REMAINDERS: FUNCTIONS OF SEVERAL VARIABLES.

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1. **Introduction.** Many of the remainders $R(x) = Rx$ in the theory of approximations related to a function $x = x(s)$, $0 \leqq s \leqq 1$, are functionals which are linear on C_n or C_{n-1} and zero whenever x is a polynomial in s of degree $n-1$. (Definitions are given in sections 2,3 below.) Accordingly the following known theorems [5; 2; 7; 11] are of importance.

Mass theorem. *Suppose that Rx is a functional linear on C_n and zero for degree $n-1$. Then there is a function $\beta(s) \in \mathcal{V}$ such that*

$$Rx = \int_0^1 x_n(s) d\beta(s) , \qquad x \in C_n .$$

Here $x_n(s)$ stands for the nth derivative of x. A fuller statement is theorem (4:3) below.

Kernel theorem. *Suppose that Rx is a functional linear on C_{n-1}, $n \geqq 1$, and zero for degree $n-1$. Then there is a function $f(s) \in \mathcal{V}$ such that*

$$Rx = \int_0^1 x_n(s) f(s) ds , \qquad x \in C_n .$$

A fuller statement is theorem (4:15).

The present paper extends the above theorems to functionals on spaces of functions of several variables. Theorems (5:11), (6:11), (9:5), and (9:11) afford direct access to integral forms of remainders in terms of partial derivatives of order n, and, furthermore, completely characterize the cases in which the integral forms

[1] The author gratefully acknowledges financial support received from the Office of Naval Research, under contract with Queens College, Flushing, N. Y.

are valid. The integral forms lead, among other things, to appraisals of remainders and to criteria of best approximation (section 8).

The character of our results is indicated by the following theorem, which is a consequence of theorem (6:11) and lemma (6:3). The space $B_{p,q}$ of functions $x = x(s, t)$ is defined in section 6.

Kernel theorem. *Let p, q be positive integers and a, b fixed numbers, $0 \leqq a, b \leqq 1$. Put $n = p+q$. Suppose that Rx is a functional linear on $B_{p-1,q-1}$ and zero whenever x is a polynomial in (s, t) of degree $n-1$. Then there are functions: $f^i(t)$, $i < p$; $g(s, t)$; $h^j(s)$, $j < q$, all in $\mathcal{V}$, such that*

$$Rx = \sum_{i<p} \int_0^1 x_{i,\,n-i}(a, t) f^i(t)dt + \int_0^1 \int_0^1 x_{p,q}(s, t)g(s, t)dsdt$$

$$+ \sum_{j<q} \int_0^1 x_{n-j,j}(s, b)h^j(s)ds\,, \qquad x \in B_{p,q}\,.$$

The terms in this equation cannot be combined since there are functions $x \in B_{p,q}$ for which one and only one of the $n+1$ terms will be different from zero.

An illustration is given in section 7.

Throughout we shall consider spaces of functions defined on the unit hyper-square U^m: the set of points in euclidean m-space with coordinates all between 0 and 1, end values included. Our theorems transform in the natural way to spaces of functions defined on a hyper-rectangle with sides parallel to the axes. The fact that our theorems refer to functions defined on U^m rather than on a general compact metric space leads to the following advantage: the relations defining the kernels and the masses are simpler than they otherwise would be [6; 10].

The functionals Rx to which our theorems apply need not involve all of the unit square U^m; for example, Rx might be defined on a space of functions x defined on a subset of U^m, for then Rx is defined, a fortiori, on a space of functions defined on U^m.

The spaces $B_{(p)}$ and $A_{(p)}$ are interesting in themselves, as the complete core of a function in either space is a unique characterization of the function in terms of independent elements.

2. **Linear functionals. Riesz's theorem.** A *functional Rx*, defined on a space X of elements x, is a correspondence from X to the real numbers. Suppose that X

is a normed linear space. The functional Rx is said to be *linear* if it is additive:

$$R(x+y) = Rx+Ry\,, \qquad x, y \in X\,,$$

and continuous at one point, say the origin:

$$Rx \to 0 \quad \text{whenever} \quad ||x|| \to 0, \qquad x \in X\,.$$

Observe that the additivity implies that $R0 = 0$. In each particular space X that we consider, we define the *norm* $||x||$ explicitly. A linear functional Rx is continous at every point, and homogeneous:

$$R(cx) = cRx\,, \qquad c \text{ a real number}, \qquad x \in X\,.$$

The above usage of the word "linear" is that of F. Riesz and Banach. [1, pp. 23, 26–27, 36–37].

All spaces X that we consider are spaces of functions of real variables; addition and scalar multiplication of elements of the space are to be understood as addition and scalar multiplication of the functions.

The space C_0^m is the space of functions $x = x(s_1, \ldots, s_m)$ continuous on the unit square U^m, with norm

$$||x|| = ||x||_{C_0^m} = \max_{(s) \in U^m} |x(s_1, \ldots, s_m)|\,.$$

A function $\gamma = \gamma(s_1, \ldots, s_m)$ is of *bounded variation* on U^m if it has the following properties: i) For all subdivisions of U^m into a finite number of rectangles with sides parallel to the axes, $\sum |\Delta_1 \ldots \Delta_m \gamma|$ is bounded. ii) For one particular value s_ν^0 of each argument s_ν, γ is a function of bounded variation in the remaining $m-1$ variables. Conditions i) and ii) imply that for any fixed value s_ν^0 of s_ν, $\nu = 1, \ldots, m$, γ is a function of bounded variation in its remaining $m-1$ variables. We use the symbol $\mathcal{V}$ to denote the class of functions of bounded variation in their variables, without regard to the number of variables.

(2:1) **Riesz's theorem.** *Suppose that Rx is a functional linear on C_0^m. Put*

$$\gamma(s_1', \ldots, s_m') = \begin{cases} 0 \quad \textit{if} \quad s_1' \ldots s_m' = 0\,, \\ R\theta_{s_1'}(s_1) \ldots \theta_{s_m'}(s_m) \quad \textit{if} \quad s_1' \ldots s_m' > 0\,, \end{cases} \tag{2:2}$$

where

$$\theta_{s'} = \theta_{s'}(s) = \begin{cases} 1 \quad \text{if} \quad s \leqq s'\,, \\ 0 \quad \text{if} \quad s > s'\,. \end{cases} \tag{2:3}$$

Then $\gamma \in \mathcal{V}$, *and*

$$(2:4) \qquad Rx = \int_0^1 \cdots \int_0^1 x(s_1, \ldots, s_m) d\gamma(s_1, \ldots, s_m), \qquad x \in C_0^m .$$

Conversely, if $\gamma(s_1, \ldots, s_m) \in \mathcal{V}$, (2:4) *defines a functional which is linear on* C_0^m.

The definition (2:2) involves an extension of the definition of R, since the argument of R is not a continuous function. Such an extension can be made in many ways; one such way, which is direct and consistent with the extension of (2:4) as a Lebesgue-Stieltjes integral, is the following. Let

$$(2:5) \qquad \theta_{s', k}(s) , \qquad k = 1, 2, \ldots$$

be, for each s', a sequence of continuous functions which converge *monotonely* to $\theta_{s'}(s)$. Then the sequence $R\theta_{s_1', k_1}(s_1) \ldots \theta_{s_m', k_m}(s_m)$ converges as $(k_1, \ldots, k_m) \to (\infty, \ldots, \infty)$; its limit is taken as the definition of γ for $s_1' \ldots s_m' > 0$. With this definition γ will be continuous from above for positive arguments:

$$(2:6) \qquad \gamma(s_1+0, \ldots, s_m+0) = \gamma(s_1, \ldots, s_m), \; s_1 \ldots s_m > 0 .$$

The following conditions determine γ in (2:4) uniquely: i) $\gamma \in \mathcal{V}$ and vanishes whenever one of its arguments vanishes. ii) (2:4) and (2:6) hold. [8; 9; 4, pp. 262–271; 1, p. 61; 3].

3. **Conventions and further notation.** We say that functions $\alpha(s_1, \ldots, s_m)$ and $\beta(s_1, \ldots, s_m)$ are equal *with countable exceptions*, and we write this:

$$\alpha = \beta , \qquad w. c. e. ,$$

if α and β are equal except when s_ν is one of a countable set of values, $\nu = 1, 2, \ldots,$ *or* m. (Thus α and β are equal except on a countable number of $(m-1)$-planes perpendicular to the axes.) "Countable" is to be understood as "countably infinite, finite, or zero".

Equations marked *, such as (4:8), are to be understood as follows. The function α or β defined in the equation is zero whenever any one of its arguments is zero; the function is as written whenever all its arguments are positive.

We say that a functional Rx *is zero for degree* $n-1$ if $Rx = 0$ whenever x is a polynomial of degree $n-1$ in all its variables.

Exponents will indicate, not powers, but powers divided by the factorial of the exponent:

$$s^0 = 1,\ s^1 = s,\ s^2 = s \cdot s/2, \ldots, s^m = s^{m-1} \cdot s/m, \ldots .$$

Indices are non-negative integers, except where otherwise indicated.

4. **Functions of one variable.** Let C_n be the space of functions $x = x(s)$ with continuous nth derivative, $s \in U^1$, the norm being

$$||x|| = ||x||_{C_n} = \max_{\substack{s \in U^1 \\ i=0,\ldots,n}} |x_i(s)| .$$

For consistency of notation with later sections, we define the spaces B_n, A_n as follows:

$$B_n = C_n;\ A_n = B_{n-1} = C_{n-1}, \qquad n \geqq 1 .$$

Let a be a fixed number $0 \leqq a \leqq 1$.

(4:1) **Taylor's formula on B_n.** *If $x \in B_n$, $n \geqq 1$, then*

$$x(s) = \sum_{i<n} (s-a)^i x_i(a) + \int_a^s (s-\bar{s})^{n-1} x_n(\bar{s}) d\bar{s} , \qquad s \in U^1 .$$

More generally (4:1) holds if x has an absolutely continuous $(n-1)$th derivative.

The *core* of a function x in B_n is the nth derivative $x_n(s)$; the *complete core* is the core together with the n numbers $x_i(a)$, $i < n$. Taylor's formula (4:1) may be used to express all the derivatives of x of order less than n in terms of the complete core of x. Hence $||x||_{B_n}$ and the maximum of the absolute values of the elements in the complete core of x are equivalent norms in the sense that each is at most a constant times the other.

The complete core of $x \in B_n$ may be thought of as the independent part of x and as an independent variable.

(4:2) **Lemma.** $B_n \subset A_n = B_{n-1}, \qquad n \geqq 1 .$
$||x||_{B_n} \geqq ||x||_{A_n}, \qquad x \in B_n .$

A functional linear on A_n is a fortiori linear on B_n.

(4:3) **Mass theorem on B_n.** *Suppose that Rx is a functional linear on B_n and zero for degree $n-1$. Then there is a function $\beta(s) \in \mathcal{V}$ such that*

(4:4) $$Rx = \int_0^1 x_n(s) d\beta(s) , \qquad x \in B_n .$$

Conversely, given a function $\beta \in \mathcal{V}$, (4:4) defines a functional which is linear on B_n and zero for degree $n-1$.

The mass β may be defined as follows:

$$\beta(s') = R\int_a^s (s-\bar{s})^{n-1}\theta_{s'}(\bar{s})d\bar{s}\,, \qquad n \geqq 1\,, \qquad *\,, \tag{4:5}$$

$$\beta(s') = R\theta_{s'}(s)\,, \qquad n = 0\,, \qquad *\,. \tag{4:6}$$

The asterisks * indicate that the formulas (4:5), (4:6) apply for $s' > 0$ and that $\beta(0) = 0$. We call β a mass even though β may be decreasing.

Proof. The theorem is a corollary to Riesz's theorem (2:1) for functions of one variable. For $n = 0$, the theorem is Riesz's theorem itself. For $n \geqq 1$, Taylor's formula (4:1) implies that

$$Rx = R\int_a^s (s-\bar{s})^{n-1}x_n(\bar{s})d\bar{s}\,, \qquad x \in B_n\,, \tag{4:7}$$

since Rx is zero for degree $n-1$. Now the second member of (4:7) is defined and additive for $x_n \in C_0^1$. Furthermore $||x_n||_{C_0^1} \to 0$ implies that

$$\left\|\int_a^s (s-\bar{s})^{n-1}x_n(\bar{s})d\bar{s}\right\|_{B_n} \to 0$$

and hence that $Rx \to 0$. Hence the second member of (4:7) is a linear functional on C_0^1 for $x_n \in C_0^1$. Hence by Riesz's theorem (4:4) and (4:5) hold.

As in Riesz's theorem the relations (4:5), (4:6) involve an extension of Rx onto a space that includes the argument of R in (4:5), (4:6). The extension may be made in many ways; for preciseness we understand (4:5) as a *definition* by monotone limits:

$$\beta(s') = \lim_{k\to\infty} R\int_a^s (s-\bar{s})^{n-1}\theta_{s',k}(\bar{s})d\bar{s}\,, \qquad *\,, \tag{4:8}$$

where $\theta_{s',k}(s)$ is a sequence (2:5). The limit (4:8) exists and $\beta(s'+0) = \beta(s')$, $s' > 0$. The relation (4:6) is understood similarly. This completes the proof, as the converse part is immediate.

If the functional Rx is linear on B_{n-1}, $n \geqq 1$, (hence on B_n), the relation (4:5) may be used without (4:8), since the argument of R in (4:5) is an element of B_{n-1}. The relations (4:5) and (4:8) will then be consistent, since

$$\lim_{k\to\infty}\left\|\int_a^s (s-\bar{s})^{n-1}\theta_{s'}(\bar{s})d\bar{s} - \int_a^s (s-\bar{s})^{n-1}\theta_{s',k}(\bar{s})d\bar{s}\right\|_{B_{n-1}} = 0\,. \tag{4:9}$$

Furthermore (4:5) implies that

$$(4:10)\qquad \beta(s') = R(s-s')^n\theta_{s'}(s)\,.$$

For, an asterisk is not needed after (4:10), since the argument of R in (4:10), when $s' = 0$, is $s^n\theta_0(s) = 0$ for all $s \geqq 0$. Also,

$$(4:11)\quad \int_a^s (s-\bar{s})^{n-1}\theta_{s'}(\bar{s})d\bar{s} = (s-s')^n\theta_{s'}(s)+[(s-a)^n-(s-s')^n]\theta_{s'}(a), \qquad n \geqq 1\,,$$

as may be seen by writing $\int_a^s = \int_{s'}^s + \int_a^{s'}$. Since the last term of (4:11) is a polynomial in s of degree $n-1$, (4:5) does indeed imply (4:10).

(4:12) **An extension of *R*.** Once the function β has been defined, we may extend the R of theorem (4:3) as follows. Put

$$(4:13)\qquad R^+x = \int_{U1} x_n(s)d\beta(s)\,, \qquad x \in B_n^+\,,$$

where B_n^+ is the space of functions x with nth derivative Lebesgue-Stieltjes integrable relative to β and, if $n \geqq 1$, with absolute by continuous $(n-1)$th derivative. Then $B_n \subset B_n^+$ and $Rx = R^+x$ for $x \in B_n$. Furthermore

$$(4:14)\qquad \beta(s') = R^+\int_a^s (s-\bar{s})^{n-1}\theta_{s'}(\bar{s})d\bar{s}\,, \qquad n \geqq 1, \quad w.\,c.\,e., \qquad *\,.$$

For, the argument of R^+ in (4:14) has the nth derivative $\theta_{s'}(s)$ for $s \neq s'$ and therefore is an element of B_n^+ except when $s = s'$ is a discontinuity of $\beta(s)$. Hence by (4:13), the second member of (4:14) equals $\int_{U1}\theta_{s'}(s)d\beta(s) = \beta(s')$ whenever s' is a continuity of β, that is, with only countable exceptions. By (4:11), (4:14) may be written

$$\beta(s') = R^+(s-s')^n\theta_{s'}(s)\,, \qquad w.\,c.\,e., \qquad *\,.$$

(Here the case $n = 0$ is valid also).

(4:15) **Kernel theorem.** *Suppose that Rx is a functional linear on $A_n = B_{n-1}$, $n \geqq 1$, and zero for degree $n-1$. Then there is a function $f(s) \in \mathcal{V}$ such that*

$$(4:16)\qquad Rx = \int_{U1} x_n(s)f(s)ds\,, \qquad x \in B_n^{++},$$

where B_n^{++} is the space of functions x with absolutely continuous $(n-1)$th *derivative,* *Conversely, given a function $f(s) \in \mathcal{V}$,* (4:16) *defines a functional which can be extended so as to be linear on A_n and zero for degree $n-1$.*

The kernel $f(s)$ may be defined as follows:

$$f(\bar{s}) = R^+(s-\bar{s})^{n-1}\psi_{a,s}(\bar{s}) , \qquad w. c. e., \qquad * , \tag{4:17}$$

where

$$\psi_{a,s}(\bar{s}) = \theta_{\bar{s}}(a) - \theta_{\bar{s}}(s) = \begin{cases} 1 & \text{if } a \leqq \bar{s} < s , \\ -1 & \text{if } s \leqq \bar{s} < a , \\ 0 & \text{otherwise} ; \end{cases} \tag{4:18}$$

$\theta_{\bar{s}}$ is defined in (2:3), *and R^+ is the extension* (4:12) *with n replaced by $n-1$.*

Observe that $B_n \subset B_n^{++} \subset A_n$; so that, in particular, (4:16) *holds for $x \in B_n$.* Actually B_n^{++} is now the same space as B_n^+ defined in (4:12), but this fact is not used in the proof or the application of the present theorem.

The relation (4:17) may be written in the following alternative forms, often useful for calculation:

$$f(\bar{s}) = -R^+(s-\bar{s})^{n-1}\theta_{\bar{s}}(s) = R^+(s-\bar{s})^{n-1}[1-\theta_{\bar{s}}(s)] , \qquad w. c. e., \qquad * , \tag{4:19}$$

since R^+x is zero for degree $n-1$.

If Rx is linear on A_{n-1}, $n \geqq 2$, the relations (4:17 and 19) hold with R^+ replaced by R and with *w. c. e.* deleted.

Proof: One proof is to apply the mass theorem (4:3) on B_{n-1} and thereafter integrate by parts [7; 11]. Here we follow Peano's original suggestion [5] as this leads to the simplest proof in the case of several variables.

By (4:18) Taylor's formula (4:1) may be written

$$x(s) = \sum_{i<n} (s-a)^i x_i(a) + \int_{U^1} (s-\bar{s})^{n-1}\psi_{a,s}(\bar{s}) x_n(\bar{s}) d\bar{s} , \qquad x \in B_n^{++} .$$

Hence

$$Rx = R\int_{U^1} d\bar{s}(s-\bar{s})^{n-1}\psi_{a,s}(\bar{s}) x_n(\bar{s}) , \qquad x \in B_n^{++} . \tag{4:20}$$

Now

$$Rx = \int_0^1 d\beta(s) x_{n-1}(s) , \qquad x \in B_{n-1} , \tag{4:21}$$

where $\beta \in \mathcal{V}$, by the mass theorem (4:3) with n replaced by $n-1$. Apply (4:21) to (4:20). Then

$$Rx = \int_0^1 d\beta(s) \int_{U^1} d\bar{s}\psi_{a,s}(\bar{s}) x_n(\bar{s}) = \int_{U^1} d\bar{s} x_n(\bar{s}) \int_{U^1} d\beta(s)\psi_{a,s}(\bar{s}) , \qquad x \in B_n^{++} ,$$

by Fubini's theorem since $x_n(\bar{s})\psi_{a,s}(\bar{s})$ is integrable $d\bar{s}d\beta(s)$. Hence

$$Rx = \int_{U^1} d\bar{s}x_n(\bar{s})f(\bar{s}) , \qquad x \in B_n^{++},$$

where

$$(4:22) \quad f(\bar{s}) = \int_{U^1} \psi_{a,s}(\bar{s})d\beta(s) = \int_{U^1} [\theta_{\bar{s}}(a) - \theta_{\bar{s}}(s)]d\beta(s) = -\int_{U^1} \theta_{\bar{s}}(s)d\beta(s) = -\beta(\bar{s}+0) ,$$

since $\int_0^1 d\beta(s) = 0$ by (4:21) with x equal to the polynomial s^{n-1}. Since $\beta \in \mathcal{V}$, $f \in \mathcal{V}$ also. This proves the first part of the theorem.

Let R^+ be the extension (4:12) of R with n replaced by $n-1$:

$$R^+x = \int_{U^1} x_{n-1}(s)d\beta(s) , \qquad x \in B_{n-1}^+ .$$

Now $\psi_{a,s}(\bar{s})$ as a function of s is constant except when $s = \bar{s}$, by (4:18). Hence $(s-\bar{s})^{n-1}\psi_{a,s}(\bar{s}) \in B_{n-1}^+$ except when $\bar{s}$ is a discontinuity of β. Hence

$$R^+(s-\bar{s})^{n-1}\psi_{a,s}(\bar{s}) = \int_{U^1} \psi_{a,s}(\bar{s})d\beta(s) = f(\bar{s}) , \qquad w.\ c.\ e. ,$$

by (4:22). This establishes (4:17).

To prove the converse, suppose that $f(s) \in \mathcal{V}$. Define the functional R as

$$Rx = \int_{U^1} f(s)dx_{n-1}(s) = f(1)x_{n-1}(1) - f(0)x_{n-1}(0) + \int_0^1 x_{n-1}(s)df(s) .$$

This functional is linear on A_n, reduces to (4:16) for $x \in B_n^{++}$, and vanishes for degree $n-1$.

This completes the proof of the theorem.

(4:23) **Remark.** Suppose that a functional R^*x is linear on B_n or A_n but not necessarily zero for degree $n-1$. One can construct a related functional Rx which will be zero for degree $n-1$ as follows:

$$Rx = R^*x - \sum_{i<n} c_i x_i(a) ,$$

where $c_i = R^*(s-a)^i$, $i < n$. That Rx is zero for degree $n-1$ follows from the fact that $x(s) = \sum_{i<n} (s-a)^i x_i(a)$ whenever x is a polynomial of degree $n-1$. Now Rx is linear on B_n or A_n whenever R^*x is. Thus the mass or kernel theorems may be

applied to Rx to give a form for R^*x:

$$R^*x = Rx + \sum_{i<n} c_i x_i(a) \,.$$

5. **Functions of two variables. The space $B_{p,q}$.** Let a, b be fixed numbers, $0 \leqq a, b \leqq 1$; and let p, q be fixed non-negative integers. Put

$$n = p+q \,.$$

We define the space $B_{p,q}$ as the space of functions $x = x(s, t)$ whose derivatives

$$\begin{aligned} &x_{i,n-i}(a,t)\,, \quad i<p\,,\\ &x_{p,q}(s,t)\,,\\ &x_{n-j,j}(s,b)\,, \quad j<q\,, \end{aligned} \tag{5:1}$$

exist and are continuous in t, (s, t), s, respectively, $(s, t) \in U^2$. Here $x_{i,j}$ stands for the partial derivative $\partial^{i+j}x/\partial s^i \partial t^j$ according to the following convention as to the order of differentiation:

$$x_{i,j} = \begin{cases} \partial^{i-p}x_{p,j}/\partial s^{i-p} & \text{if} \quad i>p,\ j \leqq q\,,\\ \partial^{j-q}x_{i,q}/\partial t^{j-q} & \text{if} \quad i \leqq p,\ j>q\,,\\ \partial^{i+j-n}x_{p,q}/\partial s^{i-p}\partial t^{j-q} & \text{if} \quad i>p,\ j>q\,, \end{cases} \tag{5:2}$$

the order of differentiation being otherwise unrestricted. Thus in the last case of (5:2), the differentiations in $x_{p,q}$ may be in any order, and the last $i+j-n$ differentiations may be in any order. Also, the order in $x_{i,j}$ is unrestricted if $i \leqq p$, $j \leqq q$. We shall describe (5:2) as the convention of $B_{p,q}$.

The *core* of a function x in $B_{p,q}$ is the set of derivatives (5:1); the *complete core* of x is the core together with the numbers $x_{i,j}(a, b)$, $i+j < n$. Thus the complete core consists of 1 function of two variables; n functions of one variable, and $n(n+1)/2$ constants. In order for the core of $x \in B_{p,q}$ to exist certain derivatives of x of lower order must exist and be continuous; we say that the latter derivatives are *covered* by the core.

Thus for $B_{p,q}$, the derivatives in the core or covered by the core are the following:

$$\begin{aligned} &x_{i,j}(s,t)\,, && i \leqq p\,, && j \leqq q\,;\\ &x_{i,j}(a,t)\,, && i+j \leqq n\,, && j>q\,;\\ &x_{i,j}(s,b)\,, && i+j \leqq n\,, && i>p\,. \end{aligned} \tag{5:3}$$

The *norm* of x in $B_{p,q}$, denoted by $||x||_{B_{p,q}}$, is defined as the maximum of the absolute values of the derivatives (5:3), $(s, t) \in U^2$.

(5:4) **Taylor's formula on $B_{p,q}$.** *If* $x \in B_{p,q}$,

$$x(s,t) = \sum_{i+j<n} (s-a)^i (t-b)^j x_{i,j}(a,b) + \sum_{i<p} (s-a)^i \int_b^t (t-\bar{t})^{n-i-1} x_{i,n-i}(a,\bar{t}) d\bar{t}$$

$$+ \sum_{j<q} (t-b)^j \int_a^s (s-\bar{s})^{n-j-1} x_{n-j,j}(\bar{s}, b) d\bar{s} + T, \qquad (s,t) \in U^2,$$

where

$$T = \begin{cases} \int_a^s (s-\bar{s})^{p-1} d\bar{s} \int_b^t (t-\bar{t})^{q-1} x_{p,q}(\bar{s},\bar{t}) d\bar{t}, & p \geqq 1, \quad q \geqq 1; \\ \int_a^s (s-\bar{s})^{p-1} x_{p,0}(\bar{s}, t) d\bar{s}, & p \geqq 1, \quad q = 0; \\ \int_b^t (t-\bar{t})^{q-1} x_{0,q}(s,\bar{t}) d\bar{t}, & p = 0, \quad q \geqq 1; \\ x_{0,0}(s,t), & p = q = 0. \end{cases}$$

Proof. Suppose that $p \geqq 1, q \geqq 1$; $x \in B_{p,q}$. By Taylor's formula (4:1),

$$x(s,t) = \sum_{j<q} (t-b)^j x_{0,j}(s,b) + \int_b^t (t-\bar{t})^{q-1} x_{0,q}(s,\bar{t}) d\bar{t}, \tag{5:5}$$

$$x_{0,j}(s,b) = \sum_{i<n-j} (s-a)^i x_{i,j}(a,b) + \int_a^s (s-\bar{s})^{n-j-1} x_{n-j,j}(\bar{s},b) d\bar{s}, \quad j < q, \tag{5:6}$$

$$x_{0,q}(s,\bar{t}) = \sum_{i<p} (s-a)^i x_{i,q}(a,\bar{t}) + \int_a^s (s-\bar{s})^{p-1} x_{p,q}(\bar{s},\bar{t}) d\bar{s}. \tag{5:7}$$

We will substitute (5:6), (5:7) in (5:5). This will lead among other things to terms involving

$$\int_b^t (t-\bar{t})^{q-1} x_{i,q}(a,\bar{t}) d\bar{t}, \qquad i < p.$$

Now this quantity vanishes at $t = b$, together with its derivatives as to t of order less than q. Its derivative of order j, $j \geqq q$, is $x_{i,j}(a,t)$. Accordingly, by Taylor's formula (4:1),

$$\int_b^t (t-\bar{t})^{q-1} x_{i,q}(a,\bar{t}) d\bar{t} = \sum_{q \leqq j < n-i} (t-b)^j x_{i,j}(a,b) + \int_b^t (t-\bar{t})^{n-i-1} x_{i,n-i}(a,\bar{t}) d\bar{t}, \qquad i < p. \tag{5:8}$$

We now obtain (5:4) by substituting (5:6), (5:7) in (5:5) and using (5:8).

The other cases of (5:4) are established similarly and more simply.

Taylor's formula (5:4) expresses x in terms of its complete core. Applied to the derivatives covered by the core, (5:4) and (4:1) express these derivatives also in terms of the complete core. Hence $||x||_{B_{p,q}}$ and the maximum of the absolute values of the elements in the complete core of x are equivalent norms in the sense that each is at most a constant times the other.

For each $x \in B_{p,q}$ there is a complete core. Conversely, *given* $n(n+1)/2$ *constants*: $c^{i,j}$, $i+j < n$; p *continuous functions of* t: $z^{i,n-i}(t)$, $i < p$; q *continuous functions of* s: $z^{n-j,j}(s)$, $j < q$; *and* 1 *continuous function of* (s, t): $z^{p,q}(s, t)$; *there is a unique function* $x \in B_{p,q}$ *having these elements as its complete core*:

$$
\begin{aligned}
x_{i,j}(a, b) &= c^{i,j}, & i+j &< n, \\
x_{i,n-i}(a, t) &= z^{i,n-i}(t), & i &< p, \\
x_{p,q}(s, t) &= z^{p,q}(s, t), \\
x_{n-j,j}(s, b) &= z^{n-j,j}(s), & j &< q.
\end{aligned}
\tag{5:9}
$$

This may be established as follows. Put the given elements into the second member of (5:4), using (5:9). The second member of (5:4) then defines the function x. The convention as to the order of differentiation enters here. To illustrate the point, consider a typical term:

$$y = y(s, t) = (s-a)^{p-1}\int_b^t (t-\bar{t})^q z^{p-1,q+1}(\bar{t})d\bar{t}, \qquad p \geqq 2,\ q \geqq 0.$$

Then $y_{0,q+1}(s, t) = (s-a)^{p-1}z^{p-1,q+1}(t)$. Since $z^{p-1,q+1}$ is merely continuous no further differentiations as to t are necessarily possible here. However $y_{p-2,q+2}(a, t)$, for example, exists. For

$$y_{p-2,q+2}(a, t) = \frac{\partial^2}{\partial t^2} y_{p-2,q}(a, t) = \frac{\partial^2}{\partial t^2}\left[(s-a)\int_b^t z^{p-1,q+1}(\bar{t})d\bar{t}\right]_{s=a} = 0.$$

The reader may verify that $y \in B_{p,q}$ and that the complete core of y vanishes, except for $y_{p-1,q+1}(a, t)$ which equals $z^{p-1,q+1}(t)$.

An essential point here is that each integral in (5:4) is differentiable at least p times as to s and q times as to t.

Thus the complete core of $x \in B_{p,q}$ may be thought of as the independent part of x and as an independent variable.

(5:10) **Lemma.**
$$B_{p,q} \subset B_{p-1,q}, \qquad p \geqq 1.$$
$$||x||_{B_{p,q}} \geqq ||x||_{B_{p-1,q}}, \qquad x \in B_{p,q}.$$

A functional linear on $B_{p-1,q}$ *is a fortiori linear on* $B_{p,q}$.

Likewise for $B_{p,q-1}$, $q \geqq 1$.

The conventions as to differentiation in $B_{p,q}$ and $B_{p-1,q}$ are consistent with one another; both are to be followed at the same time where necessary.

(5:11) **Mass theorem on** $\boldsymbol{B_{p,q}}$. *Suppose that* Rx *is a functional linear on* $B_{p,q}$ *and zero for degree* $n-1$. *Then there are functions* $\beta^{i,j} \in \mathcal{V}$ *such that*

$$Rx = \sum_{i<p} \int_0^1 x_{i,n-i}(a, t)d\beta^{i,n-i}(t) + \int_0^1\int_0^1 x_{p,q}(s, t)d\beta^{p,q}(s, t) + \sum_{j<q} \int_0^1 x_{n-j,j}(s, b)d\beta^{n-j,j}(s)\,, \qquad x \in B_{p,q}\,. \tag{5:12}$$

Conversely, given masses $\beta^{i,j} \in \mathcal{V}$, (5:12) *defines a functional which is linear on* $B_{p,q}$ *and zero for degree* $n-1$.

The masses $\beta^{i,j}$ *may be defined as follows*:

$$\beta^{i,n-i}(t') = R(s-a)^i \int_b^t (t-\bar t)^{n-i-1}\theta_{t'}(\bar t)d\bar t\,, \qquad i<p\,, \qquad *\,,$$

$$\beta^{n-j,j}(s') = R(t-b)^j \int_a^s (s-\bar s)^{n-j-1}\theta_{s'}(\bar s)d\bar s\,, \qquad j<q\,, \qquad *\,,$$

$$\beta^{p,q}(s', t') = \begin{cases} R\int_a^s (s-\bar s)^{p-1}\theta_{s'}(\bar s)d\bar s \int_b^t (t-\bar t)^{q-1}\theta_{t'}(\bar t)d\bar t\,, & p \geqq 1, q \geqq 1\,, \quad *\,, \\ R\theta_{t'}(t)\int_a^s (s-\bar s)^{p-1}\theta_{s'}(\bar s)d\bar s\,, & p \geqq 1,\ q=0\,, \quad *\,, \\ R\theta_{s'}(s)\int_b^t (t-\bar t)^{q-1}\theta_{t'}(\bar t)d\bar t\,, & p=0, q\geqq 1, \quad *\,, \\ R\theta_{s'}(s)\theta_{t'}(t)\,, & p=q=0\,, \quad *\,. \end{cases} \tag{5:13}$$

The asterisks indicate that the relations (5:13) hold for positive arguments only, and that each mass $\beta^{i,j}$ is zero if one of its arguments is zero. A theorem of C. A. Fischers's [2] is related to the present theorem for $p=q=1$ and Remark (6:18) below.

Proof. The proof is similar to that of the mass theorem on B_n. Consider the case $p \geqq 1, q \geqq 1$. By Taylor's formula (5:4),

$$Rx = \sum_{i<p} R(s-a)^i \int_b^t (t-\bar t)^{n-i-1} x_{i,n-i}(a, \bar t)d\bar t + R\int_a^s (s-\bar s)^{p-1}d\bar s \int_b^t (t-\bar t)^{q-1} x_{p,q}(\bar s, \bar t)d\bar t + \sum_{j<q} R(t-b)^j \int_a^s (s-\bar s)^{n-j-1} x_{n-j,j}(\bar s, b)d\bar s, \qquad x \in B_{p,q}\,. \tag{5:14}$$

Since the core of $x \in B_{p,q}$ may be any continuous functions, the terms of (5:14) are defined and additive on the spaces of continuous functions of t, (s, t), and s, respectively, the independent variables being the core of x. Furthermore each term of (5:14) is continuous on C_0^1 or C_0^2, since Rx is continuous on $B_{p,q}$. Hence Riesz's theorem applies to each term, and (5:12), (5:13) are established.

The other cases are treated similarly.

As in (4:5 and 6), we understand the definitions (5:13) as definitions by monotone limits. For example,

$$\beta^{p,q}(s', t') = \lim_{k, l \to (\infty, \infty)} R \int_a^s (s-\bar{s})^{p-1}\theta_{s', k}(\bar{s})d\bar{s} \int_b^t (t-\bar{t})^{q-1}\theta_{t', l}(\bar{t})d\bar{t}\,, \quad p \geqq 1, \quad q \geqq 1\,, \quad * \,, \tag{5:15}$$

where $\theta_{s, k}(s)$ is a sequence (2:5).

If the functional Rx is linear on $B_{p-1,q}$, $p \geqq 1$, (hence on $B_{p,q}$), those relations (5:13) in which the argument of R is an element of $B_{p-1,q}$ (that is, the first two relations of (5:13)) may be used directly, consistently with their interpretation as definitions by monotone limits. This is proved as was the similar fact for B_{n-1} and B_n (cf. (4:9)). Likewise for $B_{p,q-1}$, $q \geqq 1$. If Rx is linear on $B_{p-1,q-1}$, $p \geqq 1$, $q \geqq 1$, all the relations (5:13) may, equivalently, be used directly. In these cases the relations (5:13) may be transformed by the use of (4:11). In particular, the masses $\beta^{i,j}$ that are functions of a single variable take on the simpler form:

$$\begin{aligned} \beta^{i,n-i}(t') &= R(s-a)^i(t-t')^{n-i}\theta_{t'}(t)\,, \qquad i < p\,, \\ \beta^{n-j,j}(s') &= R(s-s')^{n-j}(t-b)^j\theta_{s'}(s)\,, \qquad j < q\,. \end{aligned} \tag{5:16}$$

(5:17) **An extension of *R*.** Once the masses $\beta^{i,j}$ have been defined we may extend the R of theorem (5:11) as follows. Put

$$\begin{aligned} R^+x = \sum_{i<p} \int_{U^1} x_{i,n-i}(a, t)d\beta^{i,n-i}(t) + \iint_{U^2} x_{p,q}(s, t)d\beta^{p,q}(s, t) \\ + \sum_{j<q} \int_{U^1} x_{n-j,j}(s, b)d\beta^{n-j,j}(s)\,, \qquad x \in B^+_{p,q}\,, \end{aligned} \tag{5:18}$$

where $B^+_{p,q}$ is the space of functions x for which Taylor's formula (5:4) holds with integrable nth derivatives and whose derivatives in (5:18) are Lebesgue-Stieltjes integrable relative to their corresponding masses $\beta^{i,j}$. Then $B_{p,q} \subset B^+_{p,q}$, and $Rx = R^+x$ for $x \in B_{p,q}$. Furthermore the relations (5:13), with R replaced by R^+, are valid with countable exceptions [13]; the relations (5:13) thus modified may be transformed by the use of (4:11).

6. **Functions of two variables. The space $A_{p,q}$, $p \geqq 1$, $q \geqq 1$.**

In order to obtain a kernel theorem with converse, we introduce the space $A_{p,q}$, $p \geqq 1$, $q \geqq 1$, defined as follows: $A_{p,q}$ is the space of functions $x = x(s, t)$ whose derivatives

$$(6:1) \qquad \begin{array}{ll} x_{i,\, n-i-1}(a, t)\,, & i < p-1\,, \\ x_{p-1,\, q-1}(s, t)\,, & \qquad\qquad n = p+q\,, \\ x_{n-j-1,\, j}(s, b)\,, & j < q-1\,, \end{array}$$

exist and are continuous in t, (s, t), s, respectively, $(s, t) \in U^2$. The order of differentiation in $x_{i,j}$ shall be restricted according to the convention of $B_{p-1,\, q-1}$.

The *core* of a function x in $A_{p,q}$ is the set of derivatives (6:1); the *complete core* of x is the core together with the numbers $x_{i,j}(a, b)$, $i+j < n-1$, $(i, j) \neq (p-1, q-1)$. In order for the core of $x \in A_{p,q}$ to exist, certain derivatives of x of lower order must exist and be continuous; we say that the latter derivatives are *covered* by the core. Thus in $A_{p,q}$ the derivatives covered by the core are precisely the following: All the derivatives in the core of $B_{p-1,\, q-1}$, except $x_{p-1,\, q-1}(s, t)$, and all the derivatives covered by the core of $B_{p-1,\, q-1}$.

The norm of x in $A_{p,q}$, denoted by $||x||_{A_{p,q}}$, is defined as the maximum of the absolute values of the derivatives of x in the core or covered by the core, $(s, t) \in U^2$.

(6:2) **Taylor's formula on $A_{p,q}$.** *If* $x \in A_{p,q}$,

$$x(s, t) = \sum_{\substack{i+j<n-1 \\ (i,j) \neq (p-1,\, q-1)}} (s-a)^i (t-b)^j x_{i,j}(a, b) + \sum_{i<p-1} (s-a)^i \int_b^t (t-\bar{t})^{n-i-2} x_{i,\, n-i-1}(a, \bar{t}) d\bar{t}$$

$$+ \sum_{j<q-1} (t-b)^j \int_a^s (s-\bar{s})^{n-j-2} x_{n-j-1,\, j}(\bar{s}, b) d\bar{s} + T\,, \qquad (s, t) \in U^2\,,$$

where

$$T = \begin{cases} \displaystyle\int_a^s (s-\bar{s})^{p-2} d\bar{s} \int_b^t (t-\bar{t})^{q-2} x_{p-1,\, q-1}(\bar{s}, \bar{t}) d\bar{t}\,, & p \geqq 2, \quad q \geqq 2\,, \\ \displaystyle\int_a^s (s-\bar{s})^{p-2} x_{p-1,\, 0}(\bar{s}, t) d\bar{s}\,, & p \geqq 2, \quad q = 1\,, \\ \displaystyle\int_b^t (t-\bar{t})^{q-2} x_{0,\, q-1}(s, \bar{t}) d\bar{t}\,, & p = 1, \quad q \geqq 2\,, \\ x_{0,\, 0}(s, t)\,, & p = q = 1\,. \end{cases}$$

This formula is established in the same way as Taylor's formula (5:4) on $B_{p,q}$. Taylor's formula (6:2) expresses x in terms of its complete core in $A_{p,q}$. Applied to the derivatives covered by the core, (6:2) and (4:1) express these derivatives also in terms of the complete core. Hence $||x||_{A_{p,q}}$ and the maximum of the absolute values of the elements in the complete core of x are equivalent norms in the sense that each is at most a constant times the other.

For each $x \in A_{p,q}$ there is a complete core. Conversely, *given constants*: $c^{i,j}$, $i+j < n-1$, $(i,j) \neq (p-1, q-1)$; *continuous functions of* t: $z^{i,n-i-1}(t)$, $i < p-1$; *continuous functions of* s: $z^{n-j-1,j}(s)$, $j < q-1$; *and a continuous function of* (s,t): $z^{p-1,q-1}(s,t)$; *there is a unique function* $x \in A_{p,q}$ *having these elements as its complete core.* This is understood and established precisely as was the similar fact (5:9) for $B_{p,q}$.

Thus the complete core of $x \in A_{p,q}$ may be thought of as the independent part of x and as an independent variable.

(6:3) **Lemma.**
$$B_{p,q} \subset A_{p,q} \subset B_{p-1,q-1}\,.$$
$$||x||_{B_{p,q}} \geqq ||x||_{A_{p,q}} \geqq ||x||_{B_{p-1,q-1}}\,.$$

A functional linear on $A_{p,q}$ *is a fortiori linear on* $B_{p,q}$; *a functional linear on* $B_{p-1,q-1}$ *is a fortiori linear on* $A_{p,q}$.

(6:4) **Lemma.**
$$A_{p,q} \subset A_{p-1,q}\,, \qquad p \geqq 2\,.$$
$$||x||_{A_{p,q}} \geqq ||x||_{A_{p-1,q}}\,.$$

A functional linear on $A_{p-1,q}$ *is a fortiori linear on* $A_{p,q}$.

Likewise for $A_{p,q-1}$, $\quad q \geqq 2$.

(6:5) **Mass theorem on $A_{p,q}$.** *Suppose that* Rx *is a functional linear on* $A_{p,q}$ *and zero whenever* x *is a polynomial in* (s,t) *of degree* $n-2$ *that is free of the term* $s^{p-1}t^{q-1}$. *Then there are functions* $\alpha^{i,j} \in \mathcal{V}$ *such that*

$$(6:6) \qquad Rx = \sum_{i<p-1} \int_0^1 x_{i,n-i-1}(a,t)\,d\alpha^{i,n-i-1}(t) + \int_0^1\!\!\int_0^1 x_{p-1,q-1}(s,t)\,d\alpha^{p-1,q-1}(s,t)$$
$$+ \sum_{j<q-1} \int_0^1 x_{n-j-1,j}(s,b)\,d\alpha^{n-j-1,j}(s)\,, \qquad x \in A_{p,q}\,.$$

Conversely, given masses $\alpha^{i,j} \in \mathcal{V}$, (6:6) *defines a functional which is linear on* $A_{p,q}$ *and zero whenever* $x(s,t)$ *is a polynomial of degree* $n-2$ *that is free of the term* $s^{p-1}t^{q-1}$.

The masses $\alpha^{i,j}$ *may be defined as follows*:

$$\alpha^{i,\,n-i-1}(t') = R(s-a)^i \int_b^t (t-\bar{t})^{n-i-2}\theta_{t'}(\bar{t})d\bar{t}\,, \qquad i < p-1\,, \qquad *\,,$$

$$\alpha^{n-j-1,\,j}(s') = R(t-b)^j \int_a^s (s-\bar{s})^{n-j-2}\theta_{s'}(\bar{s})d\bar{s}\,, \qquad j < q-1\,, \qquad *\,,$$

(6:7)

$$\alpha^{p-1,\,q-1}(s',t') = \begin{cases} R\int_a^s (s-\bar{s})^{p-2}\theta_{s'}(\bar{s})d\bar{s}\int_b^t (t-\bar{t})^{q-2}\theta_{t'}(\bar{t})d\bar{t}\,, & p \geqq 2,\, q \geqq 2\,, \quad *\,, \\ R\theta_{t'}(t)\int_a^s (s-\bar{s})^{p-2}\theta_{s'}(\bar{s})d\bar{s}\,, & p \geqq 2,\, q = 1\,, \quad *\,, \\ R\theta_{s'}(s)\int_b^t (t-\bar{t})^{q-2}\theta_{t'}(\bar{t})d\bar{t}\,, & p = 1,\, q \geqq 2\,, \quad *\,, \\ R\theta_{s'}(s)\,\theta_{t'}(t)\,, & p = q = 1\,, \quad *\,. \end{cases}$$

The proof is similar to that of the mass theorem (5:11) on $B_{p,q}$.

The definitions (6:7) are understood as definitions by monotone limits (cf. (5:15)).

If the functional Rx is linear on $A_{p-1,q}$, $p \geqq 2$, (hence on $A_{p,q}$), the first two relations (6:7) may be used directly, consistently with their interpretation as definitions by monotone limits. Likewise for $A_{p,q-1}$, $q \geqq 2$. If Rx is linear on $A_{p-1,q-1}$, $p \geqq 2$, $q \geqq 2$, all the relations (6:7) may, equivalently, be used directly. In these cases, the relations (6:7) may be transformed by (4:11). In particular,

$$\text{(6:8)} \qquad \begin{aligned} \alpha^{i,\,n-i-1}(t') &= R(s-a)^i(t-t')^{n-i-1}\theta_{t'}(t)\,, \qquad i < p-1\,; \\ \alpha^{n-j-1,\,j}(s') &= R(s-s')^{n-j-1}(t-b)^j\theta_{s'}(s)\,, \qquad j < q-1\,. \end{aligned}$$

(6:9) **An extension of R.** Once the masses $\alpha_{i,j}$ have been defined, we may extend the R of theorem (6:5) as follows. Put

$$\text{(6:10)} \qquad \begin{aligned} R^+x = \sum_{i<p-1}\int_{U1} x_{i,\,n-i-1}(a,t)d\alpha^{i,\,n-i-1}(t) + \iint_{U2} x_{p-1,\,q-1}(s,t)d\alpha^{p-1,\,q-1}(s,t) \\ + \sum_{j<q-1}\int_{U1} x_{n-j-1,\,j}(s,b)d\alpha^{n-j-1,\,j}(s)\,, \qquad x \in A^+_{p,q}\,, \end{aligned}$$

where $A^+_{p,q}$ is the space of functions x for which Taylor's formula (6:2) holds with integrable derivatives and whose derivatives in (6:10) are Lebesgue-Stieltjes integrable relative to their corresponding masses $\alpha^{i,j}$. Then $A_{p,q} \subset A^+_{p,q}$ and $Rx = R^+x$ for $x \in A_{p,q}$. Furthermore, the relations (6:7), with R replaced by R^+, are valid with countable exceptions; the relations (6:7) thus modified may be transformed by the use of (4:11).

(6:11) **Kernel theorem.** *Suppose that Rx is a functional linear on $A_{p,q}$ and zero for degree $n-1$. Then there are functions $f^{i,j}$ such that*

$$(6:12)\qquad Rx = \sum_{i<p} \int_{U^1} x_{i,\,n-i}(a,\,t) f^{i,\,n-i}(t)dt + \iint_{U^2} x_{p,\,q}(s,\,t) f^{p,\,q}(s,\,t)dsdt$$
$$+ \sum_{<q} \int_{U^1} x_{n-j,j}(s,\,b) f^{n-j,\,j}(s)ds\,, \qquad x \in B^{++}_{p,\,q}\,,$$

where $B^{++}_{p,q}$ is the space of functions x whose $B_{p,q}$ core derivatives are integrable and for which Taylor's formula (5:4) holds. The kernels $f^{i,j} \in \mathcal{V}$. The two particular kernels $f^{p-1,\,q+1}(t)$ and $f^{p+1,\,q-1}(s)$ vanish at 0 and 1 and are integrals of functions in $\mathcal{V}$.

Conversely, given a set of kernels $f^{i,j}$ with the properties listed in the preceding paragraph, (6:12) defines a functional Rx which can be extended so as to be linear on $A^{p,q}$ and zero for degree $n-1$.

The kernels may be defined as follows:

$$(6:13)\qquad \begin{aligned} f^{i,\,n-i}(\bar{t}) &= R^+(s-a)^i(t-\bar{t})^{n-i-1}\psi_{b,\,t}(\bar{t}), \quad i < p, \quad w.\,c.\,e., \\ f^{n-j,\,j}(\bar{s}) &= R^+(s-\bar{s})^{n-j-1}(t-b)^j\psi_{a,\,s}(\bar{s}), \quad j < q\,, \quad w.\,c.\,e., \\ f^{p,\,q}(\bar{s},\,\bar{t}) &= R^+(s-\bar{s})^{p-1}(t-\bar{t})^{q-1}\psi_{a,\,s}(\bar{s})\psi_{b,\,t}(\bar{t})\,, \quad w.\,c.\,e., \end{aligned}$$

where R^+ is the extension (6:9) of R, and ψ is defined in (4:18).

Observe that $B_{p,\,q} \subset B^{++}_{p,\,q} \subset A_{p,q}$; so that, in particular, (6:12) *holds for* $x \in B_{p,q}$. Actually $B^{++}_{p,q}$ is now the same as $B^{+}_{p,\,q}$ defined in (5:18), but this fact is not used in the proof or the application of the present theorem.

The relations (6:13) may be written in alternative forms, by the use of (4:18). In particular,

$$(6:14)\qquad f^{i,\,n-i}(\bar{t}) = -R^+(s-a)^i(t-\bar{t})^{n-i-1}\theta_{\bar{t}}(t) =$$
$$R^+(s-a)^i(t-\bar{t})^{n-i-1}[1-\theta_{\bar{t}}(t)], \quad i < p\,, \quad w.\,c.\,e\,,$$

with the dual relation for $f^{n-j,\,j}(\bar{s})$, $\quad j < q$.

If Rx is linear on $A_{p-1,\,q-1}$, $p \geqq 2$, $q \geqq 2$, the relations (6:13 and 14) hold with the $+$ and $w.\,c.\,e.$ deleted.

Observe that Rx is surely linear on $A_{p,\,q}$ if it is linear on $B_{p-1,\,q-1}$, by lemma (6:3).

Proof of theorem. Since Rx is linear on $A_{p,\,q}$, we may write Rx in the form (6:6).

Suppose that $x \in B^{++}_{p,\,q}$. Taylor's formula (5:4), (4:18), and the fact that Rx is zero for degree $n-1$ imply that

$$Rx = \sum_{i<p} R(s-a)^i \int_{U^1} (t-\bar{t})^{n-i-1} \psi_{b,t}(\bar{t}) x_{i,n-i}(a, \bar{t}) d\bar{t} \tag{6:15}$$

$$+R \int\int_{U^2} (s-\bar{s})^{p-1}(t-\bar{t})^{q-1} \psi_{a,s}(\bar{s})\psi_{b,t}(\bar{t}) x_{p,q}(\bar{s}, \bar{t}) d\bar{s} d\bar{t}$$

$$+ \sum_{j<q} R(t-b)^j \int_{U^1} (s-\bar{s})^{n-j-1} \psi_{a,s}(\bar{s}) x_{n-j,j}(\bar{s}, b) d\bar{s} , \qquad x \in B^{++}_{p,q} .$$

The derivation of (6:12) from (6:15) amounts to the interchange of R and the integral operator in each term of (6:15). Consider one of the terms, for example:

$$W = R \int\int_{U^2} d\bar{s} d\bar{t} (s-\bar{s})^{p-1}(t-\bar{t})^{q-1} \psi_{a,s}(\bar{s})\psi_{b,t}(\bar{t}) x_{p,q}(\bar{s}, \bar{t}) .$$

The argument of R in W is an element of $A_{p,q}$ whose core in $A_{p,q}$ is zero except for its derivative of index $p-1, q-1$. Hence, by (6:6) and Fubini's theorem,

$$W = \int_0^1 \int_0^1 d\alpha^{p-1,q-1}(s, t) \int\int_{U^2} d\bar{s} d\bar{t} \psi_{a,s}(\bar{s})\psi_{b,t}(\bar{t}) x_{p,q}(\bar{s}, \bar{t}) =$$

$$\int\int_{U^2} d\bar{s} d\bar{t} x_{p,q}(\bar{s}, \bar{t}) \int\int_{U^2} d\alpha^{p-1,q-1}(s, t) \psi_{a,s}(\bar{s})\psi_{b,t}(\bar{t}) =$$

$$\int\int_{U^2} x_{p,q}(\bar{s}, \bar{t}) f(\bar{s}, \bar{t}) d\bar{s} d\bar{t} ,$$

where

$$f(\bar{s}, \bar{t}) = \int\int_{U^2} \psi_{a,s}(\bar{s})\psi_{b,t}(\bar{t}) d\alpha^{p-1,q-1}(s, t) . \tag{6:16}$$

It follows from (6:16) that $f(s,t) \in \mathcal{V}$, since $\alpha^{p-1,q-1} \in \mathcal{V}$. (One may actually evaluate the integral (6:16), by using (4:18)).

Furthermore, $f^{p,q}$ defined in (6:13) equals f with countable exceptions. For, put

$$y = y(s, t) = (s-\bar{s})^{p-1}(t-\bar{t})^{q-1} \psi_{a,s}(\bar{s})\psi_{b,t}(\bar{t}) .$$

As a function of s, $\psi_{a,s}(\bar{s})$ is continuous and constant except at $s = \bar{s}$, by (4:18). Also $\psi_{a,a}(\bar{s}) = 0$. Hence the derivative $y_{p-1,q-1}(s, t)$ exists and equals $\psi_{a,s}(\bar{s})\psi_{b,t}(\bar{t})$ when $s \neq \bar{s}$, $t \neq \bar{t}$; and the other $A_{p,q}$ core derivatives of y exist and vanish. Now $\psi_{a,s}(\bar{s})\psi_{b,t}(\bar{t})$, $s \neq \bar{s}$, $t \neq \bar{t}$, is integrable $d\alpha^{p-1,q-1}(s, t)$ except when $\bar{s}$ or $\bar{t}$ is a discontinuity of $\alpha^{p-1,q-1}$. Also Taylor's formula (6:2) applies to y, since $(s-\bar{s})^{p-1}\psi_{a,s}(\bar{s})$ is an element of B^{++}_{p-1}. As $\alpha^{p-1,q-1}(s, t)$ has discontinuities at only countably many s and t [13], it follows that $y \in A^{+}_{p,q}$ with countable exceptions. Hence, by (6:10),

$$f^{p,q}(\bar{s}, \bar{t}) = R^+y = \iint_{U^2} \psi_{a,s}(\bar{s})\psi_{b,t}(\bar{t})d\alpha^{p-1,q-1}(s, t) = f(\bar{s}, \bar{t}) , \qquad w.\,c.\,e.\,.$$

A similar treatment of the other terms of (6:15) establishes (6:12), (6:13).

We now show that $f^{p-1,q+1}(t)$ vanishes at 0 and 1 and is an integral of a function in $\mathcal{V}$. The kernel $f^{p+1,q-1}(s)$ is treated similarly. Thus,

$$f^{p-1,q+1}(\bar{t}) = R(s-a)^{p-1}(t-\bar{t})^q\psi_{b,t}(\bar{t}) \; ; \tag{6:17}$$

we write R instead of R^+ as in (6:13) because the argument of R in (6:17) is an element of $A_{p,q}$; we have suppressed the $w.\,c.\,e.$, as we may. Note that

$$f^{p-1,q+1}(0) = R(s-a)^{p-1}t^q[\theta_0(b)-\theta_0(t)] = 0-0 = 0 \, ,$$

since $t\theta_0(t)$ is zero for all $t \geqq 0$. Also (4:18), (6:6), and Fubini's theorem imply that

$$f^{p-1,q+1}(\bar{t}) = -R(s-a)^{p-1}(t-\bar{t})^q\theta_{\bar{t}}(t) = -\int_0^1\int_0^1 (t-\bar{t})\theta_{\bar{t}}(t)d\alpha^{p-1,q-1}(s, t) =$$

$$\int_0^1\int_0^1 d\alpha^{p-1,q-1}(s, t)\int_0^{\bar{t}} d\tilde{t}\theta_{\tilde{t}}(t) = \int_0^{\bar{t}} d\tilde{t}h(\tilde{t}) \; ,$$

where

$$h(\tilde{t}) = \iint_{U^2} \theta_{\tilde{t}}(t)d\alpha^{p-1,q-1}(s, t) = \alpha^{p-1,q-1}(1, \tilde{t}+0) \, .$$

Thus $h \in \mathcal{V}$, since $\alpha \in \mathcal{V}$. Finally, $f^{p-1,q+1}(1) = 0$ by (6:17) since $\psi_{b,t}(1) = 0$.

It remains to prove the converse. We consider the terms of (6:12) separately and integrate by parts. For example, given that $f^{p,q} \in \mathcal{V}$, consider the term

$$W = \iint_{U^2} x_{p,q}(s, t)f^{p,q}(s, t)dsdt = \iint_{U^2} f^{p,q}(s, t)dx_{p-1,q-1}(s, t) =$$

$$f^{p,q}(1, 1)x_{p-1,q-1}(1, 1)-f^{p,q}(1, 0)x_{p-1,q-1}(1, 0)-f^{p,q}(0, 1)x_{p-1,q-1}(0, 1)$$

$$+f^{p,q}(0, 0)x_{p-1,q-1}(0, 0)-\int_0^1 x_{p-1,q-1}(1, t)df^{p,q}(1, t) + \int_0^1 x_{p-1,q-1}(0, t)df^{p,q}(0, t)$$

$$-\int_0^1 x_{p-1,q-1}(s, 1)df^{p,q}(s, 1) + \int_0^1 x_{p-1,q-1}(s, 0)df^{p,q}(s, 0)$$

$$+\int_0^1\int_0^1 x_{p-1,q-1}(s, t)df^{p,q}(s, t) \, , \qquad x \in B^{++}_{p,q} \, .$$

The last member of this equation is a linear functional on $A_{p,q}$, which reduces to W for $x \in B^{++}_{p,q}$ and vanishes for degree $n-1$.

The other terms of (6:12) are treated similarly but more simply, except for the two neighbors of W. Consider one of these. Given that $f^{p-1,q+1}(t)=\int_0^t h(\tilde{t})d\tilde{t}$ and vanishes at 1, where $h\in\mathcal{V}$. Then

$$\int_{U^1} x_{p-1,q+1}(a,t)f^{p-1,q+1}(t)dt=-\int_{U^1} x_{p-1,q}(a,t)h(t)dt=-\int_0^1 h(t)dx_{p-1,q-1}(a,t)=$$
$$-h(1)x_{p-1,q-1}(a,1)+h(0)x_{p-1,q-1}(a,0)+\int_0^1 x_{p-1,q-1}(a,t)dh(t)\,,\qquad x\in B^{++}_{p,q}\,.$$

But the last expression defines a functional linear on $A_{p,q}$ and zero for degree $n-1$.

(6:18) **Remark.** Suppose that a functional R^*x is linear on $B_{p,q}$ or $A_{p,q}$ but not necessarily zero for degree $n-1$. As in (4:23), one may construct a related functional Rx which will be zero for degree $n-1$:

$$Rx=R^*x-\sum_{i+j<n} c_{i,j}x_{i,j}(a,b)\,,$$

where $c_{i,j}=R^*(s-a)^i(t-b)^j$, $i+j<n$. The functional Rx is linear on $B_{p,q}$ if R^*x is. Note, however, that given R^*x linear on $A_{p,q}$, Rx will be linear on $A_{p,q}$ if and only if $c_{p-1,q}=c_{p,q-1}=0$.

7. **Illustration.** A simple illustration of the kernel theorem (6:11) is the following. The fundamental rectangle is $-1\leqq s,\ t\leqq 1$;

$$Rx=x(1,1)-x(-1,1)+x(1,-1)-x(-1,-1)-4x_{1,0}(0,0)\,.$$

Thus $Rx/4$ is, among other things, the remainder in the approximation of the derivative $x_{1,0}(0,0)$ by the indicated combination of the corner values of x.

The functional Rx vanishes for degree 2; Rx is linear on $B_{1,0}$, hence on $A_{2,1}$, for all (a,b); Rx is linear on $B_{0,1}$, hence on $A_{1,2}$, if and only if $b=0$.

The form of Rx on $B^{++}_{2,1}$ is .

(7:1) $$Rx=\int_{[-1,1]} x_{0,3}(a,t)f^{0,3}(t)dt+\int_{[-1,1]} x_{1,2}(a,t)f^{1,2}(t)dt+\iint_{-1\leqq s,t\leqq 1} x_{2,1}(s,t)f^{2,1}(s,t)dsdt$$
$$+\int_{[-1,1]} x_{3,0}(s,b)f^{3,0}(s)ds\,,\qquad x\in B^{++}_{2,1}\,,$$

where, for all a, b,

$$f^{0,3}(t)=0,\ f^{1,2}(t)=2(1-|t|),\ f^{3,0}(s)=2(1-|s|)^2\,,$$

and

$$f^{2,1}(s,t)=\begin{cases}(1-|s|)\text{ signum }(st) & \text{if } a=b=0\,,\\ 1-s-4\lambda & \text{if } a=b=-1\,,\end{cases}$$

λ equaling one if $s<0$, $t<0$ and zero otherwise.

The form of Rx on $B^{++}_{1,2}$, $(a,b)=(0,0)$, is

$$Rx=\int\limits_{[-1,1]} x_{0,3}(0,t)g^{0,3}(t)dt+\iint\limits_{-1\leqq s,t\leqq 1} x_{1,2}(s,t)g^{1,2}(s,t)dsdt$$
$$+\int\limits_{[-1,1]} x_{2,1}(s,0)g^{2,1}(s)ds+\int\limits_{[-1,1]} x_{3,0}(s,0)g^{3,0}(s)ds\,,\qquad x\in B^{++}_{1,2}\,, \tag{7:2}$$

where

$$g^{0,3}(t)=g^{2,1}(s)=0;\quad g^{1,2}(s,t)=1-|t|;\quad g^{3,0}(s)=2(1-|s|)^2\,.$$

The coefficients in $f^{3,0}$ and $g^{3,0}$ are 2 instead of 1 because of our convention about exponents.

8. **Appraisals and best approximations.** If Rx is a remainder, we may be interested in its appraisal. We may appraise the separate terms in (5:12) or in (6:12) in the customary ways. Such separate appraisals are efficient, since the elements of the core of x in $B_{p,q}$ are independent.

Alternatively we may appraise (6:12) as follows by a generalization of Hölder's inequality. Let $K^{i,j}$ be the set on which the kernel $f^{i,j}$ is not zero; let $|K^{i,j}|$ be the measure of $K^{i,j}$ in the t, (s,t), or s space, as is appropriate; and let $K=\sum\limits_{i+j=n}|K^{i,j}|$. Exclude the case $K=0$, for then $Rx=0$ and no appraisal is needed. Then

$$|Rx|\leqq\frac{1}{K^{1/e}}\left[\sum_{i<p}\int\limits_{K^{i,n-i}}|x_{i,n-i}(a,t)|^e dt+\iint\limits_{K^{p,q}}|x_{p,q}(s,t)|^e dsdt\right.$$
$$\left.+\sum_{j<q}\int\limits_{K^{n-j,j}}|x_{n-j,j}(s,b)|^e ds\right]^{1/e}M_e\,,\qquad x\in B^{++}_{p,q}\,, \tag{8:1}$$

where

$$M_e=K^{1/e}\left[\sum_{i<p}\int\limits_{K^{i,n-i}}|f^{i,n-i}(t)|^{e'}dt+\iint\limits_{K^{p,q}}|f^{p,q}(s,t)|^{e'}dsdt\right.$$
$$\left.+\sum_{j<q}\int\limits_{K^{n-j,j}}|f^{n-j,j}(s)|^{e'}ds\right]^{1/e'},\qquad \frac{1}{e}+\frac{1}{e'}=1\,,\qquad e>1\,,$$

and exponents are understood as powers in the ordinary sense. Now M_e is independent of x, and the multiplier of M_e in (8:1) is an average of the absolute value of the core derivatives of x in $B_{p,q}$. Fix e and n. Suppose that one has the choice of several formulas (6:12) (which may be different formulas for the same remainder Rx or formulas for different remainders, all in terms of nth derivatives.) Suppose that one intends to appraise by (8:1) and that one has no reason to prefer the average of the absolute value of the nth derivatives on one class of sets $K^{i,j}$ to the average on another class of sets $K^{i,j}$. Then it is reasonable to say that that formula (6:12) is *best* which minimizes M_e [12].

In the usual way we may admit the values $e = 1$ and $e = \infty$ in (8:1). The appraisal (8:1) may be adjusted to assign different weights to the core derivatives.

In practice, the calculation of M_2 is often decisively simpler than that of M_e, $e \neq 2$.

For the illustrations of section 7, M_2 has the following values: $(176/5)^{1/2}$ for (7:1) with $a = b = 0$; $(496/5)^{1/2}$ for (7:1) with $a = b = -1$; $(52/5)^{1/2}$ for (7:2).

In a similar fashion, we may appraise (5:12) by a generalization of the theorem of the mean:

$$(8:2)\quad |R_x| \leqq \max_{i<p,\, j<q}\ \operatorname*{ess\,sup}_{(s,t)\in U^2} \left[|x_{i,\,n-i}(a,t)|\,,\quad |x_{p,q}(s,t)|\,,\quad |x_{n-j,\,j}(s,b)|\right] M\,, \qquad x \in B^+_{p,q}\,,$$

where the essential supremum of each derivative is taken relative to its corresponding mass $\beta^{i,j}$, and

$$M = \sum_{i<p} \int_{U^1} |d\beta^{i,\,n-i}(t)| + \iint_{U^2} |d\beta^{p,\,q}(s,t)| + \sum_{j<q} \int_{U^1} |d\beta^{n-j,\,j}(s)|\,.$$

If the appraisal (8:2) is to be used on a class of formulas (5:12), it is reasonable to say that that formula is *best* for which M is least.

The appraisals (8:1 and 2) are efficient in the sense that any reduction of their second members would make the inequalities false. This is true of (8:2) because, by construction, the masses $\beta^{i,j}$ are continuous on the right for positive arguments.

9. **Functions of m variables.** The results of the previous section extend to functions of several variables. For the most part the proofs are direct generalizations of the earlier proofs. Here we state the principle facts. For $m = 1$ or 2 the concepts and theorems of the present section are those of the preceding sections.

Let $(a) = (a_1, \ldots, a_m)$ be a fixed point in U^m; and let $(p) = (p_1, \ldots, p_m)$ be fixed non-negative integers. Put

$$n = p_1 + \cdots + p_m\,.$$

We define the space $B_{(p)} = B_{p_1,\dots,p_m}$ as the space of functions $x = x(s) = x(s_1,\dots, s_m)$ whose derivatives

$$x_{i_1,\dots,i_m}(\sigma_1,\dots,\sigma_m),\ i_1+\cdots+i_m = n\ , \tag{9:1}$$

all exist and are continuous in their variables on U^m, where σ_ν, $\nu = 1,\dots, m$, is either the constant a_ν or the variable s_ν, according to the following inductive rule, which we call the rule of $B_{p_1,\dots,p_m}$:

$$\tag{9:2} \begin{cases} \text{For } m = 1,\ \sigma_1 = s_1\ . \\ \text{For } m > 1,\ (\sigma_1,\dots,\sigma_{m-1}) = (a_1,\dots,a_{m-1}) \text{ and } \sigma_m = s_m \text{ if } i_m > p_m\ ; \\ (\sigma_1,\dots\sigma_{m-1}) \text{ is determined by the rule of } B_{p_1,\dots,p_{m-1}} \text{ if } i_m \leqq p_m \text{ and} \\ \sigma_m = s_m \text{ if } i_m = p_m,\ \sigma_m = a_m \text{ if } i_m < p_m\ . \end{cases}$$

We shall use the rule (9:2) to determine $(\sigma) = (\sigma_1,\dots, \sigma_m)$, given $i_1,\dots, i_m$, in cases in which $i_1+\cdots+i_m \geqq p_1+\cdots+p_m$. The order of differentiation in $x_{j_1,\dots,j_m}$, when x is considered as an element of $B_{p_1,\dots,p_m}$, is restricted as follows. Put $q_\nu = \min\,(p_\nu, j_\nu)$, $\nu = 1,\dots, m$. Then $x_{j_1,\dots,j_m}$ is to be understood as a derivative of $x_{q_1,\dots,q_m}$; all orders of differentiations in $x_{q_1,\dots,q_m}$ are allowed and all orders of the remaining differentiations necessary to carry $x_{q_1,\dots,q_m}$ into $x_{j_1,\dots,j_m}$ are allowed.

The *core* of a function x in $B_{(p)}$ is the set of derivatives (9:1); the *complete core* of x is the core together with the numbers $x_{j_1,\dots,j_m}(a_1,\dots, a_m)$, $j_1+\cdots+j_m < n$. (Cf. Illustration (9:8) below.) In order for the core of $x \in B_{(p)}$ to exist, certain derivatives of x of lower order must exist and be continuous; we say that the latter derivatives are *covered* by the core.

The *norm* of x in $B_{(p)}$, denoted by $||x||_{B_{(p)}}$, is defined as the maximum of the absolute values of the derivatives of x in the core or covered by the core, $(s) \in U^m$.

A barycentric diagram of the derivatives of order n is useful in considering the core of x in $B_{(p)}$.

(9:3) **Taylor's formula on $B_{(p)}$.** *If* $x \in B_{(p)}$,

$$x(s) = \sum_{i_1+\cdots+i_m<n} (s_1-a_1)^{i_1}\dots(s_m-a_m)^{i_m} x_{i_1,\dots,i_m}(a) + \sum_{i_1+\cdots+i_m=n} I_1 I_2\dots I_m x_{i_1,\dots,i_m}(\tau)\ ,\qquad (s) \in U^m\ ,$$

where $(\tau) = (\tau_1,\dots, \tau_m)$ *is defined in terms of* (σ), *determined by* (9:2), *as follows:*

$$\tau_\nu = \begin{cases} \bar{s}_\nu & \textit{if}\quad \sigma_\nu = s_\nu \quad \textit{and}\quad i_\nu \geqq 1\ , \\ \sigma_\nu & \textit{otherwise}; \end{cases}$$

and the operators I_ν *are defined as follows:*

$$I_\nu = \begin{cases} (s_\nu - a_\nu)^{i_\nu} & \textit{if} \quad \tau_\nu = a_\nu , \\ \int_{a_\nu}^{s_\nu} (s_\nu - \bar{s}_\nu)^{i_\nu - 1} d\bar{s}_\nu & \textit{if} \quad \tau_\nu = \bar{s} , \qquad \nu = 1, \dots, m . \\ 1 & \textit{if} \quad \tau_\nu = s_\nu . \end{cases}$$

(9:4) **Lemma.** *Suppose that* $p_\nu \geqq 1$, $\nu = 1, \dots,$ *or* m. *Then*

$$B_{p_1, \dots, p_\nu, \dots, p_m} \subset B_{p_1, \dots, p_\nu - 1, \dots, p_m} ;$$

the norm on the larger space is at most the norm on the smaller; a functional linear on the larger space is a fortiori linear on the smaller.

(9:5) **Mass theorem on $B_{(p)}$.** *Suppose that Rx is a functional linear on $B_{(p)}$ and zero for degree $n-1$. Then there are functions $\beta^{i_1, \dots, i_m} \in \mathcal{V}$ such that*

$$\text{(9:6)} \qquad Rx = \sum_{i_1 + \cdots + i_m = n} \int_0^1 \cdots \int_0^1 x_{i_1, \dots, i_m}(\sigma_1, \dots, \sigma_m) d\beta^{i_1, \dots, i_m}(\sigma_1, \dots, \sigma_m) , \qquad x \in B_{(p)} ,$$

where each integral is relative to the independent variables in (σ) and (σ) is determined by (9:2).

Conversely, given functions $\beta^{i_1, \dots, i_m}(\sigma) \in \mathcal{V}$, (9:6) *defines a functional which is linear on $B_{(p)}$ and zero for degree $n-1$.*

The masses $\beta^{i_1, \dots, i_m}$ may be defined as follows:

$$\text{(9:7)} \qquad \beta^{i_1, \dots, i_m}(\sigma') = R\varphi_1 \varphi_2 \cdots \varphi_m , \qquad * ,$$

where

$$\varphi_\nu = \begin{cases} (s_\nu - a_\nu)^{i_\nu} & \text{if} \quad \sigma_\nu = a_\nu , \\ \int_{a_\nu}^{s_\nu} (s_\nu - \bar{s}_\nu)^{i_\nu - 1} \theta_{s_\nu'}(\bar{s}_\nu) d\bar{s}_\nu & \text{if} \quad \sigma_\nu = s_\nu \quad \text{and} \quad i_\nu \geqq 1 , \\ \theta_{s_\nu'}(s_\nu) & \text{if} \quad \sigma_\nu = s_\nu \quad \text{and} \quad i_\nu = 0 , \end{cases}$$

$\nu = 1, \dots, m$; (σ) *is determined by* (9:2), *and* $(\sigma') = (\sigma)$ *with each* s_ν *replaced by* s_ν'.

The relation (9:7) is to be understood as a definition by monotone limits. If R is linear on $B_{p_1 - 1, \dots, p_m - 1}$, $p_\nu \geqq 1$, $\nu = 1, \dots, m$, the relations (9:7) may, equivalently, be understood directly.

We define the space $A_{(p)} = A_{p_1, \dots, p_m}$, $p_\nu \geqq 1$, $\nu = 1, \dots, m$, as follows. The *tentative core* of x in $A_{(p)}$ is the set of derivatives $x_{j_1, \dots, j_m}(\sigma)$, where, for each $i_1, \dots, i_m$ such that $i_1 + \cdots + i_m = n$, (σ) is determined by (9:2) and

$$j_\nu = \begin{cases} i_\nu & \text{if} \quad \sigma_\nu = a_\nu , \\ i_\nu - 1 & \text{if} \quad \sigma_\nu = s_\nu . \end{cases}$$

Certain derivatives in the tentative core can be derived from derivatives of lower order in the tentative core by differentiations and substitutions of the type a_ν for s_ν. We exclude such derivatives. The *core* of x in $A_{(p)}$ consists of the tentative core less the excluded derivatives. Derivatives are understood according to the convention of $B_{p_1-1,\ldots,p_m-1}$ as to the order of differentiation. The space $A_{(p)}$ consists of all functions $x = x(s_1, \ldots, s_m)$ whose core in $A_{(p)}$ exists and is continuous in U^m.

(9:8) **Illustration.** For $m = 3$, we use the alphabetical notation: $(s, t, u) = (s_1, s_2, s_3)$, $(a, b, c) = (a_1, a_2, a_3)$, $(p, q, r) = (p_1, p_2, p_3)$. The core of x in $B_{1,1,2}$ consists of the derivatives:

$$\begin{array}{lllll} & x_{0,0,4}(a, b, u), & x_{0,1,3}(a, b, u), & x_{1,0,3}(a, b, u), & \\ & x_{0,2,2}(a, t, u), & x_{1,1,2}(s, t, u), & x_{2,0,2}(s, b, u), & \\ & x_{0,3,1}(a, t, c), & x_{1,2,1}(a, t, c), & x_{2,1,1}(s, t, c), & x_{3,0,1}(s, b, c), \\ x_{0,4,0}(a, t, c), & x_{1,3,0}(a, t, c), & x_{2,2,0}(a, t, c), & x_{3,1,0}(s, t, c), & x_{4,0,0}(s, b, c) . \end{array}$$

The core of x in $A_{1,1,2}$ consists of the derivatives:

$$x_{0,0,1}(s, t, u), \qquad x_{2,0,0}(s, t, c), \qquad x_{0,3,0}(a, t, c), \qquad x_{1,2,0}(a, t, c) .$$

(9:9) **Lemma.** $B_{p_1,\ldots,p_m} \subset A_{p_1,\ldots,p_m} \subset B_{p_1-1,\ldots,p_m-1}$.

The norms on these three spaces are non-increasing, from left to right. A functional linear on X, one of the above spaces, is a fortiori linear on the above subspaces of X.

(9:10) **Lemma.** *Suppose that $p_\nu \geqq 2$, $\nu = 1, \ldots,$ or m. Then*

$$A_{p_1,\ldots,p_\nu,\ldots,p_m} \subset A_{p_1,\ldots,p_\nu-1,\ldots,p_m};$$

the norm on the larger space is at most the norm on the smaller; a functional linear on the larger space is a fortiori linear on the smaller.

(9:11) **Kernel theorem.** *Suppose that Rx is a functional linear on $A_{(p)}$ and zero for degree $n-1$. Then there are functions $f^{i_1,\ldots,i_m}$ such that*

$$(9:12) \quad Rx = \sum_{i_1+\cdots+i_m=n} \int\cdots\int_{U^w} x_{i_1,\ldots,i_m}(\sigma) f^{i_1,\ldots,i_m}(\sigma)\, d\sigma_1 \ldots d\sigma_m \,, \qquad x \in B^{++}_{(p)} \,,$$

where (σ) is determined by (9:2), $d\sigma_\nu = 1$ if $\sigma_\nu = a_\nu$, w is the number of variables in (σ); and the space $B^{++}_{(p)}$ is the space of functions x whose $B_{(p)}$ core derivatives are integrable and for which Taylor's formula (9:3) holds. The kernels $f^{i_1,\ldots i_m}$ are in $\mathcal{V}$ and *have certain other properties not described here.* (*But see* (9:14) *for a complete description in the case* $m = 3$.)

Conversely, given a set of kernels with these properties, (9:12) *defines a functional* Rx *which can be extended so as to be linear on* $A_{(p)}$ *and zero for degree* $n-1$.

The kernels may be defined as follows:

$$f^{i_1,\dots,i_m}(\bar{\sigma}) = R^+\omega_1\dots\omega_m\,, \qquad w.\,c.\,e\,, \tag{9:13}$$

where

$$\omega_\nu = \begin{cases} (s_\nu - a_\nu)^{i_\nu} & \textit{if} \quad \sigma_\nu = a_\nu\,, \\ (s_\nu - \bar{s}_\nu)^{i_\nu - 1}\psi_{a_\nu, s_\nu}(\bar{s}_\nu) & \textit{if} \quad \sigma_\nu = s_\nu\,, \end{cases} \qquad \nu = 1,\dots,m\,;$$

and $(\bar{\sigma}) = (\sigma)$ *with* s_ν *replaced by* $\bar{s}_\nu$. *Here* R^+ *is the extension of* R *on* $A^+_{(p)}$, *analagous to* (6:9).

If Rx is linear on $A_{p_1-1,\dots,p_m-1}$, $p_\nu \geqq 2$, $\nu = 1,\dots,m$; the relations (9:13) hold with the $+$ and $w.\,c.\,e.$ deleted.

(9:14) **Description of the kernels $f^{i,j,k}$ in theorem (9:11) in the case $m = 3$.** Throughout $i+j+k = n$; $f^{i,j,k} \in \mathcal{V}$. If $k = r+1$, $f^{i,j,k}(\bar{u})$ vanishes at 0 and 1 and is the integral of a function in $\mathcal{V}$. Likewise for $f^{i,j,k}(\bar{t})$ if $k = r-1$, $j \geqq q+1$ or if $j = q+1$, $k < r-1$. Likewise for $f^{i,j,k}(\bar{s})$ if $k = r-1$, $j \leqq q-1$ or if $j = q-1$, $k < r-1$. Furthermore $f^{p-1,q-1,r+2}(\bar{u})$ vanishes, together with its first derivative, at 0 and 1 and is the two-fold integral of a function in $\mathcal{V}$. Likewise for $f^{p-1,q+2,r-1}(\bar{t})$ and $f^{p+2,q-1,r-1}(\bar{s})$. Lastly, put

$$\begin{aligned} h^{\mathrm{I}} &= h^{\mathrm{I}}(\bar{t}, \bar{u}) = f^{p-1,q+1,r}(\bar{t}, \bar{u}) + \psi_{b,0}(\bar{t}) f^{p-1,q,r+1}_{\bar{u}}(\bar{u})\,, \\ h^{\mathrm{II}} &= h^{\mathrm{II}}(\bar{s}, \bar{u}) = f^{p+1,q-1,r}(\bar{s}, \bar{u}) + \psi_{a,0}(\bar{s}) f^{p,q-1,r+1}_{\bar{u}}(\bar{u})\,, \\ h^{\mathrm{III}} &= h^{\mathrm{III}}(\bar{s}, \bar{t}) = f^{p+1,q,r-1}(\bar{s}, \bar{t}) + \psi_{a,0}(\bar{s}) f^{p,q+1,r-1}_{\bar{t}}(\bar{t})\,. \end{aligned}$$

Then $h^{\mathrm{I}}(1, \bar{u}) = 0$, $h^{\mathrm{I}}(0, \bar{u}) = -f^{p-1,q,r+1}_{\bar{u}}(\bar{u})$, and $h^{\mathrm{I}}(\bar{t}, \bar{u})$ is the integral with respect to $\bar{t}$ of a function in $\mathcal{V}$ as regards $(\bar{t}, \bar{u})$. Likewise for h^{II} and h^{III}.

(9:15) **Remark.** *For* $m \geqq 3$, *there are spaces similar to* $B_{(p)}$, $A_{(p)}$, *but different from them, for which mass and kernel theorems are valid.* In fact, if $m = 3$, there are $(p+1)!\,(q+1)!\,(r+1)!$ different spaces, all analagous to $B_{p,q,r}$ with origin (a, b, c). The core in these spaces consists of $x_{p,q,r}(s, t, u)$ and other derivatives as follows. Throughout $i+j+k = n$. For each $k < r$ there is a pair of indices i_k, j_k such that $i_k \geqq p$, $j_k \geqq q$. The core includes $x_{i_k, j_k, k}(s, t, c)$; $x_{i,j,k}(a, t, c)$ if $j > j_k$; $x_{i,j,k}(s, b, c)$ if $i > i_k$. Likewise for each $j < q$ and for each $i < p$. These specifications are consistent and lead to $(p+1)!\,(q+1)!\,(r+1)!$ spaces, of which $B_{p,q,r}$ is one. Of these spaces two are symmetrical in their relations (which $B_{p,q,r}$ is not). For $m = 4$ there is no space symmetrical in its relations.

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