) & \text{if } n < 0. \end{cases}$$

Thus, by Proposition I.1.2, A_1 is, under the strong order, a lattice-ordered ring. In A_1, the identity element $(1, 0)$ is a strong order unit: if $(n, a) \in A_1$, then $(|n| + 1)(1, 0) \geqslant (n, a)$. Hence, by Proposition I.3.4, i), A_1 is an f-ring.

To complete the proof of i), we must show that A is an l-ideal in A_1. It is a ring ideal, so we must show that $|(0, a)| = (0, |a|) \geqslant (n, b) \geqslant 0$ implies $(n, b) \in A$ (that is, $n = 0$). Now, $(n, b) \geqslant 0$ implies $n \geqslant 0$, and $(0, |a|) \geqslant (n, b)$ implies $(-n, |a| - b) \geqslant 0$, whence $-n \geqslant 0$. Thus $n = 0$.

We have shown that A_1 is an f-ring and A is an l-ideal in A_1. If, now, A is an ordered ring and $(n, a) \in A_1$, then it is clear that $(n, a) \in A_1^+$ or $(n, a) \in (-A_1^+)$, so A_1 is an ordered ring.

3. Imbedding as a sub-f-ring

We consider the following condition for an f-ring A:

for all $a, b, c \in A^+$ and all integers n,

$$(ab - nb) \wedge (nc - ac) \leqslant 0, \ (ab - nb) \wedge (nc - ca) \leqslant 0, \qquad (*)$$
$$(ba - nb) \wedge (nc - ac) \leqslant 0, \ (ba - nb) \wedge (nc - ca) \leqslant 0.$$

This condition is clearly equivalent to:

for all $a, b, c \in A^+$ and all integers n,

$$[(ab - nb) \vee (ba - nb)] \wedge [(nc - ac) \vee (nc - ca)] \leqslant 0. \qquad (*)$$

LEMMA 3.1. *If A is an ordered ring that satisfies (*), then A can be imbedded in an ordered ring with identity.*

Proof. We show that the strong order for A_1 is a total order. Let $\overline{(n, a)} \in A_1$; we wish to show that $\overline{(n, a)} \in A_1^+ \cup (-A_1^+)$. It is clearly sufficient to consider the case $n \geqslant 0$. If $n = 0$, then $\overline{(n, a)} \in A_1^+ \cup (-A_1^+)$, since A is an ordered ring. If $n > 0$ and $a \in A^+$, then $\overline{(n, a)} \in A_1^+$. If $n > 0$ and $-a \in A^+$, and if $\overline{(n, a)} \neq 0$, then there is a $b \in A^+$ such that either $nb + ab \neq 0$ or $nb + ba \neq 0$. We suppose $nb + ab = nb - (-a)b > 0$; the other cases can be handled similarly. For any $c \in A^+$ we have, by (*),

$$(nb - (-a)b) \wedge ((-a)c - nc) \leqslant 0 \text{ and } (nb - (-a)b) \wedge (c(-a) - nc) \leqslant 0.$$

Since $nb + ab > 0$, we have $(c(-a) - nc) \vee ((-a)c - nc) \leqslant 0$, whence $nc + ac \in A^+$ and $nc + ca \in A^+$ for each $c \in A^+$. Thus, in this case, $\overline{(n, a)} \in A_1^+$.

THEOREM 3.2.[1] *If A is an f-ring, then the following are equivalent:*

i) *A can be imbedded as a sub-f-ring of an f-ring with identity;*

ii) *A satisfies condition (*);*

iii) *if $A = \sum_s \{A_\alpha : \alpha \in \Gamma\}$ is any representation of A as a subdirect union of ordered rings A_α, then each A_α can be imbedded in an ordered ring A_α^* with identity.*

Proof. Suppose A is a sub-f-ring of the f-ring B with identity. If $a, b, c \in A^+$ and if n is any integer then, in B, $(a - n\cdot 1) \wedge (n\cdot 1 - a) \leqslant 0$, and it is easily seen that $(ab - nb) \wedge (nc - ac) = (a - n\cdot 1)b \wedge (n\cdot 1 - a)c \leqslant 0$. The other relations in (*) follow similarly. Thus, i) implies ii).

That iii) implies i) is obvious. To see that ii) implies iii), observe that condition (*) is preserved under homomorphism. Hence, if A satisfies (*), then each A_α satisfies (*), whence each A_α can be imbedded in an ordered ring A_α^* with identity by Lemma 3.1.

As an immediate consequence of the equivalence of i) and iii) above, we have:

COROLLARY 3.3. *If A is an ordered ring, then A can be imbedded as a sub-f-ring of an f-ring with identity if and only if A can be imbedded in an ordered ring with identity.*

It is now easy to give an example of an f-ring that cannot be imbedded as a sub-f-ring of an f-ring with identity. For, by Theorem 3.2, we have only to display an f-ring in which condition (*) fails to hold.

Example 3.4. Let A be the algebra over the ordered field Q of rational numbers generated by two elements e, z, where $e^2 = e, z^2 = ze = 0$, and $ez = z$. Order A lexicographically with e dominating: $ae + bz \geqslant 0$ if and only if $a > 0$ or $a = 0$ and $b \geqslant 0$. With this ordering, A becomes an ordered ring. The element $2e \in A^+$ satisfies $2e\cdot e = 2e$ and $z\cdot 2e = 0$. Thus, there are elements $a, b, c \in A^+$, $(a = 2e, b = e, c = z)$ and an integer n $(n = 1)$ such that $(ab - nb) \wedge (nc - ca) > 0$. Thus, A is an f-ring which does not satisfy condition (*).

THEOREM 3.5. *If A is an ordered ring without non-zero divisors of zero, then A can be imbedded in an ordered ring with identity that contains no non-zero divisors of zero.*

Proof. We first show that A can be imbedded in an ordered ring with identity by showing that condition (*) holds in A.

Let $a, b, c \in A^+$, and let n be an integer. We may assume that $b \neq 0$, since otherwise the relations of (*) hold trivially. If $ab - nb \in A^+$ then, for $d \in A^+$, $(da - nd)b = d(ab - nb) \in A^+$. Since $0 \neq b \in A^+$, this implies that $da - nd \in A^+$ for each $d \in A^+$. In particular, $ba - nb \in A^+$, so $b(ac - nc) = (ba - nb)c \in A^+$, whence $ac - nc \in A^+$. Thus, if $ab - nb \in A^+$,

[1] The author wishes to thank the referee, whose observations led to the present form of Theorem 3.2.

then $(nc - ca) \vee (nc - ac) \leqslant 0$. If, on the other hand, $ab - nb \in -A^+$, then $(ba - nb)b = b(ab - nb) \in -A^+$, whence $ba - nb \in -A^+$. Thus, in this case, $(ba - nb) \vee (ab - nb) \leqslant 0$. We have shown that, in either case, the relations of condition (*) hold.

Thus, under the strong order, A_1 is an ordered ring. To show that A_1 contains no non-zero divisors of zero, it is sufficient, by Theorem I.4.1, to show that A_1 contains no non-zero nilpotent elements. Hence, suppose $\overline{(n, a)} \in N(A_1)$, the set of nilpotent elements of A_1. Since $N(A_1)$ is an l-ideal in A_1, $\overline{(n, a)}\,\overline{(0, b)} = \overline{(0, nb + ab)} \in N(A_1)$ for each $b \in A$. Since A contains no non-zero nilpotent elements, we have $nb + ab = 0$ for each $b \in A$. Similarly, $nb + ba = 0$ for each $b \in A$. Hence, $-a$ is an n-fier of A and it follows that $\overline{(n, a)} \in \varrho(k, x)$ (Lemma 2.1), so $\overline{(n, a)} = 0$. Hence, $N(A_1) = \{0\}$.

Theorem 3.5 is the analogue of a result of Szendrei ([21]) for abstract rings without non-zero divisors of zero. In fact, Szendrei's ring construction is the same one that we have used. In [15], Johnson remarked, without proof, that Szendrei's result also applied to ordered rings.

There are two important classes of f-rings for which special imbeddings are always available:

THEOREM 3.6. *If A is a J-semisimple f-ring, then A can be imbedded as a sub-f-ring of a J-semisimple f-ring with identity.*

Proof. By Theorem II.4.12, A is isomorphic to a subdirect union of l-simple ordered rings with identity; their complete direct union is a J-semisimple f-ring with identity.

THEOREM 3.7. *If A is an f-ring with zero l-radical, then A can be imbedded as a sub-f-ring of an f-ring with zero l-radical that contains an identity.*

Proof. By Theorem I.4.8, A is isomorphic to a subdirect union of ordered rings A_α which contain no non-zero divisors of zero. Each of these can be imbedded in an ordered ring A_α^* with identity that contains no non-zero divisors of zero by Theorem 3.5. The complete direct union of the A_α^* is the required f-ring with zero l-radical.

4. Imbedding as an l-ideal

The main result of this section is a strengthening of Theorem 2.5 for imbedding as a (right) l-ideal: if an f-ring A without identity can be imbedded as a (right) l-ideal in an f-ring B with identity, then there is a standard imbedding available (i.e., under the strong order, A_1 is an f-ring).

The following result immediately eliminates many f-rings from consideration:

PROPOSITION 4.1. *If A is an f-ring without identity that contains elements a, b such that $|ac| \geqslant |c|$ and $|cb| \geqslant |c|$ for all $c \in A$, then A cannot be imbedded as a right l-ideal in any f-ring B with identity.*

Proof. Note that it is sufficient to assume that there is an $a' \in A$ such that $|a'c| \geqslant |c|$ and $|ca'| \geqslant |c|$ for all $c \in A$ (e.g. $a' = |a| \vee |b|$).

Now suppose that A is a right l-ideal in the f-ring B with identity. Then $1 \wedge |a'| \in A$. But, $(1 \wedge |a'|)|c| = |c| \wedge |a'c| = |c|$ for every $c \in A$. Thus, for any $c \in A$, $(1 \wedge |a'|)c = (1 \wedge |a'|)(c^+ - c^-) = c^+ - c^- = c$. Similarly, $c(1 \wedge |a'|) = c$ for every $c \in A$. Thus, $1 \wedge |a'|$ is an identity in A, a contradiction.

For the sake of economy, we introduce the following notation:

DEFINITION 4.2. If A is an f-ring that satisfies

1) A does not contain an identity element, and

2) A contains no element a' such that $|a'c| \wedge |ca'| \geqslant |c|$ for all $c \in A$,

then A is called an f_D-ring.

In view of Proposition 4.1, we need concern ourselves only with f_D-rings. They are the only f-rings without identity that can possibly be imbedded as right l-ideals in f-rings with identity. However, not every f_D-ring can be so imbedded:

Example 4.3. Let N denote the space of non-negative integers in the usual (discrete) topology. Then $C(N)$ consists of all sequences of real numbers. Let A denote the sub-f-ring of $C(N)$ consisting of those null sequences $a = (a_n : n = 0, 1, 2, \ldots)$ with $a_0 = 2k$, an even integer. Then A is easily seen to be an f_D-ring.

We show that A cannot be imbedded as an l-ideal in an f-ring with identity by proving that if A is imbedded as an l-ideal in a lattice-ordered ring B with identity, then the identity element of B is not a weak order unit (cf. Proposition I.3.3). For, let $b \in A$ be defined by $b_0 = 2$ and $b_n = 0$ for $n \neq 0$. Now $b^2 = 2b \nleqslant b$, so $b \nleqslant 1$, whence $b \wedge 1 \neq b$. But, since A is an l-ideal in B and $b \in A^+$, we must have $b \wedge 1 \in A$. The only $a \in A$ with $0 \leqslant a \leqslant b$ and $a \neq b$ is $a = 0$, so $b \wedge 1 = 0$. Since $b \neq 0$, 1 is not a weak order unit in B.

Thus, A is an example of a commutative f_D-ring without non-zero nilpotent elements that can be imbedded as a sub-f-ring of an f-ring with identity but cannot be imbedded as a right l-ideal in any f-ring with identity.

By condition 2) of Definition 4.2, every f_D-ring has mode $k = 0$, so $A_1 = A'$. We notice that if $(n, a) \in A'$ is positive in the strong order, then $n \geqslant 0$. For, if $n < 0$, then $-|d| + |ad| \geqslant n|d| + a|d| \geqslant 0$ and $-|d| + |da| \geqslant 0$, whence $|ad| \wedge |da| \geqslant |d|$, for every $d \in A$, contrary to Definition 4.2.

Now, by Theorem 2.5, if an f_D-ring A is imbedded as a sub-f-ring of an f-ring B with identity, then there is a subring C of B that contains A and is ring isomorphic to A' under a correspondence that leaves A elementwise fixed. Moreover, the partial order induced in A', via this ring isomorphism, by the partial order in B is the strong order. As noted above, the mode of A is zero. Moreover, A contains no $y \geqslant 0$ which satisfies $ya \wedge ay \geqslant a$ for every $a \in A^+$. Thus, in the language of Section 2, A belongs to case III. We recall, from the proof of Theorem 2.5, that, in case III, the identity element of the subring C of B can be taken to be the identity element of B. We now have:

THEOREM 4.4. *If an f_D-ring A can be imbedded as a right l-ideal in an f-ring B with identity, then B contains a sub-f-ring C that contains A and the identity of B and such that C is isomorphic to A', with the strong order for A', under a correspondence that keeps A elementwise fixed.*

Proof. By the remarks made above, we need only show that the subring C of B provided by Theorem 2.5 is a sub-f-ring of B. We will denote the elements of C by (n, a), etc. (i.e., as the elements of A'). We will show that $(n, a)^+ = (n, a) \vee 0 \in C$ for each $(n, a) \in C$. Then, by Proposition I.1.2, C is a lattice-ordered ring, hence a sub-f-ring of B.

Let $(n, a) \in C$. We consider three cases:

i) $n = 0$. Then, $(0, a)^+ = (0, a^+) \in C$.

ii) $n > 0$. $(n, a) = (n, a^+) - (0, a^-)$ expresses (n, a) as the difference of positive elements of B. Hence, $(n, a)^+ = (n, a^+) - (n, a^+) \wedge (0, a^-)$, by Proposition I.1.3, vii). Now, since A is an l-ideal in B and $(0, a^-) \in A$, we have $(n, a^+) \wedge (0, a^-) \in A$, since $(0, a^-) \geqslant (n, a^+) \wedge (0, a^-) \geqslant 0$. Thus, $(n, a)^+ \in C$.

iii) $n < 0$. $(n, a) = (0, a^+) - (-n, a^-)$ expresses (n, a) as the difference of positive elements. As before, $(0, a^+) \wedge (-n, a^-) \in A$, so $(n, a)^+ = (0, a^+) - (0, a^+) \wedge (-n, a^-) \in C$.

Since, if A is a right l-ideal in B, then A is a right l-ideal in the sub-f-ring C of B, we have:

COROLLARY 4.5. *If A is an f-ring without identity, then A can be imbedded as a right l-ideal in an f-ring with identity if and only if A is an f_D-ring and A' is an f-ring under the strong order.*

Since A is a two-sided (ring) ideal in A', this yields:

COROLLARY 4.6. *An f-ring A can be imbedded as a right l-ideal in an f-ring B with identity if and only if A can be imbedded as an l-ideal in some f-ring B^* with identity.*

Our central problem, that of characterizing those f-rings that can be imbedded as right l-ideals in f-rings with identity, is now reduced to that of characterizing those f_D-rings A for which A' is an f-ring under the strong order. For ordered rings, we have:

THEOREM 4.7. *If A is an ordered ring without identity, then the following are equivalent:*

i) *A can be imbedded as an l-ideal in an f-ring with identity;*

ii) *A can be imbedded as an l-ideal in an ordered ring with identity;*

iii) for every pair a, b of elements of A, $|ab| \leqslant |a| \wedge |b|$;

iv) *for every pair a, b of non-zero elements of A, $|ab| < |a| \wedge |b|$.*

Proof. That iv) implies iii) and ii) implies i) are trivial, and iii) implies ii) by Theorem 2.8. Now, it is easily seen that iii) implies iv). For suppose $a, b \in A$ are non-zero elements and that $|ab| \not< |a| \wedge |b|$. Since A is an ordered ring, $|a| \wedge |b|$ is $|a|$ or $|b|$. If $|a| \wedge |b| = |a|$ and $|ab| \leqslant |a| \wedge |b|$, then we must have $|ab| = |a|$, whence $|a(2b)| = 2|a| \not\leqslant |a| \wedge |2b| = |a|$.

The proof will be completed by showing that i) implies iii). If A can be imbedded as an l-ideal in an f-ring with identity, then A is an f_D-ring (Corollary 4.5) and A satisfies condition (*) (Theorem 3.2). If $a, b \in A^+$ are such that $ab > b$ (or $ba > b$) then, by condition (*), $[(ab - b) \vee (ba - b)] \wedge [(c - ac) \vee (c - ca)] \leqslant 0$ for each $c \in A^+$. Hence, the element a of A satisfies $(c - ac) \vee (c - ca) = c - ac \wedge ca \leqslant 0$ for each $c \in A^+$, contradicting the fact that A is an f_D-ring. Hence, iii) is satisfied.

For arbitrary f-rings, no such characterization is known. In fact, it is not known whether it is sufficient to require only that A' be a lattice under the strong order. It is not inconceivable that, if A' is a lattice-ordered ring, then the special nature of A' and the presence in A' of the maximal l-ideal A, which is itself an f-ring, force A' to be an f-ring.

We note the following result, the proof of which is omitted:

PROPOSITION 4.8. *Let A be an f_D-ring. Then A', under the strong order, is a lattice-ordered ring if and only if $(1, 0) \wedge (0, a)$ exists for every $a \in A^+$.*

CHAPTER IV. ONE-SIDED l-IDEALS IN f-RINGS

In Chapter II, a question was raised concerning the existence in f-rings of one-sided l-ideals that are not two-sided. At that time, we presented an example of an ordered ring containing a right l-ideal that is not two-sided. In this chapter, we present an example of an ordered ring A without non-zero divisors of zero with this property, and we show that if an ordered ring without non-zero divisors of zero has a right l-ideal that is not two-sided, then it contains a subring isomorphic to A. In Section 2, we consider the status of this question for the larger classes of f-rings that have been considered in this paper.

1. An example

Example 1.1. Let S denote the free semigroup without identity generated by two elements a, x: the elements of S are "words" of the form

$$z = a^{n_1}x^{k_1}a^{n_2}x^{k_2}\ldots a^{n_r}x^{k_r}, \tag{1}$$

where the n_i and k_j are non-negative integers, and n_1 and k_1 are not both zero. With each element z of S, where z has the form (1), associate the non-negative integer $N(z) = \sum_{i=1}^{r} n_i$.

We define a total order in S as follows. If $z \in S$ is as in (1), and if $z' = a^{n'_1}x^{k'_1}a^{n'_2}x^{k'_2}\ldots a^{n'_s}x^{k'_s} \in S$, then we will say that $z > z'$ if and only if

i) $N(z) < N(z')$, or

ii) $N(z) = N(z')$ and either

1) $n_1 = n'_1$, $k_1 = k'_1$, $n_2 = n'_2$, $k_2 = k'_2$, $\ldots$, $k_{m-1} = k'_{m-1}$, and $n_m < n'_m$ for some $m = 1, 2, \ldots, r-1$, or

2) $n_1 = n'_1$, $k_1 = k'_1$, $n_2 = n'_2$, $k_2 = k'_2$, $\ldots$, $k_{m-1} = k'_{m-1}$, $n_m = n'_m$, and $k_m > k'_m$ for some $m = 1, 2, \ldots, r$.

Thus, if $z \neq z'$ and if $N(z) = N(z')$, then the order relation between z and z' is determined by observing the first exponent at which the expressions of z and z' in the form (1) differ.

It is easily seen that this is a total order on S. Moreover, this order is preserved under multiplication; that is, if z, z', $z'' \in S$, with $z > z'$, then $zz'' > z'z''$ and $z''z > z''z'$. For, if $N(z) < N(z')$, then $N(zz'') = N(z''z) < N(z''z') = N(z'z'')$, and if $N(z) = N(z')$, then $N(zz'') = N(z'z'') = N(z''z) = N(z''z')$ and whichever of the conditions 1) and 2) of ii) above holds for z, z' will also hold for zz'', $z'z''$ and $z''z$, $z''z'$.

Now let A denote the semigroup ring of S over the ring of integers. The elements of A are finite formal sums:

$$A = \left\{ \sum_{i=1}^{n} m_i z_i : m_i \text{ integers, } z_i \in S \right\}.$$

In A, addition is defined coordinatewise, and multiplication is defined by:

$$\left(\sum_{i=1}^{n} m_i z_i\right)\left(\sum_{j=1}^{n'} m'_j z'_j\right) = \sum_k \Big(\sum_{z_i z'_j = z_k} m_i m'_j\Big) z_k.$$

In order to define an order in A, we write the elements of A in the form $\sum_{i=1}^{n} m_i z_i$, where $i < j$ implies $z_i > z_j$. Now, say that $\sum_{i=1}^{n} m_i z_i \geqslant 0$ if and only if $m_1 > 0$, or $m_1 = 0$ and $m_2 > 0$, or ..., or $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $m_n \geqslant 0$. It is readily verified that this is a total order in A, under which A is an ordered ring. Clearly, A contains no non-zero divisors of zero.

In A, $\langle a\rangle_r$ is a right l-ideal that is not two-sided, since $\langle a\rangle_r =$

$$\{d \in A : |d| \leqslant m_0 a + a \sum_{i=1}^{n} m_i z_i,\ m_0 \text{ an integer}, \sum_{i=1}^{n} m_i z_i \in A\} \text{ and, clearly, } xa \notin \langle a\rangle_r.$$

The rest of this section is devoted to the proof of the following result.

THEOREM 1.2. *If A is an ordered ring without non-zero divisors of zero that contains a right l-ideal that is not two-sided, then A contains a subring that is isomorphic to the ring of Example 1.1.*

By Theorem III.3.5, the strong order makes the ring extension A_1 of A into an ordered ring without non-zero divisors of zero in which the ordering extends that on A. Recalling the details of this imbedding, we note that if I is any right l-ideal in A, then it is a right (ring) ideal in A_1, and hence that if I is proper in A, then $\langle I\rangle_r$ is a proper right l-ideal in A_1.

Now, if I is a right l-ideal in A that is not two-sided, then there are positive elements $a \in I$ and $x \in A$ such that $xa \notin I$. Then, in A, $\langle a\rangle_r$ is a right l-ideal that is not two-sided. Hence, as noted above, the right l-ideal $J = \langle a\rangle_r$ in A_1 is a right l-ideal of A_1 that is not two-sided.

Since A_1 contains an identity element, J is a modular right l-ideal. Hence, J can be extended to a maximal (modular) right l-ideal M of A_1 (Proposition II.1.2). By Corollary II.4.7, M is a two-sided l-ideal in A_1. Thus, $J^* = \langle a\rangle$ is a proper l-ideal in A_1, since $J^* \subseteq M$; it is the smallest (two-sided) l-ideal of A_1 that contains J.

Now let J_* denote the largest l-ideal of A_1 that is contained in J. Since the right l-ideals of A_1 form a chain, J^* is the smallest l-ideal of A_1 that properly contains J_*. Hence, A_1/J_* is a subdirectly irreducible f-ring, with smallest non-zero l-ideal J^*/J_*. By Theorem II.5.2, J^*/J_* consists entirely of nilpotent elements of order two; that is, $z \in J^*$ implies $z^2 \in J_*$.

Now, let $I^* = J^* \cap A$ and $I_* = J_* \cap A$. We have: $I_* \subset \langle a\rangle_r \subseteq I \subset I^*$. Thus, we have proved:

LEMMA 1.3. *If A is an ordered ring without non-zero divisors of zero that contains a right l-ideal I that is not two-sided, then A is not l-simple. More precisely, there is an l-ideal $I_* \subseteq I$ and an l-ideal $I^* \supseteq I$, with $I^{*2} \subseteq I_*$.*

For the sake of economy, we will write, for $b, c \in A_1$, $b \gg c$ in case $b \geqslant nc$ for every integer n. Note that, since $xa \notin J$, we have $xa \geqslant na$ for every integer n, whence $x \gg 1$ in A_1.

The proof of Theorem 1.2 will proceed as follows. We consider the sub-semigroup T of the multiplicative semigroup of A generated by a, x, and show that the order induced in T by that in A is the same as the order defined on the semigroup S of Example 1.1. Moreover, we will show that if $z, z' \in T$ and $z > z'$, then $z \gg z'$. Then the subring B of A generated by T is just the semigroup ring of T over the ring of integers and the order on B is the same as the order on the ring A of Example 1.1. For, if $\sum_{i=1}^{n} m_i z_i \in B$, where each $z_i \in T$, each m_i s an integer, and $i < j$ implies $z_i > z_j$, then $\sum_{i=1}^{n} m_i z_i \geqslant 0$ if and only if $m_1 > 0$, or $m_1 = 0$ and $m_2 > 0$, or $\ldots$, or $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $m_n \geqslant 0$.

In what follows, we will make tacit use of the following facts:

1) The elements of T are non-zero positive elements of A.

2) A contains no non-zero divisors of zero, hence T is cancellative. Thus, if z, z', $z'' \in T$ and $zz' > zz''$, then $z' > z''$.

Let z, $z' \in T$, say $z = a^{n_1} x^{k_1} a^{n_2} \ldots a^{n_r} x^{k_r}$ and $z' = a^{n'_1} x^{k'_1} a^{n'_2} \ldots a^{n'_s} x^{k'_s}$. As before, let $N(z) = \sum_{i=1}^{r} n_i$ and $N(z') = \sum_{j=1}^{s} n'_j$. That $N(z)$ is uniquely determined for each $z \in T$ is shown by I) below.

If $N(z) = N(z') = 0$, then $z = x^{k_1}$ and $z' = x^{k'_1}$. Since, in A_1, $x \gg 1$, we have $z > z'$ if and only if $k_1 > k'_1$, in which case $z \gg z'$. If $N(z) = 0$ and $N(z') \neq 0$, then, in A_1, $z' \in J^*$, whence $z' \ll 1 \ll z = x^{k_1}$. Thus, if either $N(z)$ or $N(z')$ is zero, then the order relation between z and z' is the same as that defined in Example 1.1. In what follows, we will consider those $z \in T$ with $N(z) \geqslant 1$.

I) If $N(z) < N(z')$, then $z \gg z'$.

We prove this by induction on $N(z)$. If $N(z) = 1$ and $N(z') > 1$, then $z \geqslant a$, so $z \notin I_*$, and z' may be written as the product of two elements of I^*, whence $z' \in I_*$. Since I_* is an l-ideal in A, $z \gg z'$.

Now suppose that $t > 1$ and that I) is true whenever $N(z) < t$. If $N(z) = t$ and $N(z') > t$, then we consider two cases:

i) $k_r \geqslant k'_s$. For any integer n,

$$\begin{aligned} z - nz' &= (a^{n_1} x^{k_1} \ldots a^{n_r} x^{k_r - k'_s} - n a^{n'_1} x^{k'_1} \ldots a^{n'_s}) x^{k'_s} \\ &\geqslant (a^{n_1} x^{k_1} \ldots a^{n_r} - n a^{n'_1} x^{k'_1} \ldots a^{n'_s}) x^{k'_s} \\ &= (a^{n_1} x^{k_1} \ldots a^{n_r - 1} - n a^{n'_1} x^{k'_1} \ldots a^{n'_s - 1}) a x^{k'_s} \geqslant 0 \end{aligned}$$

by the induction hypothesis.

ii) $k_r < k'_s$. For any integer n,

$$z - nz' = (a^{n_1} x^{k_1} \ldots a^{n_r} - n a^{n'_1} x^{k'_1} \ldots a^{n'_s} x^{k'_s - k_r}) x^{k_r}.$$

Now, since $xa > ax$, we have

$$a^{n'_1} x^{k'_1} \dots a^{n'_s} x^{k'_s - k_r} < a^{n'_1} x^{k'_1} \dots a^{n'_s - 1} x^{k'_s - k_r} a,$$

so $$z - nz' > (a^{n_1} x^{k_1} \dots a^{n_r - 1} - n a^{n'_1} x^{k'_1} \dots a^{n'_s - 1} x^{k'_s - k_r})\, a x^{k_r} \geqslant 0$$

by the induction hypothesis.

Thus, in each case, $z \gg z'$, so I) is established.

II) If $N(z) = N(z')$, $n_1 = n'_1$, $k_1 = k'_1$, $n_2 = n'_2$, ..., $k_{m-1} = k'_{m-1}$, and $n_m < n'_m$, then $z \gg z'$.

The proof is again by induction on $N(z)$. If $N(z) = 1$, then there is only one case: $z = x^{k_1} a x^{k_2}$ and $z' = a x^{k'_1}$, where $k_1 \neq 0$. Then $z \geqslant xa \notin I$, while $z' \in I$, whence $z \gg z'$.

Now suppose that $t > 1$, and that II) holds whenever $N(z) < t$. Let $z, z' \in S$ satisfy the hypothesis of II), with $N(z) = t$. There are four cases:

i) $m \neq 1$. For any integer n,

$$z - nz' = a^{n_1} x^{k_1} a (a^{n_2 - 1} x^{k_2} \dots x^{k_r} - n a^{n'_2 - 1} x^{k'_2} \dots x^{k'_s}) \geqslant 0$$

by the induction hypothesis.

ii) $m = 1$ and $n_1 \neq 0$. For any integer n,

$$z - nz' = a (a^{n_1 - 1} x^{k_1} \dots x^{k_r} - n a^{n'_1 - 1} x^{k'_1} \dots x^{k'_s}) \geqslant 0$$

by the induction hypothesis.

iii) $m = 1$, $n_1 = 0$, and $n'_1 \neq 1$. For any integer n,

$$z - nz' = x^{k_1} a^{n_2} x^{k_2} \dots x^{k_r} - n a^{n'_1} x^{k'_1} \dots x^{k'_s}.$$

Now, since $xa \gg ax$, we have

$$x^{k_1} a^{n_2} \dots x^{k_r} > a x^{k_1} a^{n_2 - 1} \dots x^{k_r},$$

whence $$z - nz' > a (x^{k_1} a^{n_2 - 1} \dots x^{k_r} - n a^{n'_1 - 1} x^{k'_1} \dots x^{k'_s}) \geqslant 0$$

by the induction hypothesis, since $n'_1 - 1 > 0$.

iv) $m = 1$, $n_1 = 0$, and $n'_1 = 1$. As in the proof of I), we consider two cases:

1) $k_r \geqslant k'_s$. For any integer n,

$$\begin{aligned} z - nz' &= (x^{k_1} a^{n_2} x^{k_2} \dots a^{n_r} x^{k_r - k'_s} - n a x^{k'_1} a^{n'_2} \dots a^{n'_s})\, x^{k'_s} \\ &\geqslant (x^{k_1} a^{n_2} x^{k_2} \dots a^{n_r} - n a x^{k'_1} a^{n'_2} \dots a^{n'_s})\, x^{k'_s} \\ &= (x^{k_1} a^{n_2} x^{k_2} \dots a^{n_r - 1} - n a x^{k'_1} a^{n'_2} \dots a^{n'_s - 1})\, a x^{k'_s} \geqslant 0 \end{aligned}$$

by the induction hypothesis.

2) $k_r < k'_s$. For any integer n,

$$\begin{aligned} z - n z' &= (x^{k_1} a^{n_2} \dots a^{n_r} - n a x^{k'_1} a^{n'_2} \dots a^{n'_s} x^{k'_s - k_r}) x^{k_r} \\ &> (x^{k_1} a^{n_2} \dots a^{n_r} - n a x^{k'_1} a^{n'_2} \dots a^{n'_s - 1} x^{k'_s - k_r} a) x^{k_r} \\ &= (x^{k_1} a^{n_2} \dots a^{n_r - 1} - n a x^{k'_1} a^{n'_2} \dots a^{n'_s - 1} x^{k'_s - k_r}) a x^{k_r} \geqslant 0 \end{aligned}$$

by the induction hypothesis.

Thus, in each case, we have $z \gg z'$, so II) is established.

III) If $N(z) = N(z')$, $n_1 = n'_1$, $k_1 = k'_1$, $n_2 = n'_2$, ..., $n_m = n'_m$, and $k_m > k'_m$, then $z \gg z'$. The proof is again by induction on $N(z)$. For $N(z) = 1$, there are two cases;

i) $z = x^{k_1} a x^{k_2}$, $z' = x^{k_1} a x^{k'_2}$, with $k_2 > k'_2$. For any integer n,

$$z - n z' = x^{k_1} a x^{k'_2} (x^{k_2 - k'_2} - n) > 0.$$

ii) $z = x^{k_1} a x^{k_2}$, $z' = x^{k'_1} a x^{k'_2}$, with $k_1 > k'_1$. For any integer n,

$$z - n z' = x^{k'_1} (x^{k_1 - k'_1} a x^{k_2} - n a x^{k'_2}) > 0, \text{ since } x^{k_1 - k'_1} a x^{k_2} \notin I, \text{ while } a x^{k'_2} \in I.$$

Now suppose that $t > 1$ and III) holds whenever $N(z) < t$. Let $z, z' \in S$ satisfy the hypothesis of III), with $N(z) = t$. There are two cases:

i) $n_1 \neq 0$. For any integer n,

$$z - n z' = a (a^{n_1 - 1} x^{k_1} a^{n_2} \dots x^{k_r} - n a^{n_1 - 1} x^{k'_1} \dots x^{k'_s}) \geqslant 0$$

by the induction hypothesis.

ii) $n_1 = 0$. We consider two cases:

1) $k_1 = k'_1$, For any integer n,

$$z - n z' = x^{k_1} a (a^{n_2 - 1} x^{k_2} \dots x^{k_r} - n a^{n_2 - 1} x^{k'_2} \dots x^{k'_s}) \geqslant 0$$

by the induction hypothesis.

2) $k_1 > k'_1$. For any integer n,

$$z - n z' = x^{k'_1} (x^{k_1 - k'_1} a^{n_2} \dots x^{k_r} - n a^{n'_2} x^{k'_2} \dots x^{k'_s}) \geqslant 0 \text{ by II)}.$$

Thus, in each case, $z \gg z'$ so III) is established.

I), II), and III) show that the order in T is exactly the order given for the semigroup S in Example 1.1. As noted earlier, this completes the proof of Theorem 1.2.

2. The general question

The result stated in Lemma 1.3 can be stated in the following more general form:

PROPOSITION 2.1. *Every ordered ring that contains a non-zero proper one-sided l-ideal contains a non-zero proper two-sided l-ideal.*

Proof. For ordered rings without non-zero divisors of zero, this follows from Lemma 1.3 and its left analogue. If A is an ordered ring containing non-zero divisors of zero but no non-zero proper l-ideals, then A contains non-zero nilpotents by Theorem I.4.1, hence $Z_2(A) = \{a \in A : a^2 = 0\} = A$, since $Z_2(A)$ is an l-ideal in A (cf. the discussion following Corollary I.3.17). Then, since every l-subgroup of the additive group of A is an l-ideal in A, there are no non-zero proper l-subgroups of the additive group of A. Hence, there are no non-zero proper one-sided l-ideals in A.

As an immediate consequence, we have:

COROLLARY 2.2. *Every maximal l-ideal in an f-ring A is a maximal right (left) l-ideal of A.*

In the remainder of this section, we summarize what is known about the existence of one-sided l-ideals that are not two-sided in the general classes of f-rings that have been considered in this paper.

Recall that the ring A of Example II.6.2 is an ordered ring containing only one non-zero proper two-sided l-ideal (Fz). Hence, A satisfies the descending chain condition for l-ideals, and A is subdirectly irreducible, so we have:

1) There exist f-rings satisfying the descending chain condition for l-ideals that contain one-sided l-ideals that are not two-sided.

2) There exist subdirectly irreducible f-rings that contain one-sided l-ideals that are not two-sided.

Since the ring of Example 1.1 contains no non-zero divisors of zero, we have:

3) There exist f-rings with zero l-radical that contain one-sided l-ideals that are not two-sided.

If A is an f-ring with zero l-radical that satisfies the descending chain condition for l-ideals, then, by Lemma II.5.7, A is isomorphic to a (finite) direct union of l-simple ordered rings $A_1, \dots, A_n$. By Proposition 2.1, each A_i contains no non-zero proper one-sided l-ideals. If I is a right (left) l-ideal in A, then $I \cap A_i$ is a right (left) l-ideal in A_i for $i = 1, \dots, n$. Hence, each $I \cap A_i$ is $\{0\}$ or A_i, so I is clearly a two-sided l-ideal in A. Thus, we have:

4) If A is an f-ring with zero l-radical which satisfies the descending chain condition for l-ideals, then every one-sided l-ideal of A is two-sided.

Since the J-radical of an f-ring always contains the l-radical, we have:

5) If A is a J-semisimple f-ring which satisfies the descending chain condition for l-ideals, then every one-sided l-ideal of A is two-sided.

Our final example, which is due to Professor Henriksen, shows the following:

6) There exist J-semisimple f-rings that contain one-sided l-ideals that are not two-sided.

Example 2.3. Let A be an ordered ring with identity and without non-zero divisors of zero that contains a right l-ideal I that is not two-sided (cf. Section 1). Let $A[\lambda]$ denote the ring of polynomials in one indeterminate λ with coefficients in A, ordered lexicographically with the leading coefficient dominating: $a_0 + a_1\lambda + \cdots + a_n\lambda^n \geqslant 0$ if and only if $a_n > 0$, or $a_n = 0$ and $a_{n-1} > 0$, or ..., or $a_n = a_{n-1} = \cdots = a_1 = 0$ and $a_0 \geqslant 0$. With this ordering, $A[\lambda]$ is an ordered ring with identity and without non-zero divisors of zero; it is clearly l-simple, since $|a\lambda| > 1$ for every $0 \neq a \in A$. Hence, $A[\lambda]$ is an l-primitive f-ring. Observe that A is an ordered sub-ring of $A[\lambda]$ and that, if $y \in A[\lambda]$, then $y \notin A$ implies $|y| > |a|$ for every $a \in A$.

Now let B denote the set of all sequences in $A[\lambda]$ that are eventually in A: if $z \in B$, then $z = (z_1, z_2, \ldots)$, where each $z_i \in A[\lambda]$, and there is an integer n such that $i \geqslant n$ implies $z_i \in A$. Under coordinatewise ring and lattice operations, B is an f-ring; it is clearly a subdirect union of copies of the l-primitive f-ring $A[\lambda]$. Hence, B is a J-semisimple f-ring with identity.

Let $J = \{z \in B: z \text{ is eventually in } I\}$. Since I is a right l-ideal in A that is not two-sided, it is clear that J is a right l-ideal in B that is not two-sided.

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